A CHARACTERIZATION OF REFLEXIVE BANACH SPACES

EVA MATOUŠKOVÁ AND CHARLES STEGALL

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Abstract. A Banach space $Z$ is not reflexive if and only if there exist a closed separable subspace $X$ of $Z$ and a convex closed subset $Q$ of $X$ with empty interior which contains translates of all compact sets in $X$. If, moreover, $Z$ is separable, then it is possible to put $X = Z$.

We consider the following problem: When does a Banach space contain a closed convex set $Q$ with empty interior which contains a translate of any compact set in $X$? The basic example of such a Banach space is the space $C(K)$ of continuous functions on a compact infinite space $K$. Indeed, it is enough to choose a point $p \in K$ which is not isolated and define $Q$ as the set of all functions in $C(K)$ which attain their minima at $p$. Since $p$ is not isolated, $Q$ has empty interior. If $K$ is a compact subset of $C(K)$, then by the Banach-Dieudonné theorem [3] there exists a sequence $\{f_n\}$ of functions in $C(K)$ converging to zero such that $K$ is contained in its closed convex hull. If we define the function $g$ by

$$g(t) := \sup\{|f_n(t) - f_n(p)| : n \in \mathbb{N}\}$$

for $t \in K$, then it is easy to check that $g$ is continuous and each function $g + f_n$ is in $Q$. Consequently, since $Q$ is convex, the translate $g + K$ is contained in $Q$.

If a Banach space $Z$ can be mapped linearly onto a Banach space $X$ containing the required set $Q$, then $Z$ also contains such a set. Namely, by the open mapping theorem, it is enough to take the preimage of $Q$. Therefore, for example, $\ell_1$ contains the required set because it can be mapped onto any separable Banach space, in particular, $C[0,1]$.

In this note we show that, in fact, any separable nonreflexive Banach space $X$ contains a closed convex set with empty interior which contains a translate of any compact set in $X$.

Borwein and Noll observed in [1] that there exist a convex continuous function on the space $c_0$ of null sequences and a closed subset $Q$ of $c_0$ which is not a Haar null set so that $f$ fails to be Fréchet differentiable on $Q$. They define $f$ as the distance from the positive cone $Q := \{\{x_n\} \in c_0; x_n \geq 0, n = 1, 2, \ldots\}$. As $Q$ has no interior points, $f$ fails to be Fréchet differentiable at all points of $Q$. The set $Q$ contains a translate of any compact set in $c_0$, and, therefore, for any

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probability Borel measure $\mu$ on $c_0$ there exists some $x \in c_0$ such that $\mu(Q+x) > 0$. Consequently, $Q$ is not Haar null (for the definition see [2]). They conjecture in [1] that also in $\ell_2$ there exists a closed convex set $C$ with empty interior which contains a translate of any compact set. We show that this is not the case in any reflexive Banach space, but on the other hand every nonreflexive Banach space has a closed subspace containing such a set.

By $B_X$ we denote the open unit ball of a Banach space $X$, and $B_X(x,r)$ is the usual notation for the open ball with center $x$ and diameter $r$; the subscript will be often omitted. We denote the closure of a set $A$ by $\overline{A}$ or $\operatorname{cl}A$.

We will make use of the following variation of the Banach-Dieudonné theorem: Let $X$ be a Banach space, $K$ a compact subset of $B_X(0,c)$ and $E$ a dense subset of $B_X(0,2c)$. Then there exists a sequence $\{F_n\}$ of finite sets in $E$ so that

$$(1) \quad K \subset \operatorname{cl}\left(\sum_{n=1}^{\infty} 2^{-n} F_n\right).$$

This follows from the fact that there exist a sequence $\{F_n\}$ of finite sets in $E$ and a sequence of compact sets $\{K_n\}$ in $B_X(0,c)$ so that

$$(2) \quad K \subset \sum_{i=1}^{n} 2^{-i} F_i + 2^{-n} K_n \quad \text{for } n \in \mathbb{N}.$$ 

