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THE CLASS OF CO-NAMIOKA COMPACT SPACES IS STABLE UNDER PRODUCT

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Abstract. In an earlier paper we have established that the cartesian product of a family of co-Namioka compact spaces is co-Namioka if and only if all finite cartesian products of this family are co-Namioka. The purpose of this note is to show that the product of two co-Namioka compact spaces is always co-Namioka. The class of co-Namioka compact spaces is consequently stable under arbitrary products.

Let X be a compact space. The space X is said to be a co-Namioka compact space if, for any Baire space B the following property $\mathcal{N}(B,X)$ is satisfied:

$\mathcal{N}(B,X)$: For each separately continuous mapping $f\colon\ B\times X\to\mathbb{R}$ there is a residual set A of B such that f is continuous at each point of $A\times X$.

This class of compact spaces, which was introduced by G. Debs in [3], has been extensively studied. (Cf. [4] and [5] for more information related to this topic.) In a previous paper [1], we have been interested in the study of the stability of this class under some topological operations; in particular, whether this class is stable under products. We have shown in [1, Théorème 2.2] that this class is stable under arbitrary products if and only if it is stable under finite products; and, concerning finite products, we have established the following [1, Théorème 2.1]: For every Valdiva compact space X and for every co-Namioka compact space Y, the cartesian product $X\times Y$ is co-Namioka. Of course, these results set the problem of whether or not the product of two arbitrary co-Namioka compact spaces is co-Namioka. The aim of this note is to clarify this situation by giving a positive answer to this question. More precisely we establish the following somewhat stronger result:

Theorem 1. *Let X and Y be two compact spaces and B a Baire space. Suppose that $\mathcal{N}(B,X)$ and $\mathcal{N}(B,Y)$ are satisfied. Then the property $\mathcal{N}(B,X\times Y)$ is also satisfied.*

Proof. Let $f\colon B\times(X\times Y)\to\mathbb{R}$ be a separately continuous mapping. Let $O\subset B$ be a nonvoid open set and $\varepsilon>0$. Let us show that there exists $b\in O$ such that the oscillation of f is less than or equal to ε at each point of $\{b\}\times(X\times Y)$. This will imply the theorem. Put $\eta=\varepsilon/18$.

Fact 1. There is a nonvoid open set $U\subset O$ and points $y_1,\dots,y_n$ of Y satisfying the following condition:

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For each $(b,y) \in U \times Y$ *there exists* $i \in \{1,\dots,n\}$ *such that*

$$|f(b,x,y) - f(b,x,y_i)| \le \eta \quad \textit{for all } x \in X.$$

Assume that Fact 1 is false. We shall define a strategy σ for player β in the Choquet game on the space O (see [2] or [6] for the definition of this game). Put $\sigma(\emptyset) = O$ and let $(x_0, y_0) \in X \times Y$. At the n-th stroke, if player α has played $V_0, \dots, V_n$, then player β chooses $b_n \in V_n, y_n \in Y$ and $x_{n,i} \in X$, $0 \le i < n$, such that

$$|f(b_n, x_{n,i}, y_n) - f(b_n, x_{n,i}, y_i)| > \eta \quad \text{for all } 0 \le i < n;$$

and then β plays the nonvoid open subset $\sigma(V_1, \dots, V_n)$ of V_n given by

$$\bigcap_{i<n} \{b \in V_n / |f(b, x_{n,i}, y_n) - f(b, x_{n,i}, y_i)| > \eta\}.$$

As O is a Baire space, σ cannot be a winning strategy (cf. [6]); hence there is for α a winning game $(V_n)_{n\in\mathbb{N}}$ against σ. Let $b \in \bigcap_{n\in\mathbb{N}} V_n$. By the Stone-Weirstrass theorem there exist finitely many continuous real functions $g_1, \dots, g_k$ defined on X and $h_1, \dots, h_k$ defined on Y, such that $|f(b,x,y) - \sum_1^k g_i(x)h_i(y)| < \eta/3$ for every $(x,y) \in X \times Y$. Let $(\phi(n))_{n\in\mathbb{N}}$ be a subsequence of the natural numbers such that the sequences $(h_i(y_{\phi(n)}))$ are convergent for all $i \in \{1, \dots, k\}$. (Such a subsequence exists because all subspaces $h_i(Y)$ of $\mathbb{R}$ are compact.) Let $M \in \mathbb{R}_+^*$ such that $|g_i(x)| \le M$ for each $i \in \{1, \dots, k\}$ and for each $x \in X$. Take $n \in \mathbb{N}$ such that $|h_i(y_{\phi(n)}) - h_i(y_{\phi(n+1)})| \le \eta/(3kM)$ for all $i \in \{1, \dots, k\}$. It follows that

$$\begin{aligned} &|f(b, x_{\phi(n+1),\varphi(n)}, y_{\phi(n)}) - f(b, x_{\phi(n+1),\varphi(n)}, y_{\phi(n+1)})| \\ &\le 2\eta/3 + \Big| \sum_1^k g_i(x_{\phi(n+1),\varphi(n)})[h_i(y_{\phi(n+1)}) - h_i(y_{\phi(n)})] \Big| \\ &\le 2\eta/3 + \eta/3 = \eta, \end{aligned}$$

which is a contradiction since b belongs to $V_{\phi(n+2)}$. The proof of Fact 1 is complete.

