EVERY ČECH-ANALYTIC BAIRE SEMITOPOLOGICAL GROUP IS A TOPOLOGICAL GROUP

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Abstract. Among other things, we prove the assertion given in the title. This solves a problem of Pfister.

1. Introduction

A semitopological (respectively, paratopological group) is a group endowed with a topology for which the product is separately (respectively, jointly) continuous. In 1957 R. Ellis [6] showed that every locally compact paratopological group is a topological group. This answered a question posed by A. D. Wallace in [19]. Moreover, in [20] W. Zelazko has shown that every completely metrizable paratopological group is a topological group. Later, in 1982, N. Brand [3] generalized both Ellis’ and Zelazko’s results by proving that every Čech-complete paratopological group is a topological group. A new and short proof of this result was given by H. Pfister [17] three years later. It had been well known for many years that every locally compact or completely metrizable semitopological group is a paratopological group ([7], [16] respectively). These results motivated Pfister in [17, Remarks] to ask whether every Čech-complete semitopological group is a paratopological group and, hence by Brand’s result, a topological group.

In [2, Theorem 4.3] we show the following: Let \( G \) be a Čech-complete semitopological group. Then \( G \) is a topological group if and only if \( G \) is paracompact. (The ‘only if’ part is a result by L. G. Brown [4].) Then, to answer Pfister’s question it suffices to prove that every Čech-complete semitopological group is paracompact. The purpose of the present paper is to give a complete answer to Pfister’s problem by a different method and in a general form. We prove that every Čech-analytic Baire semitopological group is a topological group (Theorem 3.3). The class of Čech-analytic spaces was introduced by D. H. Fremlin in an unpublished note of 1980 (cf. [12]). This class is sufficiently large to include all completely metrizable or locally compact spaces, and more generally, all Čech-complete spaces.

Theorem 3.3 is stated in Section 3 as a corollary of a somewhat more general statement (Theorem 3.2). Theorem 3.2 and Theorem 3.1 are settled in terms of \( p-\sigma \)-fragmentability. In Section 2, the concept of \( p-\sigma \)-fragmentability is introduced and some auxiliary results are established.
2. Definitions and preliminaries

All topological spaces considered in this paper are supposed to be completely regular. Let \( X \) be a topological space. We shall say that a subset \( Y \) of \( X \) is fragmented by a collection \( \mathcal{U} \) of subsets of \( X \) if each nonvoid subset of \( Y \) has a nonvoid relatively open subset contained in some member of \( \mathcal{U} \). The space \( X \) is \( \sigma \)-fragmented by a cover \( \mathcal{U} \) of \( X \) (not necessarily related to the topology of \( X \)) if we can write \( X = \bigcup_{n \in \mathbb{N}} X_n \) where, for each \( n \in \mathbb{N} \), \( X_n \) is fragmented by \( \mathcal{U} \).

Recall that a sequence of covers \( (\mathcal{U}_n)_{n \in \mathbb{N}} \) of \( X \) is complete (cf. \cite[p. 278]{12}) if, whenever \( \mathcal{F} \) is a filter base on \( X \) such that each \( \mathcal{U}_n \) has a member containing some member of \( \mathcal{F} \), then \( \bigcap \{ \mathcal{F} : F \in \mathcal{F} \} \neq \emptyset \). We shall say that a sequence of covers \( (\mathcal{U}_n)_{n \in \mathbb{N}} \) of \( X \) is point-complete if, whenever \( \mathcal{F} \) is a filter base on \( X \) such that each \( \mathcal{U}_n \) has a member \( U_n \) containing some member of \( \mathcal{F} \) such that \( \bigcap_{n \in \mathbb{N}} U_n \neq \emptyset \), then \( \bigcap \{ \mathcal{F} : F \in \mathcal{F} \} \neq \emptyset \). It is clear that a complete sequence of covers of \( X \) is a point-complete sequence of covers of \( X \). We now introduce a concept which plays a fundamental role in this paper.

**Definition 2.1.** The space \( X \) is called \( p\)-\( \sigma \)-fragmentable if \( X \) has a point-complete sequence of covers \( (\mathcal{U}_n)_{n \in \mathbb{N}} \) such that:

1. \( X \) is \( \sigma \)-fragmented by each \( \mathcal{U}_n \),
2. the collection \( \{ \overline{U} : U \in \mathcal{U}_{n+1} \} \) is a refinement of \( \mathcal{U}_n \) for each \( n \in \mathbb{N} \).