Indeed, if $n = 1$, choose $F_1 \subset E$ so that $2^{-1}F_1$ is a $\frac{2}{3}$-net for $K$. Then the set

$$K_1 := 2\left((K - 2^{-1}F_1) \cap \overline{B}(0,\frac{c}{2})\right)$$

is a compact subset of $\overline{B}(0,c)$ and $K \subset 2^{-1}F_1 + 2^{-1}K_1$. Now we can continue by induction. Suppose that $F_i$ and $K_i$ for $i = 1, \ldots, n$ so that (2) holds have been already constructed. Choosing $F_{n+1} \subset E$ so that $2^{-1}F_{n+1}$ is a $\frac{2}{3}$-net for the set $K_n$ and defining

$$K_{n+1} := 2\left((K_n - 2^{-1}F_{n+1}) \cap \overline{B}(0,\frac{c}{2})\right)$$

completes the proof. The following lemmata are possibly not the most efficient way to our main result, but we think that they may be of independent interest.

**Lemma 1.** Let $Z$ be a Banach space, $U$ an open convex subset of $Z$ and $f$ a continuous real valued function defined on $U$. Then, either $f$ is affine or the convex hull $G$ of the graph of $f$ has nonempty interior.

**Proof.** Suppose that $f$ is not affine. Then there exist $x$ and $y$ in $U$ such that $1/2(f(x) + f(y)) \neq f((x+y)/2)$. Define $z_0 := (x+y)/2$ and $c := (f(x) + f(y))/2$. We can suppose by replacing $f$ by $-f$ and adding a constant, if necessary, that

$$f(x) + f(y) - 2(f(z_0)) - \alpha > 0 \quad \text{and} \quad f(z_0) > 0.$$ 

Choose some $\varepsilon > 0$ so that $0 < f(v) < f(z_0) + \alpha/2$ for every $v \in Z$ for which $\|v - z_0\| < \varepsilon$. Clearly the interior of the cone cap

$$M := \{x_{z,t} = t(z,0) + (1-t)(z_0,c) : z \in Z, \|z-z_0\| < \varepsilon, \quad 0 \leq t \leq \alpha/(2c)\}$$

is nonempty. Let some $x_{z,t} \in M$ be given, we will show that $x_{z,t} \in G$. Consider the function

$$g(s) := (1-s)c - f((1-s)z_0 + sz), \quad 0 \leq s \leq 1.$$
The function $g$ is continuous, $g(t) > 0$, and $g(1) < 0$. Therefore, there exists some $r \in (t, 1)$ for which $g(r) = 0$. Hence, $x_{x,r}$ is contained in the graph of $f$ and since

$$x_{x,t} = \frac{t}{r}x_{x,r} + (1 - \frac{t}{r})(z_0, c)$$

we have $x_{x,t} \in G$.

We say that a convex subset $Q$ of a Banach space $X$ is spanning if it contains a line segment in every direction, that is $X = \bigcup_{t \geq 0} t(Q - Q)$. Observe that if a convex set $Q$ contains a translate of every finite subset of the unit ball, then $Q$ is spanning. If $Q$ contains translates of all compact sets in $X$ (or, for that matter, of all line segments), then $X = Q - Q$. Indeed, if $x \in X$ is given, then there exists $z \in X$ so that $[z, z + x] \subset Q$, and $x = z + x - z \in Q - Q$.

**Lemma 2.** Suppose that $X$ is a Banach space and $Q \subseteq X$ is a bounded, closed and convex set with empty interior that is also spanning. Then for any compact subset $K$ of $X$ it follows that $Q + K$ also has empty interior.

