

TOPOLOGICALLY CONJUGATE KLEINIAN GROUPS

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ABSTRACT. Two Kleinian groups Γ_1 and Γ_2 are said to be topologically conjugate when there is a homeomorphism $f : S^2 \rightarrow S^2$ such that $f\Gamma_1f^{-1} = \Gamma_2$. It is conjectured that if two Kleinian groups Γ_1 and Γ_2 are topologically conjugate, one is a quasi-conformal deformation of the other. In this paper generalizing Minsky's result, we shall prove that this conjecture is true when Γ_1 is finitely generated and freely indecomposable, and the injectivity radii of all points of \mathbf{H}^3/Γ_1 and \mathbf{H}^3/Γ_2 are bounded below by a positive constant.

1. INTRODUCTION

Kleinian groups Γ_1 and Γ_2 are said to be topologically conjugate when there is a homeomorphism $f : S^2 \rightarrow S^2$ such that $\Gamma_2 = f\Gamma_1f^{-1}$. In particular, if f can be taken to be a quasi-conformal homeomorphism, we say that Γ_1 and Γ_2 are quasi-conformally conjugate. It is conjectured that topologically conjugate Kleinian groups are also quasi-conformally conjugate. (See Minsky [4].) As quasi-conformal deformations of Kleinian groups are fairly well understood, if this conjecture is solved, it will help much to understand the topological property of the actions of Kleinian groups on S^2 .

Marden proved that this conjecture is true (as a corollary of his isomorphism theorem) when the Kleinian group Γ_1 is geometrically finite ([3]). Minsky proved in [4], using his strong result on the rigidity of freely indecomposable Kleinian group with bounded injectivity radius, that when both groups are surface Kleinian groups such that there is a positive lower bound of the injectivity radii at all points of their quotient hyperbolic manifolds, then the conjecture is true. In this paper, we shall generalize Minsky's result to general freely indecomposable Kleinian groups with the same condition on the injectivity radius as above. Our result heavily depends on Minsky's theorem on the rigidity relative to the end invariant of freely indecomposable Kleinian group with bounded injectivity radius and is also motivated by his paper [4].

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2. PRELIMINARIES

Kleinian groups are discrete subgroups of the Lie group $PSL_2\mathbf{C}$. In this paper, we always assume that Kleinian groups are finitely generated and torsion free. As $PSL_2\mathbf{C}$ is the group of conformal automorphisms of S^2 , a Kleinian group acts on S^2 conformally. On the other hand, as $PSL_2\mathbf{C}$ is also the full group of orientation preserving isometries of the hyperbolic 3-space \mathbf{H}^3 (whose ideal boundary is identified with S^2 and denoted by S_∞^2), a Kleinian group Γ acts on \mathbf{H}^3 discontinuously, and the quotient \mathbf{H}^3/Γ is a complete hyperbolic 3-manifold.

We say two Kleinian groups Γ_1 and Γ_2 are topologically conjugate when there is a homeomorphism $f : S^2 \rightarrow S^2$ such that $\Gamma_2 = f\Gamma_1f^{-1}$. A homeomorphism $f : S^2 \rightarrow S^2$ is said to be quasi-conformal when there is a function $\mu : S^2 \rightarrow \mathbf{C}$ whose L^∞ -norm is less than 1 such that $f_{\bar{z}} = \mu f_z$ for any $z \in S^2$.

For a Kleinian group Γ its limit set $\Lambda_\Gamma \subset S_\infty^2$ is the closure of the set of fixed points in S_∞^2 of elements of Γ . The complement of Λ_Γ is called the region of discontinuity of Γ . Let C_Γ be the intersection of \mathbf{H}^3 and the convex hull of Λ_Γ in $\mathbf{H}^3 \cup S_\infty^2$. Then C_Γ is Γ -invariant since Λ_Γ is Γ -invariant. The quotient C_Γ/Γ is a deformation retract of \mathbf{H}^3/Γ . A Kleinian group Γ is said to be geometrically finite when C_Γ/Γ has finite volume.