We shall say that a sequence of covers of \( X \) satisfying Definition 2.1 is associated to the \( p\)-\( \sigma \)-fragmentable space \( X \).

**Examples 2.2.** (1) All metrizable and all \( \check{\text{C}} \)ech-complete spaces are \( p\)-\( \sigma \)-fragmentable. More generally all \( p \)-spaces are \( p\)-\( \sigma \)-fragmentable. Recall that a space \( X \) is a \( p \)-space \((\text{[10]})\) if \( X \) has a sequence \( (\mathcal{U}_n)_{n \in \mathbb{N}} \) of open covers such that if \( x \in X \) and for each \( n \in \mathbb{N} \) there is \( U_n \in \mathcal{U}_n \) such that \( x \in U_n \), then the set \( K = \bigcap_{n \in \mathbb{N}} U_n \) is compact and the sequence \( (\bigcap_{i \leq n} \overline{U}_i)_{n \in \mathbb{N}} \) is an outer network for \( K \). A family \( \mathcal{N} \) of subsets of \( X \) is an outer network for \( K \) if for any open subset \( U \) such that \( K \subset U \) there exists \( N \in \mathcal{N} \) such that \( K \subset N \subset U \). The concept of a \( p \)-space was introduced by A. V. Arhangel’skiĭ \([1]\) in a different but equivalent form.

(2) Let \( X \) be a space \( \sigma \)-fragmented by a lower semicontinuous metric (cf. \cite{13}). If the metric topology is finer than the original one, then \( X \) is \( p\)-\( \sigma \)-fragmented. One can use Lemma 2.4 below to show that not all spaces fragmented by a metric are \( p\)-\( \sigma \)-fragmented; this answers a question asked by the referee. This is the case of the Sorgenfrey line (see the paragraph after Lemma 2.4).

(3) \( \check{\text{C}} \)ech-analytic spaces are \( p\)-\( \sigma \)-fragmentable. (Recall that a space \( X \) is \( \check{\text{C}} \)ech-analytic \([12, \text{Theorem } 5.3]\) if \( X \) is the projection on some compactification \( X^* \) of \( X \) of the intersection of a closed set and a \( G_\delta \) subset of \( X^* \times \mathbb{N}^\mathbb{N} \).) In fact, Theorem 5.7 of \([12]\) says that a \( \check{\text{C}} \)ech-analytic space \( X \) has a complete sequence of covers \( (\mathcal{U}_n)_{n \in \mathbb{N}} \) satisfying (2) of 2.1; moreover, for each \( n \in \mathbb{N} \) one can write \( \mathcal{U}_n = \bigcup_{m \in \mathbb{N}} \mathcal{U}_{n,m} \), where for each \( m \in \mathbb{N} \), \( \mathcal{U}_{n,m} \) is an open cover of the subspace \( X_m = \bigcup_{m \in \mathbb{N}} \mathcal{U}_{n,m} \) of \( X \). Hence the space \( X \) is \( \sigma \)-fragmented by each \( \mathcal{U}_n \).

Following \([15]\) a point \( x \in X \) is called a q-point if it has a sequence of neighborhoods \( (U_n)_{n \in \mathbb{N}} \) such that if \( x_n \in U_n \), then the sequence \( (x_n)_{n \in \mathbb{N}} \) has a cluster point in \( X \). The space \( X \) is called a q-space if every \( x \in X \) is a q-point. We shall also need the following generalizations of continuity. Let \( X \) and \( Y \) be topological spaces, and let \( f : X \to Y \). The mapping \( f \) is called quasicontinuous at \( x \in X \) \([14]\)
Take a sequence of covers \( \{U_n\}_{n \in \mathbb{N}} \) associated to the p-\( \sigma \)-fragmentable space \( Y \). For each \( n \in \mathbb{N} \) let \( A_n \) be the set of points \( x \in X \) for which there is a nonvoid open set \( U \in \mathcal{U} \) such that \( f(V) \subseteq \overline{U} \). Let \( A = \bigcap_{n \in \mathbb{N}} A_n \); by Lemma 2.3 the set \( A \) is a dense \( G_\delta \) subset of \( X \). We show that \( f \) is subcontinuous at every \( x \in A \). Let \( x \in A \) and \( (x_n)_{n \in \mathbb{N}} \) be a net in \( X \) which converges to \( x \). Let \( F = \{ \{ f(x_n) : \beta \leq \alpha \} : \beta \in \Lambda \} \); we must verify that \( \bigcap \{ \overline{F} : F \in \mathcal{F} \} \neq \emptyset \). For each \( n \in \mathbb{N} \) pick \( U_n \in U_{n+1} \) and \( \beta_n \in \Lambda \) such that \( f(x) \in U_n \) and \( \{ f(x_n) : \beta_n \leq \alpha \} \subseteq U_n \); since, for each \( n \in \mathbb{N} \), the collection \( \{ U : U \in U_{n+1} \} \) is a refinement of \( U_n \), and since \( (U_n)_{n \in \mathbb{N}} \) is point-complete, we have \( \bigcap \{ \overline{U} : U \in U_{n+1} \} \neq \emptyset \). \( \square \)