**Proof.** First, we show that $Q \cap H$ is nowhere dense in $H$ if $H$ is any closed hyperplane. Suppose that $x^* \neq 0$, $w \in H = \{x^* = a\}$, $\delta > 0$ and

$$B(w, \delta) \cap H \subseteq Q \cap H.$$ 

Choose some $y \in X$ such that $(x^*, y) > 0$. Since $Q$ is spanning there exist $t > 0$, $u$ and $v$, both in $Q$, so that $t(u - v) = y$. It follows that $(x^*, u - v) > 0$ and one of $u$ or $v$, say $u$, is not in $H$. It is routine to check that the convex hull of $\{u\} \cup (B(w, \delta) \cap H)$ has an interior point relative to $X$ (try $\frac{1}{2}(u + v)$) which contradicts the fact that $Q$ has no interior. Suppose that $H \subseteq X$ is a closed hyperplane, $u \in X$, $x \notin H$ and suppose that $h^* \in H^*$. Then the set $\{y + (h^*, y)x + u : y \in H\}$ is a hyperplane in $X$ and the transformation $y \mapsto y + (h^*, y)x + u$ is an affine homeomorphism. If $x \in X$, then the set $Q' = Q + [-x, x]$ is also bounded, closed, convex and spanning. We will show that it has empty interior. Suppose that the interior of $Q'$ is nonempty. Then $x \neq 0$; choose $x^* \in X^*$ so that $(x^*, x) > 0$. Let $P$ be the projection on $X$ whose image is the kernel $H$ of $x^*$ and whose kernel is the span of $x$. The open mapping theorem says that $P(Q) = P(Q')$ has nonempty interior in $H$. Suppose that $w \in H$, $\delta > 0$ and $B(w, \delta) \cap H \subseteq P(Q)$. For $z \in B(w, \delta) \cap H$ define

$$f(z) := \inf\{t : z + tx \in Q\}.$$ 

It is easy to see that $f$ is bounded and convex, hence continuous. The mapping $(z, t) \mapsto z + tx$ is an isomorphism from $H \times R$ onto $X$ which maps the graph of $f$ onto the set $\{z + f(t)x : z \in B(w, \delta) \cap H\} \subseteq Q$. Because $Q$ has empty interior, Lemma 1 implies that $f$ must be affine, and we shall show that this leads to a contradiction. Since it is defined on an open convex subset of $H$, there exists an $h^* \in H^*$ and a real number $b$ such that $f(z) = (h^*, z) + b$. Finally,

$$\{z + (h^*, z)x + bx : z \in B(w, \delta) \cap H\} \subseteq Q$$

and this means that $Q$ contains a relatively open subset of a hyperplane, which is a contradiction. By induction, given $x_1, x_2, \ldots, x_n \in X$ we have that

$$Q + [-x_1, x_1] + \cdots + [-x_n, x_n]$$

has no interior point. The case of an arbitrary compact set $K$ can be verified by an application of (1). We give a few details. Suppose the interior of $Q + K$ is nonempty. By translating $Q + K$ if necessary we can suppose that $B(0, r) \subseteq Q + K$. 

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for some $r > 0$. Choose a sequence $\{F_n\}$ of finite subsets of a ball in $X$ so that (1) holds. Choose $n_0 \in N$ so that

$$\sum_{i=n_0}^{\infty} 2^{-i} F_i \subset B(0, r/4).$$

Because the interior of the closed and convex set $Q_0 := Q + \sum_{i=1}^{n_0} 2^{-i} \text{co} F_i$ is empty, there exists $v \in B_X(0, r)$ so that\n
$$\text{dist}(v, Q_0) > r/2.$$

To see this choose a point $y \in B(0, r/4) \setminus Q_0$ and $x^*$ in the unit sphere of $X^*$ which separates $y$ from $Q_0$, namely $r/4 \geq \langle x^*, y \rangle \geq \langle x^*, u \rangle$ for any $u \in Q_0$. Choose $x \in B(0, r)$ so that $\langle x^*, x \rangle > 3r/4$. Then $v = x$ satisfies the required inequality. From (3) and (4) follows that

$$\text{dist}(v, Q + \sum_{i=1}^{\infty} 2^{-i} F_i) \geq r/4,$$

which is a contradiction.

With the hypothesis above, observe that if $T : X \to Y$ is a surjective linear operator with finite-dimensional kernel $F$, then $T(Q)$ is a bounded, closed and convex set with empty interior that is also spanning; this is because $T^{-1}(T(X)) = Q + F$ is a first category set.