It is clear that, similarly, we have the following:

Fact 2. For each nonempty open subset O' of B there are a nonempty open set $V \subset O'$ and points $x_1, \dots, x_m$ of X satisfying the following condition:

For each $(b,x) \in V \times X$ *there exists* $i \in \{1, \dots, m\}$ *such that*

$$|f(b,x,y) - f(b,x_i,y)| \le \eta \quad \textit{for all } y \in Y.$$

Having established these two facts, to conclude the proof we shall combine them with the properties $\mathcal{N}(B,X)$ and $\mathcal{N}(B,Y)$. First, by applying $\mathcal{N}(B,X)$ to the separately continuous functions $(b,x) \in B \times X \to f(b,x,y_i) \in \mathbb{R}$, $i \in \{1, \dots, n\}$, take a nonvoid open set $U_1 \subset U$ so that the set

$$W_1 = \bigcap_{b\in U_1} \left(\bigcap_{i\le n} \{(x,x') \in X \times X / |f(b,x,y_i) - f(b,x',y_i)| < \eta\} \right)$$

is a neighborhood of the diagonal of $X \times X$. Now, let $U_2 \subset U_1$ be a nonvoid open set and $x_1, \dots, x_m$ be in X, satisfying Fact 2. Finally, apply the property $\mathcal{N}(B,Y)$ to the functions $(b,y) \in B \times Y \to f(b,x_i,y) \in \mathbb{R}, i \in \{1, \dots, m\}$, to get a nonvoid open set $U_3 \subset U_2$ such that the set

$$W_2 = \bigcap_{b \in U_3} \left(\bigcap_{i \leq m} \{(y, y') \in Y \times Y / |f(b, x_i, y) - f(b, x_i, y')| < \eta\} \right)$$

is a neighborhood of the diagonal of $Y \times Y$.

Let us now show that the oscillation of f is less than or equal to ε at each point of $U_3 \times X \times Y$. Let $(b, x, y) \in U_3 \times X \times Y$ and put $V_1 = W_1[x]$ and $V_2 = W_2[y]$. By Fact 1, we can fix an integer $j \in \{1, \dots, n\}$ such that $|f(b, t, y) - f(b, t, y_j)| \leq \eta$ for all $t \in X$; let

$$\begin{aligned} V = U_3 &\cap \{b' \in B / |f(b', x, y) - f(b, x, y)| < \eta\} \\ &\cap \{b' \in B / |f(b', x, y) - f(b', x, y_j)| < 2\eta\} \\ &\cap \left(\bigcap_{i \leq m} \{b' \in B / |f(b', x_i, y_j) - f(b', x_i, y)| < 2\eta\} \right). \end{aligned}$$

The set V is a neighborhood of b in B. Let $(b', x', y') \in V \times V_1 \times V_2$. Since $b' \in U_2$, there exists $i \in \{1, \dots, n\}$ such that $|f(b', x', z) - f(b', x_i, z)| \leq \eta$ for each $z \in Y$; it follows that

$$\begin{aligned} |f(b, x, y) - f(b', x', y')| < &|f(b, x, y) - f(b', x, y)| \\ &+ |f(b', x, y) - f(b', x, y_j)| \\ &+ |f(b', x, y_j) - f(b', x', y_j)| \\ &+ |f(b', x', y_j) - f(b', x_i, y_j)| \\ &+ |f(b', x_i, y_j) - f(b', x_i, y)| \\ &+ |f(b', x_i, y) - f(b', x_i, y')| \\ &+ |f(b', x_i, y') - f(b', x', y')| \\ \leq &9\eta = \varepsilon/2. \end{aligned}$$

This shows that the oscillation of f at (b, x, y) is less than or equal to ε. □

A consequence of Theorem 1 is that the product of two (hence finitely many) co-Namioka compact spaces is co-Namioka. From [1, Théorème 2.2] we then get the stability result mentioned in the title:

Theorem 2. *The class of co-Namioka compact spaces is stable under arbitrary product.*

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