**Remarks 2.5.** Let \( X \) be a Baire p-\( \sigma \)-fragmentable space and let \( \{ U_n \} \) be a sequence of covers of \( X \) associated to this space. For each \( n \in \mathbb{N} \), let \( A_n \) be the union of interiors of all elements in \( U_n \). By Lemma 2.3, the set \( A = \bigcap_{n \in \mathbb{N}} A_n \) is a dense \( G_\delta \) subset of \( X \).
(1) Every point \( x \in A \) is a q-point of \( X \). This follows from the point-completeness of the sequence \((U_n)_{n \in \mathbb{N}}\).

(2) Suppose moreover that the sequence \((U_n)_{n \in \mathbb{N}}\) is complete. Then the subspace \( A \) of \( X \) is Čech-complete. To show this fact, for each \( n \in \mathbb{N} \) and for each \( x \in A \) pick by Lemma 2.3 a neighborhood \( V_n^x \) of \( x \) in \( X \) and \( U_n^x \in \mathcal{U} \) such that \( V_n^x \subset U_n^x \subset A_n \); then the sequence of open covers \((V_n)_{n \in \mathbb{N}}\) of the space \( A \), defined by \( V_n = \{ V_n^x \cap A : x \in A \} \), is complete.

In this paper we investigate continuity of separately continuous group operations. The following lemma implies in particular that every separately continuous mapping \( f : X \times Y \to Z \), where \( X \) is a Baire p-space and \( Y \) a q-space, is quasicontinuous. Other results of the same type have been obtained in the past (see [11] and the bibliography in this paper). To establish this lemma we use the following topological game.

**Christensen’s game** (cf. [5]). Let \( X \) be a topological space. The game \( \mathcal{G}_\sigma \) is a two-player game. An instance of \( \mathcal{G}_\sigma \) is a sequence of triplets \( ((U_n,V_n,x_n))_{n \in \mathbb{N}} \) defined inductively as follows: Player \( \beta \) begins and chooses a nonempty open set \( U_0 \) of \( X \); player \( \alpha \) then chooses a nonempty open set \( V_0 \subset U_0 \) and a point \( x_0 \in X \). When \( (U_i,V_i,x_i), \, 0 \leq i \leq n - 1 \), have been defined, player \( \beta \) picks a nonempty open set \( U_n \subset V_{n-1} \) and player \( \alpha \) chooses a nonempty open set \( V_n \subset U_n \) and a point \( x_n \in X \). Player \( \alpha \) wins if \[
\left( \bigcap_{n=0}^{\infty} U_n \right) \cap \{ x_n : n \in \mathbb{N} \} \neq \emptyset.
\]

In [2, Proposition 3.6] we demonstrate that every Baire p-space \( X \) is \( \sigma \)-\( \beta \)-defavorable, which means player \( \beta \) has no winning strategy in the game \( \mathcal{G}_\sigma \) on \( X \). By Lemma 2.3 (see also 2.5) the same proof of [2, Proposition 3.6] allows more generally that every p-\( \sigma \)-fragmentable Baire space is \( \sigma \)-\( \beta \)-defavorable. We shall use this fact in this paper.