In connection with the next theorem observe that the positive cone of $\ell_2$ is a closed convex set with empty interior which contains a translate of any finite subset $F$ of $\ell_2$. (Indeed, if for $x = \{x_n\} \in \ell_2$ we define $x^- = \{x_n^*\}$ so that $x_n^* = -x_n$ if $x_n < 0$ and $x_n^* = 0$ otherwise, then the set $F + \sum_{x \in F} x^-$ is contained in the positive cone.) However, as we will see later, because $\ell_2$ is reflexive it does not contain a closed convex set with empty interior containing a translate of every compact set. Hence the boundedness hypothesis in (iv) of the next theorem is needed.

**Theorem 3.** Let $X$ be a Banach space. Then the following are equivalent:

(i) there exists a convex and closed subset $Q$ of $X$ with empty interior which contains translates of all compact sets in $X$; i.e. whenever $K$ is a compact subset of $X$ there exists $x_K \in X$ so that $K + x_K \subset Q$;

(ii) there exists a convex and closed subset $P$ of $X$ with empty interior such that if $K$ is a compact subset of the unit ball of $X$, then there exists $x_K \in X$ so that $K + x_K \subset P$;

(iii) there exists a convex, closed and bounded subset $C$ of $X$ with empty interior such that if $K$ is a compact subset of the unit ball of $X$, then there exists $x_K \in X$ so that $K + x_K \subset C$; and

(iv) there exist a dense subset $E$ of the unit ball of $X$ and a convex, closed and bounded subset $D$ of $X$ with empty interior so that whenever $F$ is a finite set contained in $E$, there exists $x_F \in X$ so that $F + x_F \subset D$.

**Proof.** Clearly (i) implies (ii). To prove that (ii) implies (iii), it is enough to show that there exists $1 \geq r > 0$ and $c > 0$ so that for any compact set $K \subset B(0, r)$ there exist $z_K \in B(0, c)$ so that $K + z_K \subset P$, for then we may define

$$C := \frac{1}{r} \left( P \cap \bar{B}(0, r + c) \right).$$
For a contradiction, suppose that for every \( n \in \mathbb{N} \) there exists a compact set \( K_n \subset B(0, 1/n) \) so that
\[
(5) \quad \text{if } K_n + x \subset P, \text{ then } \|x\| \geq n.
\]
Define
\[
K := \bigcup_{n=1}^{\infty} K_n \cup \{0\}.
\]
The set \( K \) is a compact subset of the unit ball, hence there exists \( z \in X \) such that \( K + z \subset P \). Because \( K_n \subset K \) for \( n \in \mathbb{N} \), we have \( \|z\| \geq n \) for all \( n \), which is nonsense.

Let us show now that (iii) implies (i). We can suppose that zero is not contained in \( C \) and define
\[
Q := \bigcup_{\lambda \geq 0} \lambda C.
\]
The set \( Q \) is convex and contains translates of all compact sets in \( X \). To show that \( Q \) is closed, let \( z \in X \), \( x_n \in C \) and \( \lambda_n \geq 0 \) such that \( \lim_{n \to \infty} \lambda_n x_n = z \) be given. Because the sequence \( \{x_n\} \) is bounded away from zero, the sequence \( \{\lambda_n\} \) is bounded, and consequently it has a converging subsequence \( \lambda_{n_k} \to \lambda \geq 0 \). If \( \lambda = 0 \), then from the boundedness of \( C \) it follows that \( z = 0 \in Q \). Otherwise the sequence \( \{x_{n_k}\} \) converges to \( z/\lambda \). Because \( C \) is closed we get that \( z = \lim_{k \to \infty} \lambda_{n_k} x_{n_k} = z \in \lambda C \). Finally, let us show that the set \( Q \) has empty interior. Choose some \( z \in C \).

