RIGHT ADJOINT FOR THE SMASH PRODUCT FUNCTOR

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Abstract. The smash-product functor \((-) \wedge (X, x_0)\) in the category Top, of pointed topological spaces has a right adjoint for any choice of the base point \(x_0\), if and only if the topological space \(X\) is quasi-locally compact, that is, if and only if the product functor \((-) \times X\) has a right adjoint in the category Top of topological spaces.

Introduction

A space \(X\) is cartesian in the category of topological spaces and continuous maps if the product functor \((-) \times X\) has a right adjoint. This means that there exists a proper and admissible topology on the space of maps \(Y^X\) between \(X\) and \(Y\) (for any topological space \(Y\)) [D]. Cartesian objects in Top were characterized by Day and Kelly [D-K]. They are the quasi-locally compact spaces [H-L].

The problem of the existence of a proper and admissible topology on the function space \((Y, y_0)\)\(^{(X,x_0)}\) consisting of the maps of \(Y\) to \(X\) preserving base points is related to the adjointness of the smash-product functor. It is known that this functor has a right adjoint whenever \(X\) is locally compact and Hausdorff; in this case the topology on \((Y, y_0)\)\(^{(X,x_0)}\) is the compact open topology [M].

In this paper, it is proved that the spaces \((X, x_0)\) for which the functor \((-) \wedge (X, x_0)\) has a right adjoint are exactly the spaces \(X\) which are cartesian in Top, independently of the choice of \(x_0\). That is, the existence of a proper and admissible topology on \((Y, y_0)\)\(^{(X,x_0)}\) for any \((Y, y_0)\) is equivalent to the existence of a proper and admissible topology on the whole space of maps from \(X\) to \(Y\), for any \(Y\).

Smash-product and adjunction

We can consider, in Top, the endofunctor \((-) \wedge (X, x_0)\) and ask when it has a right adjoint. When it exists we will call it \(\mathcal{G}_X(x_0)\).

In the case of \(X\) cartesian in Top, we indicate by \(Y^X\) the power object and by \((Y, y_0)\)\(^{(X,x_0)}\) the subspace of \(Y^X\) given by \(f \in Y^X\{f(x_0) = y_0\}\) with base point the constant \(y_0\)-valued map.

Theorem 1. If \(X\) is cartesian in Top, then \((-) \wedge (X, x_0)\) has a right adjoint, for every \(x_0\) in \(X\). Moreover \(\mathcal{G}_X(x_0)(Y, y_0) = (Y, y_0)\)\(^{(X,x_0)}\).
**Proof.** Suppose $X$ is cartesian in $\textbf{Top}$ and take any space $Y$; let $\hat{e}: Y^X \times X \to Y$ be the evaluation map. Consider $Y^X$ the subspace $\mathcal{G}_{(X,x_0)}(Y,y_0) = \{f \in Y^X | f(x_0) = y_0\}$ and the restriction $e_1$ of $\hat{e}$ to $\mathcal{G}_{(X,x_0)}(Y,y_0) \times (X,x_0)$ which is a map in $\textbf{Top}$. The map $e_1$ is compatible with the quotient in the definition of the smash-product, and so we can consider the map $e: \mathcal{G}_{(X,x_0)}(Y,y_0) \wedge (X,x_0) \to (Y,y_0)$ induced by $e_1$. We note that $e(f, x) = f(x)$.

Let $f: (Z, z_0) \wedge (X,x_0) \to (Y,y_0)$ be a map in $\textbf{Top}$, and consider the quotient $p: (Z, z_0) \times (X,x_0) \to (Z, z_0) \wedge (X,x_0)$ which gives the smash-product. Since $X$ is cartesian in $\textbf{Top}$, related to $fp: (Z, z_0) \times (X,x_0) \to (Y,y_0)$ there is an $(fp)_1: Z \to Y^X$ such that $\hat{e}(fp)_1 \times \text{id}_X = fp$. The map $(fp)_1$ preserves the base points and its image is a subspace of $\mathcal{G}_{(X,x_0)}(Y,y_0)$, so we can factor $(fp)_1$ through the inclusion of $\mathcal{G}_{(X,x_0)}(Y,y_0)$ in $Y^X$ and consider the first factor $(fp)_2$ as a map in $\textbf{Top}$. In such a way we obtain $(fp)_2 \times \text{id}_X: (Z, z_0) \times (X,x_0) \to (Y,y_0)$. By construction $(fp)_2 \times \text{id}_X$ is compatible with the quotient $p$, and the proof is complete.