**Lemma 2.6.** Let \( X \) be a \( \sigma \)-\( \beta \)-defavorable space, \( Y \) a space with a dense subset of q-points, \( Z \) a topological space and \( f : X \times Y \to Z \) a separately continuous mapping. Then \( f \) is quasicontinuous.

**Proof.** Let us suppose the opposite. Choose a point \((a,b) \in X \times Y \) such that \( f \) is not quasicontinuous at \((a,b)\). Let \( W \) be an open set of \( Z \) which contains \( f(a,b) \) and let \( U \times V \) be an open paving of \( X \times Y \) which contains \((a,b)\), such that for each nonvoid open set \( O \subset U \times V \) one has \( f(O) \not\subset W \). Without loss of generality we may assume that \( b \) is a q-point of \( Y \). Let \( \varphi : Z \to \mathbb{R} \) be a continuous mapping such that \( \varphi(f(a,b)) = 1 \) and \( \varphi(W^c) \subset \{0\} \) (recall that \( Z \) is a completely regular space). Let \( \psi \) denote the separately continuous map \( f \circ \varphi : X \times Y \to \mathbb{R} \). Pick a sequence \((O_n)_{n \in \mathbb{N}} \) of neighborhoods associated to the q-point \( b \in Y \). We shall define a strategy \( \tau \) for player \( \beta \) in the game \( \mathcal{G}_\sigma \) on the space \( X \) as follows: To begin \( \beta \) plays the nonempty open subset \( \tau(\emptyset) = U \setminus \{x \in X : \psi(x,b) > 0\} \) of \( X \) and chooses a point \((x_0,y_0) \in \tau(\emptyset) \times (O_0 \cap V) \) such that \( \psi(x_0,y_0) = 0 \). At the \((n+1) \) th stroke, if player \( \alpha \) has played \( ((V_0,a_0), \ldots, (V_n,a_n)) \), then \( \beta \) chooses \( x_{n+1} \in V_n \) and \( y_{n+1} \in O_{n+1} \cap V \), satisfying the conditions

\[
\begin{align*}
|\psi(a_i, b) - \psi(a_i, y_{n+1})| &\leq 1/(n+1) \quad \text{for each } i = 0, \ldots, n, \\
\psi(x_{n+1}, y_{n+1}) & = 0,
\end{align*}
\]

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and β plays the nonempty open set
\[ \tau((V_0,a_0),\ldots,(V_n,a_n)) = \{ x \in V_n \mid |\psi(x,y_{n+1})| < 1/(n+1) \}. \]
Since the space X is σ-β-defeasible, there is for α a winning game \((V_n,a_n)_{n \in \mathbb{N}}\) against the strategy \(\tau\). Let \(x \in (\bigcap_{n \in \mathbb{N}} V_n) \cap \{a_n : n \in \mathbb{N}\}\) and let \(y \in Y\) be some cluster point of the sequence \((y_n)_{n \in \mathbb{N}}\) in Y. Then we easily obtain the contradiction 0 = \(\psi(x,y) = \psi(x,b) = 1\). \(\square\)

The key for the proof of our main result (Theorem 3.1) is

**Theorem 2.7.** Let \(X\) be a σ-β-defeasible space, \(Y\) a space with a dense set of q-points and \(Z\) a p-σ-fragmentable space. Suppose that the product \(X \times Y\) is a Baire space. Then every separately continuous mapping \(f : X \times Y \to Z\) is subcontinuous at each point of a dense \(G_δ\) subset of \(X \times Y\).

**Proof.** This follows from Lemma 2.6 and Lemma 2.4. \(\square\)

### 3. Applications to Semitopological Groups

Let \(G\) be a group endowed with a topology. Let us recall that the group \(G\) is said to be semitopological if for every \(g \in G\) the mappings \(h \in G \to hg \in G\) and \(h \in G \to gh \in G\) are continuous. It is called paratopological if the mapping \((g,y) \in G \times G \to gh \in G\) is continuous.

**Theorem 3.1.** Let \(G\) be a semitopological group and suppose that the product \(G \times G\) is a Baire space. If \(G\) is p-σ-fragmentable, then \(G\) is a paratopological group.