The set \( \tilde{C} := C + [-z, 0] \) is closed and convex, and because \( C \) is spanning \( \tilde{C} \) has empty interior by Lemma 2. Since
\[
Q = \bigcup_{\lambda \geq 0} \lambda C \subset \bigcup_{n \in \mathbb{N}} n\tilde{C},
\]
it follows from the Baire theorem that the interior of \( Q \) is empty.

Clearly (iii) implies (iv), so let us show that the opposite implication also holds. Let \( K \) be a compact subset of \( B_X(0, 2^{-1}) \). We will show that \( K \) can be translated into \( D \). Then \( C := 2D \) will satisfy (iii). Let \( \{F_n\} \) be a sequence of finite sets in \( E \) so that (1) holds. Choose \( z_n \in X \) so that \( z_n + F_n \subset D \). Because \( D \) is bounded, the sequence \( \{z_n\} \) is bounded. If we define \( z := \sum_{n=1}^{\infty} (1/2^n)z_n \), we get
\[
z + K \subset z + \operatorname{cl} \sum_{n=1}^{\infty} 2^{-n}F_n \subset \operatorname{cl} \left( \sum_{n=1}^{\infty} 2^{-n}z_n + 2^{-n}F_n \right) \subset D,
\]
where the last inclusion follows from the fact that \( D \) is convex and closed.

It should be remarked here that from the proof of equivalence of (i) and (iii) of the previous theorem it follows that if a Banach space \( X \) contains a closed and convex set with empty interior containing the translates of all compacts, then \( X \) contains a closed and convex cone \( Q \) with empty interior which contains the translates of all compacts.

**Corollary 4.** Let \( Z \) be a Banach space and \( Y \) be a separable subspace of \( Z \). Let \( Z \) contain a convex closed set \( Q \) with empty interior which contains translates of all compact sets in \( Z \). Then there exist a closed, separable and linear subspace \( X \) of \( Z \) containing \( Y \) and a convex closed subset \( P \) of \( X \) with empty interior which contains translates of all compact sets in \( X \).
Proof. By Theorem 3 there exists a convex closed bounded subset $C$ of $Z$ with empty interior which contains translates of all compact subsets of $B_Z$. Using induction we construct an increasing sequence $\{X_n\}$ of closed separable subspaces of $Z$. Define $X_1 := Y$ and choose a dense countable subset $S_1$ of the unit ball of $X_1$. Choose a countable set $T_1 \subset Z$ such that whenever $F$ is a finite subset of $T_1$ there exists $v \in T_1$ for which $v + F \subset C$. Choose a countable set $C_1 \subset Z \setminus C$ such that $C_1 \supset C \cap X_1$. Suppose $X_n$, $S_n$, $T_n$ and $C_n$ for some $n \in N$ have been already constructed. Define

$$X_{n+1} := \overline{\operatorname{span}}(X_n \cup T_n \cup C_n),$$

and choose a countable dense subset $S_{n+1} \supset S_n$ of the unit ball of $X_{n+1}$. Choose a countable set $T_{n+1} \subset Z$ such that whenever $F$ is a finite subset of $S_{n+1}$ there exists $v \in T_{n+1}$ for which $v + F \subset C$. Choose a countable set $C_{n+1} \subset Z \setminus C$ such that $C_{n+1} \supset C \cap X_{n+1}$. Define

$$X := \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad D := \bigcup_{n=1}^{\infty} (X_n \cap C).$$

The set $E := \bigcup_{n=1}^{\infty} S_n$ is dense in $\bar{B}_X$ and from the construction it follows that any finite set contained in $E$ can be translated into $D$. The set $D$ is closed and convex, and it has empty interior because $\bigcup_{n=1}^{\infty} C_n \subset X \setminus D$ and $\bigcup_{n=1}^{\infty} C_n \supset D$. An application of Theorem 3 completes the proof.

The following lemma is essentially due to James [4].