**Theorem 2.** If the functor $(-) \wedge (X,x_0)$ has a right adjoint, then $\mathcal{G}_{(X,x_0)}(Y,y_0)$ is a space whose underlying set is in natural bijective correspondence with $(Y,y_0)^{(X,x_0)}$, the count of the adjunction is the map $e: \mathcal{G}_{(X,x_0)}(Y,y_0) \wedge (X,x_0) \to (Y,y_0)$ such that $e(f,x) = f(x)$ and the base point corresponds to the constant function valued at $y_0$.

**Proof.** Let $D_2$ be the space with two points $a, b$ and the discrete topology. By the adjunction, there is a bijection between $(Y,y_0)^{(D_2, a)} \wedge (X,x_0)$ and $(\mathcal{G}_{(X,x_0)}(Y,y_0))^{(D_2, a)}$, and on the other side $(D_2, a) \wedge (X,x_0)$ is homeomorphic to $(X,x_0)$ and $(\mathcal{G}_{(X,x_0)}(Y,y_0))^{(D_2, a)}$ is in bijection with $\mathcal{G}_{(X,x_0)}(Y,y_0)$; so the first part of the theorem is proved.

Any map $f: (X,x_0) \to (Y,y_0)$ can be considered as a map from $(D_2, a) \wedge (X,x_0)$ into $(Y,y_0)$. As a consequence, by the adjunction, for any $f$, there is an $f_1: (D_2, a) \to \mathcal{G}_{(X,x_0)}(Y,y_0)$ such that $e(f_1 \wedge \text{id}_X) = f$; so $e(f,x) = f(x)$. Finally, given the one point space $\bullet$, and the map $h: \bullet \wedge (X,x_0) \to (Y,y_0)$, there is a map $h_1: \bullet \to \mathcal{G}_{(X,x_0)}(Y,y_0)$ such that $h_1(\bullet)$ is the base point of $\mathcal{G}_{(X,x_0)}(Y,y_0)$. This completes the proof.

We denote by $S$ the Sierpinski space with the two points 0 and 1 and $\{0\}$ the nontrivial open set. If the functor $(-) \wedge (X,x_0)$ has a right adjoint, as a consequence of Theorem 2, $\mathcal{G}_{(X,x_0)}(S,0)$ can be identified with the set of the open sets $U$ of $X$ such that $x_0 \in U$ and base point the open set $X$. On the other hand, $\mathcal{G}_{(X,x_0)}(S,1)$ can be identified with the set of the open sets $U$ of $X$ such that $x_0 \notin U$ and base point the empty set.

The following Lemma characterizes convergent nets of the spaces $\mathcal{G}_{(X,x_0)}(S,0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S,1)$), while Lemma 4 proves that the open sets of these spaces are Scott-open [H-L].

**Lemma 3.** Suppose $(-) \wedge (X,x_0)$ admits a right adjoint. A net $U_i$ converges to $U$ in $\mathcal{G}_{(X,x_0)}(S,0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S,1)$) if and only if:

\[ (*) \quad \text{for each } x \in U \text{ and for each net } x_\alpha \text{ converging to } x \text{ in } X, \text{ there is an } i' \text{ and a } X' \text{ such that } x_\lambda \in U_i, \text{ for every } i > i' \text{ and } x > X'. \]

**Proof.** Let $U_i$ converge to $U$ in $\mathcal{G}_{(X,x_0)}(S,0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S,1)$), $x \in U$ and $x_\lambda$ converge to $x$ in $X$. Consider the count of the adjunction $e: \mathcal{G}_{(X,x_0)}(S,0) \wedge
(X,x₀) → (S,0) (respectively, (S,1)) and the quotient map p: G_{(X,x₀)}(S,0) × (X,x₀) → G_{(X,x₀)}(S,0) ∧ (X,x₀) (respectively, (S,1)). Since (Uᵢ, xₙ) converges to (U, x) and the map ep is continuous with ep(U, x) = 0, then ep(Uᵢ, xₙ) converges to 0. In S the nets converging to 0 are eventually constant, so there is an i' and a λ' such that, for every i > i' and λ > λ', ep(Uᵢ, xₙ) = 0, that is xₙ ∈ Uᵢ.