**Proof.** We must show that the product mapping \((g,h) \in G \times G \to gh \in G\) is continuous. It suffices to prove that for every net \(((g_\alpha,h_\alpha))_{\alpha \in \Lambda}\) in \(G \times G\) which converges to the point \((g,h)\), the point \(gh\) is a cluster point of the net \((g_\alpha h_\alpha)_{\alpha \in \Lambda}\) in \(G\). Let \((g_\alpha,h_\alpha)_{\alpha \in \Lambda}\) be such a net. Consider by Remarks 2.5 and Theorem 2.7 a point \((a,b) \in G \times G\) of subcontinuity of the product mapping in \(G\). We have \(\lim g_\alpha a = a\) and \(\lim h_\alpha b = b\), hence the net \((a g_\alpha^{-1} h_\alpha)_{\alpha \in \Lambda}\) has a cluster point in \(G\); as the multiplication in \(G\) is separately continuous the net \((g_\alpha h_\alpha)_{\alpha \in \Lambda}\) must have a cluster point \(y \in G\). To end the proof we show that \(y = gh\). Since \(G\) is completely regular, it is sufficient to show that \(f(gh) = f(y)\) for any continuous real function on \(G\). Let \(f : G \to \mathbb{R}\) be such a function. Let \(B\) denote the set of q-points of \(G\); \(B\) is a dense subset of \(G\) by Remarks 2.5. (Since \(G\) is homogeneous, we have \(B = G\); but we do not use this fact.) Pick \(c \in B\). The mapping \(\varphi : (u,v) \in G \times G \to f(uv) \in \mathbb{R}\) is separately continuous, hence by [2, Theorem 2.3] there is a dense subset \(A\) of \(G\) such that \(\varphi\) is continuous at every point of \(A \times \{c\}\). Let \(u \in A\). We have \(\lim u g_\alpha^{-1} a = u\) and \(\lim h_\alpha h_\alpha^{-1} c = c\), hence \(\lim f(u g_\alpha^{-1} g_\alpha h_\alpha c) = f(u c)\). Then \(f(u g_\alpha^{-1} y h_\alpha^{-1} c) = f(u c)\) for each \(u \in A\). Since \(A\) is a dense subset of \(G\), it follows that \(f(y h_\alpha^{-1} c) = f(gc)\); and since \(B\) is also dense in \(G\), we obtain \(f(y) = f(gh)\). This completes the proof. \(\square\)

Let \(G\) be a p-σ-fragmentable semitopological group. Suppose that \(G\) is p-σ-fragmented by a complete sequence of covers. If \(G\) is a Baire space, then by Remarks 2.5 \(G\) has a dense Čech-complete subspace. It follows that \(G \times G\) is a Baire space, and then by Theorem 3.1 \(G\) is a paratopological group. Now, by [2, Theorem 4.2] \(G\) is a topological group. This proves the following result.

**Theorem 3.2.** Let \(G\) be a semitopological Baire group. If \(G\) is p-σ-fragmentable by a complete sequence of covers, then \(G\) is a topological group.
Since every Čech-analytic space is p-σ-fragmentable by a complete sequence of covers (cf. 2.2(3)), the following is a corollary of 3.2.

**Theorem 3.3.** Every Čech-analytic Baire semitopological group is a topological group.

The next particular case of 3.3 answers affirmatively Pfister’s problem mentioned in the introduction.

**Corollary 3.4.** Every Čech-complete semitopological group is a topological group.

**Remark 3.5.** Brand proves in [3] that every locally Čech-complete paratopological group is a topological group. Hence, as asked by the referee, is it natural to try to prove that every locally Čech-complete semitopological group is paratopological and hence a topological group? In a private conversation J. P. Troallic solves this question as follows: Let $G$ be a locally Čech-complete semitopological group and note that $G$ is a q-space and a $σ$-$β$-defavorable space. Let $W$ be a nonvoid open Čech-complete subspace of $G$. By Lemma 2.6 the group multiplication $π : (g, h) ∈ G × G → gh ∈ G$ is quasicontinuous, hence there exists a nonvoid open paving $U × V$ of $G × G$ such that $π(U × V) ⊂ W$. Then, by Theorem 2.7, the mapping $π : U × V → W$ has at least a point of subcontinuity. Now, by the proof of Theorem 3.1, it follows that the group multiplication is continuous.

**Note.** In his (her) comments on a second version of this paper, the referee pointed out to us that E. A. Reznichenko has announced without proof in [18] our Corollary 3.4.

**References**


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