Lemma 5. Let $X$ be a nonreflexive Banach space. Then there exists a sequence $\{x_n\}$ in the unit ball of $X$ and $\varepsilon > 0$ so that for any finite-dimensional subspace $Y$ of $X$ there exists $n \in N$ so that

$$\operatorname{dist}(Y, \overline{\operatorname{co}}\{x_i\}_{i=1}^{\infty}) > \varepsilon.$$ 

Proof. The unit ball $\bar{B}_X$ of $X$ is not weakly compact, therefore by the Gantmacher-Smulian theorem [3] there exists a decreasing sequence $\{C_n\}$ of nonempty, closed and convex subsets of $\bar{B}_X$ such that $\bigcap_{n=1}^{\infty} C_n = \emptyset$. We will show that there exist $\varepsilon > 0$ and a decreasing sequence of convex nonempty sets $\{D_n\}$ so that $D_n \subset C_n$ for $n \in N$ and for any compact set $K \subset X$ there exists $m \in N$ such that

$$(K + B(0, \varepsilon)) \cap D_m = \emptyset.$$ 

Suppose for a contradiction that the required sequence $\{D_n\}$ does not exist. Let $C_{1,n} := C_n$ for $n \in N$. There exists a compact convex set $K_1$ so that

$$C_{1,n} \cap (K_1 + B(0, 2^{-1})) \neq \emptyset \quad \text{for} \quad n \in N.$$ 

Let $C_{2,n} := C_{1,n} \cap (K_1 + B(0, 2^{-1}))$ for $n \in N$. In general, if the sequence $\{C_{k,n}\}$ and the compact convex set $K_k$ have been already constructed, define

$$C_{k+1,n} := C_{k,n} \cap (K_k + B(0, 2^{-k})) \quad \text{for} \quad n \in N,$$

and choose a compact convex set $K_{k+1}$ so that

$$C_{k+1,n} \cap (K_{k+1} + B(0, 2^{-k+1})) \neq \emptyset \quad \text{for} \quad n \in N.$$ 

Then $C_{k+1,n} \subset C_{k,n}$, and by induction $C_{k,n+1} \subset C_{k,n}$. In particular if we define $G_n := C_{n,n},$ then the sequence $\{G_n\}$ is decreasing, $G_n \subset C_n$ and

$$G_{n+1} \subset K_n + B(0, 2^{-n}).$$
Choose some $y_n \in G_n$. The sequence $\{y_n\}$ has a finite $\delta$-net for any $\delta > 0$. Therefore it has a converging subsequence. The limit point of this subsequence is contained in $\bigcap_{n=1}^{\infty} C_n$, which is a contradiction. Now that we have shown the existence of the sequence $\{D_n\}$, to finish the proof simply choose any $x_n \in D_n$.

Theorem 6. Let $Z$ be a Banach space. The following are equivalent:

(i) $Z$ is not reflexive;
(ii) there exist a nontrivial closed subspace $X$ of $Z$ and a convex closed subset $Q$ of $X$ with empty interior which contains translates of all compact sets in $X$, i.e. whenever $K$ is a compact subset of $X$ there exists $x_K \in X$ so that $K + x_K \subset Q$.

Moreover, if $Z$ is separable, then (ii) holds with $X = Z$.

Proof. To show that (i) implies (ii) choose any separable nonreflexive subspace $X$ of $Z$; such a space exists by the Eberlein-Smulian theorem. If $Z$ is separable let $X := Z$. Choose an increasing sequence $\{X_n\}$ of finite-dimensional subspaces of $X$ so that $X = \bigcup_{n=1}^{\infty} X_n$. Choose a sequence $\{x_n\}$ in the unit ball of $X$ and $\varepsilon > 0$ as in Lemma 5. By passing to a subsequence of $\{x_n\}$ if necessary we may suppose that

\[ \text{dist} (\text{span}(X_n \cup \{x_i\}_{i=1}^{n}), \text{co}\{x_i\}_{i=n+1}^{\infty}) > \varepsilon \quad \text{for} \quad n \in N. \]

Put $K_n := X_n \cap B_Z$, and define

\[ D := \text{cl} \bigcup_{i=1}^{\infty} (x_i + (\varepsilon/4)K_i). \]