We say a Kleinian group Γ is freely indecomposable when Γ is not decomposed into a non-trivial free product of its subgroups. By Bonahon's theorem, freely indecomposable Kleinian groups are geometrically tame. For the definition of geometrical tameness, refer to Thurston [7] or Bonahon [1]. In particular, every end of \mathbf{H}^3/Γ with a geometrically tame Γ has a neighbourhood which is homeomorphic to the product of a closed surface and \mathbf{R} .

For an open 3-manifold M , we call a compact 3-submanifold $C \subset M$ a core when C is a deformation retract of M . P. Scott proved that any 3-manifold with finitely generated fundamental group contains a core. If Γ is freely indecomposable, a core of \mathbf{H}^3/Γ is boundary-irreducible, i.e., every boundary component of C is incompressible. In this case moreover, a core is unique up to isotopy.

Let M be a hyperbolic 3-manifold. The injectivity radius at $z \in M$ is the infimum of the length of a non-contractible loop based at z . We shall deal with hyperbolic 3-manifolds such that the injectivity radii at all points are bounded below by a positive constant. In particular, for such a hyperbolic 3-manifold \mathbf{H}^3/Γ , the Kleinian group Γ has no parabolic elements, i.e., elements conjugate to matrices of the form $\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$.

3. THE MAIN THEOREM AND ITS PROOF

Theorem 1. *Let Γ_1 and Γ_2 be freely indecomposable Kleinian groups. Assume that there is a lower bound $\epsilon > 0$ of the injectivity radii at all points in \mathbf{H}^3/Γ_1 and \mathbf{H}^3/Γ_2 . Suppose that there exists a homeomorphism $f : S^2 \rightarrow S^2$ such that $f\Gamma_1f^{-1} = \Gamma_2$. Then there exists a quasi-conformal homeomorphism $\omega : S^2 \rightarrow S^2$ such that $\omega\gamma\omega^{-1} = f\gamma f^{-1}$ for every $\gamma \in \Gamma_1$, hence especially $\omega\Gamma_1\omega^{-1} = \Gamma_2$.*

We shall deduce Theorem 1 from the following theorem of Minsky [4]. We state his theorem in a form slightly different from the original for our convenience.

Theorem 2. *(Minsky) Let G_1 and G_2 be freely indecomposable Kleinian groups. Assume that there is a lower bound $\epsilon > 0$ of the injectivity radii at all points of \mathbf{H}^3/G_1 and \mathbf{H}^3/G_2 . Suppose that there is a homeomorphism $h : \mathbf{H}^3/G_1 \rightarrow$*

\mathbf{H}^3/G_2 such that both h and h^{-1} map ending laminations to ending laminations. (In other words, if λ is a measured lamination on a boundary component of a core S which cannot be realized by a pleated surface homotopic to the inclusion $\iota : S \rightarrow \mathbf{H}^3/G_1$, then it is not realized by a pleated surface homotopic to $f \circ \iota$ either, and the same holds for $\iota' : S \rightarrow \mathbf{H}^3/G_2$ and $f^{-1} \circ \iota'$.) Then there is a quasi-conformal homeomorphism $w : S^2 \rightarrow S^2$ such that $wgw^{-1} = h_{\#}(g)$ for every $g \in G_1$, hence especially $wG_1w^{-1} = G_2$.

We want to show that Γ_1 and Γ_2 in Theorem 1 satisfy the conditions in Theorem 2 above. For that, we shall first prove the following lemma which is an essential part of the proof.

Lemma 3. *For Γ_1 and Γ_2 in Theorem 1, there exists a homeomorphism $h : \mathbf{H}^3/\Gamma_1 \rightarrow \mathbf{H}^3/\Gamma_2$ such that $h_{\#} : \pi_1(\mathbf{H}^3/\Gamma_1) \cong \Gamma_1 \rightarrow \pi_1(\mathbf{H}^3/\Gamma_2) \cong \Gamma_2$ induces an isomorphism equivalent to the conjugation by f .*

Proof. By assumption, there is a homotopy equivalence $g : \mathbf{H}^3/\Gamma_1 \rightarrow \mathbf{H}^3/\Gamma_2$ induced by the isomorphism $\Gamma_2 \cong f\Gamma_1f^{-1}$. We have to prove that g can be homotoped to a homeomorphism.