Vice versa, suppose Uᵢ is a net in G_{(X,x₀)}(S,0) and U ∈ G_{(X,x₀)}(S,0) (respectively, G_{(X,x₀)}(S,1)) fulfilling condition (∗). Consider the space I ∪ {•}, where I is the direct set of the net Uᵢ. • is a maximum point whose base-neighbourhoods are the sets of the form I_j = {•} ∪ {i ∈ I | i ≥ j}, j ∈ I, and the points of I are isolated. We can assume, without changing the nature of the net Uᵢ, that there exists a point i₀ ∈ I, such that Uᵢ₀ = X (respectively, Uᵢ₀ = 0). Moreover, we consider the map α: (I ∪ {•}, i₀) → G_{(X,x₀)}(S,0) (respectively, G_{(X,x₀)}(S,1)) so defined by α(0) = Uᵢ, α(•) = U. We prove that the map α is continuous, which implies the convergence of the net Uᵢ to U. By the existence of the right adjoint, α is continuous if and only if the corresponding map e(α ∧ Id_X) = π: (I ∪ {•}, i₀) ∧ (X,x₀) → (S,0) (respectively, (S,1)) is continuous; so we will prove the continuity of π. To this aim we can consider the quotient map q: (I ∪ {•}, i₀) × (X,x₀) → (I ∪ {•}, i₀) ∧ (X,x₀) induced by the smash-product and prove the continuity of πq. By the adjunction (πq)⁻¹(0) = {(i, x) | x ∈ Uᵢ, i ∈ I} ∪ {(•, x) | x ∈ U}. Since every point of I is isolated and each Uᵢ is open, the set {(i, x) | x ∈ Uᵢ, i ∈ I} is open since it is union of open sets, therefore each (i, x) of (πq)⁻¹(0) belongs to its interior. Take now (•, x) ∈ (πq)⁻¹(0) (that is x ∈ U); the topology defined on (I ∪ {•}) and condition (∗) implies that any net converging to (•, x) in the product space (I ∪ {•}, i₀) × (X,x₀) is eventually in (αq)⁻¹(0), that is, (•, x) belongs to the interior of (πq)⁻¹(0). We can conclude that (αq)⁻¹(0) is open and so, α is continuous.

Lemma 4. Suppose (−) ∧ (X,x₀) admits a right adjoint, and suppose H is open in G_{(X,x₀)}(S,0) (respectively, G_{(X,x₀)}(S,1)); then H is Scott-open [H-L], i.e.: 

(a) If U, U' are open in X, U ⊆ H and U' ⊇ U (with x₀ ∉ U' when H is open in G_{(X,x₀)}(S,1)), then U' ∈ H.
(b) If V = {Uᵢ | i ∈ I} is a family of open subsets of X and U = ∪(Uᵢ | i ∈ I) ∈ H, there exists a finite subfamily of V whose union belongs to H.

Proof. (a) Suppose U, U' open in X, U ⊆ H and U' ⊇ U (x₀ ∉ U' in the case when H is open in G_{(X,x₀)}(S,1)). The constant sequence whose value is U' converges to U in G_{(X,x₀)}(S,0) (respectively, G_{(X,x₀)}(S,1)) because it fulfils the condition (∗) in Lemma 3. Since H is open and U belongs to H, there must be an element of the constant sequence belonging to H; consequently U' ∈ H.