The convex, closed and bounded set $D := (4/\varepsilon)D$ contains a translate of any finite subset of $B_X \cap \bigcup_{n=1}^{\infty} X_n$. By Theorem 3, it only remains to show that the interior of $D$ is empty. For a contradiction, suppose that the interior of $D$ is nonempty. Because $\text{co}\bigcup_{i=1}^{\infty} (x_i + (\varepsilon/4)K_i)$ is dense in $D$ there exist $n \in N$, $\alpha_i \geq 0$ and $u_i \in (\varepsilon/4)K_i$, $i = 1, \ldots, n$, so that $\sum_{i=1}^{n} \alpha_i = 1$ and the point $z := \sum_{i=1}^{n} \alpha_i (x_i + u_i)$ is contained in the interior of $D$. From (6) it follows that there exists a point $x^*$ in the unit sphere of $X^*$ so that

\[ \langle x^*, x \rangle = 0 \quad \text{for} \quad x \in X_n, \]
\[ \langle x^*, x \rangle \leq -\varepsilon/2 \quad \text{for} \quad x \in \text{co}\{x_i\}_{i=n+1}^{\infty}. \]

Choose a point $w$ in the unit sphere of $X$ for which

\[ \langle x^*, w \rangle \geq 1/2. \]

Since $z$ is an interior point of $D$, there exists an $r > 0$ so that $z + rw \in D$. Consequently, there exist $m \in N$, $m > n$, $\beta_i \geq 0$ and $v_i \in (\varepsilon/4)K_i$, $i = 1, \ldots, m$, so that $\sum_{i=1}^{m} \beta_i = 1$ and if we define $y := \sum_{i=1}^{m} \beta_i (x_i + v_i)$, then

\[ \|z + rw - y\| < r/2. \]

From the definition of $x^*$ it follows that

\[ \langle rw + z - y, x^* \rangle = r \langle w, x^* \rangle + \sum_{i=1}^{m} \alpha_i (x_i + u_i) - \beta_i (x_i + v_i), x^* \rangle - (\sum_{i=n+1}^{m} \beta_i (x_i + v_i), x^*) \]
\[ \geq r/2 + 0 - \sum_{i=n+1}^{m} \beta_i (\langle x_i, x^* \rangle + \langle v_i, x^* \rangle) \]
\[ \geq r/2 - \sum_{i=n+1}^{m} \beta_i (\varepsilon/2 + \varepsilon/4) \]
\[ \geq r/2, \]

which is a contradiction.
Now, let us prove that (ii) implies (i). By Corollary 4, we may suppose that $X$ is separable. We will show that $X$ is nonreflexive and therefore $Z$ is also nonreflexive. For a contradiction suppose that $X$ is reflexive. Choose a sequence $\{x_i\}_{i=1}^{\infty} \subset X$ that is dense in the unit sphere of $X$. Denote $K_n := \text{span}\{x_i\}_{i=1}^{n} \cap \bar{B}_X$.

Clearly $\{K_n\}$ is an increasing sequence of compact subsets of the unit ball of $X$ for which

$$\bigcup_{n=1}^{\infty} K_n = \bar{B}_X. \tag{7}$$

By Theorem 3 there exists a closed, convex and bounded subset $C$ of $X$ with empty interior which contains translates of all compact subsets of the unit ball of $X$. For $n \in \mathbb{N}$ choose $z_n \in X$ so that $z_n + K_n \subset C$. The sequence $\{z_n\}$ is bounded, therefore it has a weakly converging subsequence $\{z_{n_k}\}$. Denote $z := \text{w-lim}_{k \to \infty} z_{n_k}$. Because the set $C$ is convex and closed, it is also weakly closed. Consequently, because the sets $K_n$ are increasing, if there exists a $k \in \mathbb{N}$ so that $y \in K_{n_k}$, then $y + z \in C$. Hence,

$$z + \bar{B}_X = z + \bigcup_{k=1}^{\infty} K_{n_k} \subset C, \tag{8}$$

which, of course, means that the interior of $C$ is nonempty, which is a contradiction.

\[\square\]

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