Since we assumed that both Γ_1 and Γ_2 are freely indecomposable, there are cores C_1 of \mathbf{H}^3/Γ_1 and C_2 of \mathbf{H}^3/Γ_2 such that every component of the complement of C_i is homeomorphic to the product of the boundary component of C_i facing the component and \mathbf{R} for $i = 1, 2$. We shall prove that g can be homotoped so that $g|_{C_1}$ is a homeomorphism to C_2 .

We can assume that $\bar{g} = g|_{C_1}$ is a homotopy equivalence to C_2 because both C_1 and C_2 are cores. Therefore the point is to prove that \bar{g} preserves the peripheral structure. Then by applying Waldhausen's theorem [8], we can prove that \bar{g} is homotopic to a homeomorphism to C_2 .

Now let S be a boundary component of C_1 . Then S is incompressible since Γ_1 is freely indecomposable. We want to prove that $\bar{g}|_S$ is homotopic to a map to a boundary component of C_2 . Let Γ^S be a subgroup of Γ_1 corresponding to $\pi_1(S)$. We shall consider the following two possibilities;

- (1) Γ^S is geometrically finite,
- (2) Γ^S is geometrically infinite.

(1) *The case when Γ^S is geometrically finite.* Since Γ^S corresponds to a peripheral subgroup of $\pi_1(C_1)$, there is a component Ω^S of the region of discontinuity of Γ_1 whose stabilizer is exactly Γ^S . Then $f\Omega^S$ is a component of the region of discontinuity of Γ_2 whose stabilizer is $f\Gamma^Sf^{-1}$. It follows that $f\Gamma^Sf^{-1}$ is a component subgroup, hence corresponds to a peripheral subgroup. Therefore $g_{\#}\pi_1(S)$ is a peripheral subgroup, which implies that $\bar{g}|_S$ can be homotoped to a homeomorphism to a boundary component of C_2 .

(2) *The case when Γ^S is geometrically infinite.* Let Γ_2^S be $f\Gamma^Sf^{-1} \subset \Gamma_2$. We shall show that Γ_2^S is also geometrically infinite. Suppose that Γ_2^S is geometrically finite on the contrary. Then Γ_2^S is a quasi-Fuchsian group, hence the limit set of Γ_2^S is a Jordan curve K . Then as $\Gamma^S = f^{-1}\Gamma_2^Sf$, the limit set of Γ^S is also a Jordan curve $f^{-1}(K)$. This implies that Γ^S is also quasi-Fuchsian, and contradicts our assumption. Thus Γ_2^S is also geometrically infinite.

Since Γ_2^S is freely indecomposable (because it is isomorphic to a surface group), it must be geometrically tame by Bonahon's theorem. Let e be a geometrically infinite tame end of \mathbf{H}^3/Γ_2^S . Let $q : \mathbf{H}^3/\Gamma_2^S \rightarrow \mathbf{H}^3/\Gamma_2$ be the covering projection associated

to the inclusion $\Gamma_2^S \subset \Gamma_2$. By Theorem 9.2.2 in Thurston [7] or Lemma 2.2 in Ohshika [5], there exists a neighbourhood E of e which is homeomorphic to $S \times \mathbf{R}$ such that $q|E$ is a finite-sheeted covering to its image. This implies that there exist an end \bar{e} of \mathbf{H}^3/Γ_2 and its neighbourhood \bar{E} homeomorphic to $\Sigma \times \mathbf{R}$ such that $q|S \times \{t\}$ covers $\Sigma \times \{t\}$. We can assume then that the component of $\mathbf{H}^3/\Gamma_2 - C_2$ containing \bar{e} is \bar{E} . Now, $g|S$ lifts to a homotopy equivalence $\tilde{g} : S \rightarrow \mathbf{H}^3/\Gamma_2^S$. We can homotope g so that \tilde{g} is a homeomorphism to $S \times \{t\} \subset E$. It follows that $g|S = q \circ \tilde{g}$ is homotopic to a finite-sheeted covering map onto $\Sigma \times \{t\}$, hence to a covering map onto a boundary component of C_2 .