(b) Let U = ∪(Uᵢ | i ∈ I) ∈ H. If H is open in G_{(X,x₀)}(S,0), it follows that x₀ ∈ U; therefore there exists an i' such that x₀ ∈ Uᵢ. Consider in G_{(X,x₀)}(S,0) the net whose direct set is {(i₁, i₂, ..., iₙ) | i₁, i₂, ..., iₙ ∈ I} with the relation (i₁, i₂, ..., iₙ) ≻ (j₁, j₂, ..., jₙ) if Uᵢ₁ ∪ Uᵢ₂ ∪ ... ∪ Uᵢₙ ⊇ Uᵢ_j₁ ∪ Uᵢ_j₂ ∪ ... ∪ Uᵢ_jₙ and image of (i₁, i₂, ..., iₙ) equal to Uᵢ₁ ∪ Uᵢ₂ ∪ ... ∪ Uᵢₙ. This net, according to (∗), converges to U in G_{(X,x₀)}(S,0); as a consequence, since H is an open set which contains the limit of the net, there exists (i₁, i₂, ..., iₙ) such that Uᵢ₁ ∪ Uᵢ₂ ∪ ... ∪ Uᵢₙ ∈ H. When H is open in G_{(X,x₀)}(S,1), x₀ ∉ U and therefore, x₀ ∉ Uᵢ, for each i ∈ I. In this case, we can consider the net with the same direct set as above and with the image of (i₁, i₂, ..., iₙ) equal to Uᵢ₁ ∪ Uᵢ₂ ∪ ... ∪ Uᵢₙ. The same argument of the first case proves that there is (i₁, i₂, ..., iₙ) such that Uᵢ₁ ∪ Uᵢ₂ ∪ ... ∪ Uᵢₙ ∈ H.
Definition 5. A space $X$ is quasi-locally compact if for every $x$ in $X$ and for every neighbourhood $U$ of $x$ there is a neighbourhood of $V \subseteq U$ of $x$ such that every open cover of $U$ has a finite subcover of $V$.

Theorem 6. Let $X$ be a topological space. The following statements are equivalent:

(1) There exists $x_0$ in $X$ such that the functor $(-) \cap (X,x_0): \text{Top}, \rightarrow \text{Top}$ has a right adjoint.

(2) $X$ is cartesian in $\text{Top}$, that is $X$ is quasi-locally compact.

(3) For any $x_0$ in $X$, the functor $(-) \cap (X,x_0): \text{Top}, \rightarrow \text{Top}$ has a right adjoint.

Proof. (1) $\rightarrow$ (2) Since the continuity of the counit of the adjunction $c: \mathcal{G}(X,x_0)(S,0) \cap (X,x_0) \rightarrow (S,0)$ implies the continuity of the evaluation map $c': \mathcal{G}(X,x_0)(S,0) \times (X,x_0) \rightarrow (S,0)$, it follows that $(c')^{-1}(0)$ is an open set with $c'(W,x_0) = 0$, for any $W \in \mathcal{G}(X,x_0)(S,0)$.

Take a point $x \in X$ and fix $U$ open in $X$ with $x \in U$.

Suppose $x \in c(x_0)$; then $x_0 \in U$. Consequently $U$ belongs to $\mathcal{G}(X,x_0)(S,0)$ and $c(U,x) = 0$. Since $(c')^{-1}(0)$ is open, there is an $H$, open in $\mathcal{G}(X,x_0)(S,0)$, with $U \in H$ and a neighbourhood $V$ of $x$ such that $(c')^{-1}(0) \supseteq H \times V$. Since $c'(W,y) = 0$ when $y \in W$, any element of $H$ contains $V$. Now, consider an open cover of $U$, $\{U_i \mid i \in I\}$ and consider $U' = \bigcup\{U_i \mid i \in I\}$. Since $U' \supseteq U$, by Lemma 4a, $U' \in H$ and by Lemma 4b there are $t_1, t_2, \ldots, t_n \in I$ such that $U_{t_1} \cup U_{t_2} \cup \cdots \cup U_{t_n} \in H$ and this implies that $U_{t_1} \cup U_{t_2} \cup \cdots \cup U_{t_n} \supseteq V$.

Suppose now $x \notin c(x_0)$. Therefore, there is an open $A$ of $X$, such that $x \in A$ and $x_0 \notin A$. The open set $U \cap A \in \mathcal{G}(X,x_0)(S,1)$; the map $c: \mathcal{G}(X,x_0)(S,1) \cap (X,x_0) \rightarrow (S,1)$ is continuous and $c(U \cap A,x) = 0$. Replacing $U' = \bigcup\{U_i \mid i \in I\}$ by $U' = \bigcup\{U_i \cap A \mid i \in I\}$ in the argument used before proves that there exists a neighbourhood $V$ of $x$ with $U \cap A \supseteq V$, such that any open cover of $U$ (and then of $U \cap A$), admits a finite subcover for $V$.

(2) $\rightarrow$ (3) Theorem 1.

(3) $\rightarrow$ (1) Trivial.

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References


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