

ON TWO-BLOCK-FACTOR SEQUENCES AND ONE-DEPENDENCE

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ABSTRACT. The distributions of two-block-factors $(f(\eta_i, \eta_{i+1}); i \geq 1)$ arising from i.i.d. sequences $(\eta_i; i \geq 1)$ are observed to coincide with the distributions of the superdiagonals $(\zeta_{i,i+1}; i \geq 1)$ of jointly exchangeable and dissociated arrays $(\zeta_{i,j}; i, j \geq 1)$. An inequality for superdiagonal probabilities of the arrays is presented. It provides, together with the observation, a simple proof of the fact that a special one-dependent Markov sequence of Aaronson, Gilat and Keane (1992) is not a two-block factor.

1. INTRODUCTION

Let $\eta = (\eta_i; i \geq 1)$ be an i.i.d. sequence with the variables ranging in a measurable space (U, \mathcal{U}) and let f be a measurable function defined on the product space $(U^{m+1}, \mathcal{U}^{m+1})$, $m \geq 0$, with its values in a measurable space (S, \mathcal{S}) . The random sequence $(f(\eta_i, \eta_{i+1}, \dots, \eta_{i+m}); i \geq 1)$ is called an $(m+1)$ -block-factor of the sequence η . The problem of determining which random sequence $\xi = (\xi_i; i \geq 1)$ is equal in distribution to an $(m+1)$ -block-factor has been likely contemplated by many probabilists during the last decades.

A sequence ξ having the distribution of an $(m+1)$ -block-factor must be, trivially, strictly stationary and m -dependent in the sense that $(\xi_i; j > i \geq 1)$ is stochastically independent of $(\xi_i; i \geq j+m)$ for every $j > 1$. The fact that these two necessary conditions are not sufficient, or in other words, that the distributions of $(m+1)$ -block-factors do not exhaust the distributions of all m -dependent stationary sequences, $m > 0$, was already mentioned in Ibragimov and Linnik [9] (see the footnote on p. 469). The first published examples of one-dependent stationary sequences which are not two-block-factors appeared much later, see [2]. Markov sequences were examined from this point of view in [1] where an example of a five-state one-dependent homogeneous Markov sequence which is not a two-block-factor is presented. Moreover, it was proved that there is no such example with less than five states. More sophisticated examples of one-dependent stationary sequences which are even not m -block-factors for any $m \geq 0$ were introduced recently in [5]. For lists of further references on the topic we refer the reader to [7] and [11].

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The aim of the present note is to relate the problem of which sequences have the distributions of $(m + 1)$ -block-factors to known results on representations of partially exchangeable random arrays. This will provide a characterization of $(m + 1)$ -block-factors in terms of jointly exchangeable and dissociated arrays. Since the generalization for $m > 1$ is straightforward, we confine our discussion, for simplicity, to the case $m = 1$. If S is a Polish space, then as a consequence of the characterization we derive that the distributions of two-block-factors form a close set under weak topology. This fact combined with the example from [2] implies that they are not dense in the set of one-dependent stationary distributions provided S has at least two elements.

Further we employ the standpoint of arrays to simplify and strengthen the results of Aaronson, Gilat and Keane [1] on the five-state Markov sequence. By transparent graphical manipulations we prove an inequality concerning the superdiagonals of arrays. This inequality enables us to modify slightly their example to obtain a five-state one-dependent Markov sequence which is not a two-block-factor and has positive probabilities of all cylinders.

2. BLOCK-FACTORS IN RANDOM ARRAYS

An array $\zeta = (\zeta_{i,j}; i, j \geq 1)$ of random variables with values in (S, \mathcal{S}) is called *jointly* (or also weakly) *exchangeable*, see [10] and [6], if its distribution coincides with the distribution of every array $(\zeta_{\pi(i), \pi(j)}; i, j \geq 1)$. Here both indices are permuted simultaneously by any permutation π which moves only a finite number of positive integers. An array ζ is *dissociated* if $(\zeta_{i,j}; k > i \geq 1, k > j \geq 1)$ is independent of $(\zeta_{i,j}; i \geq k, j \geq k, k > 1)$.

Proposition. *Let (S, \mathcal{S}) be a Borel space. A sequence $\xi = (\xi_i; i \geq 1)$ is equal in distribution to a two-block-factor if and only if there exists a jointly exchangeable and dissociated array $\zeta = (\zeta_{i,j}; i, j \geq 1)$ with the superdiagonal $(\zeta_{i,i+1}; i \geq 1)$ equal in distribution to ξ .*

Proof. If ξ has the distribution of a two-block-factor, in symbols

$$\mathcal{D}(\xi) = \mathcal{D}(f(\eta_i, \eta_{i+1}); i \geq 1),$$

then it has the distribution of the superdiagonal of the array $(f(\eta_i, \eta_j); i, j \geq 1)$ which is trivially jointly exchangeable and dissociated.