Thus in either case $\bar{g}|S$ is homotopic to a covering map to a boundary component of C_2 . Although the homotopy is in \mathbf{H}^3/Γ_2 , we can assume that it is in C_2 as C_2 is a core. Hence $\bar{g} : C_1 \rightarrow C_2$ can be homotoped so that $g|\partial C_1$ is a covering map to the union of components of ∂C_2 . Since \bar{g} is a homotopy equivalence, this implies that \bar{g} is homotopic to a homeomorphism to C_2 preserving $\bar{g}|\partial C_1$ by Waldhausen's theorem [8] (see also Hempel [2], Corollary 13.7). Obviously this extends to a homotopy from g to a homeomorphism from \mathbf{H}^3/Γ_1 to \mathbf{H}^3/Γ_2 as each component of $\mathbf{H}^3/\Gamma_i - C_i$ is the product of a surface and \mathbf{R} . \square

Proof of Theorem 1. By Lemma 3, there is a homeomorphism $h : \mathbf{H}^3/\Gamma_1 \rightarrow \mathbf{H}^3/\Gamma_2$ which induces an isomorphism equivalent to the conjugation by f . It remains to prove that h and h^{-1} map ending laminations to ending laminations.

Let e be an end of \mathbf{H}^3/Γ_1 for which λ is an ending lamination. Let S_1 be a boundary component of the core C_1 of \mathbf{H}^3/Γ_1 facing the end e . We can assume that λ lies on S_1 . The homeomorphism h maps C_1 to a core $h(C_1) = C_2$ of \mathbf{H}^3/Γ_2 . Let S_2 be $h(S_1)$. Let Γ_1^S be a subgroup of Γ_1 corresponding to $\pi_1(S_1)$ and let Γ_2^S be $f\Gamma_1^S f^{-1}$. The homeomorphism h is covered by a homeomorphism $\tilde{h} : \mathbf{H}^3/\Gamma_1^S \rightarrow \mathbf{H}^3/\Gamma_2^S$. The surfaces S_1 and S_2 are lifted homeomorphically to $\tilde{S}_1 \subset \mathbf{H}^3/\Gamma_1^S$ and $\tilde{S}_2 \subset \mathbf{H}^3/\Gamma_2^S$ such that $\tilde{h}(\tilde{S}_1) = \tilde{S}_2$.

Let $\tilde{\lambda}$ be a homeomorphic lift of λ into \tilde{S}_1 . Let $p_1 : \mathbf{H}^3/\Gamma_1^S \rightarrow \mathbf{H}^3/\Gamma_1$ and $p_2 : \mathbf{H}^3/\Gamma_2^S \rightarrow \mathbf{H}^3/\Gamma_2$ be the covering projections associated to the inclusions. As $f\Gamma_1^S f^{-1} = \Gamma_2^S$, by Theorem E in Minsky [4], the topological rigidity for surface groups, $\tilde{h}(\tilde{\lambda})$ is an ending lamination, which means that there is no pleated surface homotopic to the inclusion of $\tilde{S}_2 \subset \mathbf{H}^3/\Gamma_2^S$ realizing $\tilde{h}(\tilde{\lambda})$. If $h(\lambda)$ were not an ending lamination, there would be a pleated surface in \mathbf{H}^3/Γ_2 homotopic to the inclusion of S_2 realizing $h(\lambda)$, which would be lifted to a pleated surface in \mathbf{H}^3/Γ_2^S homotopic to the inclusion of \tilde{S}_2 realizing $\tilde{h}(\tilde{\lambda})$. This is a contradiction. Therefore $h(\lambda)$ is also an ending lamination.

By the same argument we can prove that if $\mu \subset S_2$ is an ending lamination, so is $h^{-1}(\mu) \subset S_1$. Thus all the conditions of Theorem 2 are satisfied for our Γ_1 and Γ_2 and it follows from Theorem 2 that there exists a quasi-conformal homeomorphism $\omega : S^2 \rightarrow S^2$ such that $\omega\gamma\omega^{-1} = h_\#(\gamma) = f\gamma f^{-1}$. This completes the proof of Theorem 1. \square

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