In the opposite direction, let ξ be equal in distribution to the superdiagonal of such an array ζ . Then there are real random variables $(\beta_i; i \geq 1)$ and $(\gamma_{i,j}; j > i \geq 1)$ which are altogether mutually independent and distributed uniformly on the unit interval $[0, 1]$ and there exists a measurable function g on $[0, 1]^3$ with values in S such that

$$\mathcal{D}(\zeta_{i,j}; j > i \geq 1) = \mathcal{D}(g(\beta_i, \beta_j, \gamma_{i,j}); j > i \geq 1).$$

This claim is a simple consequence of Theorem 3.1 from [10] (cf. also [3], [4] and [8]). If we take now $U = [0, 1]^2$ and $\eta_i = (\beta_i, \gamma_{i,i+1})$, then $\eta = (\eta_i; i \geq 1)$ is an i.i.d. sequence and $\mathcal{D}(\xi) = \mathcal{D}(f(\eta_i, \eta_{i+1}); i \geq 1)$ for the measurable function f given by $f((x_1, x_2), (y_1, y_2)) = g(x_1, y_1, x_2)$. \square

Consequence. *Let (S, \mathcal{S}) be a Polish space with its Borel σ -algebra of subsets. The family $\mathcal{P}_{2bf}(S)$ of all distributions of two-block-factors ranging in S is closed in the usual weak topology.*

Proof. It is a matter of an elementary Polish space calculation to verify that the mapping

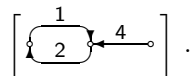
$$\lambda : \mathcal{D} (\zeta_{i,j}; j > i \geq 1) \rightarrow \mathcal{D} (\zeta_{i,i+1}; i \geq 1)$$

is continuous w.r.t. the weak topologies on the distributions of arrays and sequences. If S is compact, then the family $\mathcal{P}_{jed}(S)$ of the distributions $\mathcal{D} (\zeta_{i,j}; j > i \geq 1)$ for ζ jointly exchangeable and dissociated is compact and then by the proposition $\mathcal{P}_{2bf}(S) = \lambda(\mathcal{P}_{jed}(S))$ is compact, too. To see that $\mathcal{P}_{2bf}(S)$ is closed for noncompact S we can identify S with a $(G_\delta$ and dense) subspace of a compact Polish space T . Now, if $Q_n \in \mathcal{P}_{2bf}(S)$ converges to the distribution Q of a sequence with values in S , then also $Q_n \rightarrow Q$ when considered over the space T . But for some $P_n \in \mathcal{P}_{jed}(S)$ we have $\lambda(P_n) = Q_n$ and passing to a subsequence, if necessary, we can assume that $P_n \in \mathcal{P}_{jed}(T)$ converges to a distribution $P \in \mathcal{P}_{jed}(T)$. As $\lambda(P) = Q$, the triangular array with the distribution P ranges, in fact, in S . Namely, all its variables are equidistributed. Thus $P \in \mathcal{P}_{jed}(S)$ and then $Q \in \mathcal{P}_{2bf}(S)$. \square

Knowing from [2] that $\mathcal{P}_{2bf}(S)$ is a proper subset of the family $\mathcal{P}_{1ds}(S)$ of all distributions of one-dependent stationary sequences ranging in S one can conclude immediately that $\mathcal{P}_{2bf}(S)$ is not dense in $\mathcal{P}_{1ds}(S)$ for nontrivial S .

3. INEQUALITY FOR SUPERDIAGONALS

Let ζ be a fixed jointly exchangeable and dissociated array with values in the space $S = \{1, 2, 3, 4, 5\}$; we will have in mind mainly the “trivial” arrays $(f(\eta_i, \eta_j); i, j \geq 1)$. For our purposes it is advantageous to denote the probabilities of events like $(\zeta_{i,j} = 1, \zeta_{j,i} = 2, \zeta_{k,j} = 4)$, i, j, k distinct, by means of oriented graphs and brackets. For example, this event is represented by the 1-graph with three vertices i, j and k and with three edges $(i, j), (j, i)$ and (k, j) labeled by the elements 1, 2 and 4 of S , respectively. Its probability will be written as



The notation of vertices by i, j and k can be omitted here due to the joint exchangeability. Further, we shall abbreviate these two kinds of probabilities concerning the superdiagonal

$$\left[\overset{1}{\circ} \longrightarrow \circ \right] = [1] = P(\zeta_{i,i+1} = 1)$$

and

$$\left[\overset{1}{\circ} \longrightarrow \overset{3}{\circ} \longrightarrow \circ \right] = [13] = P(\zeta_{i,i+1} = 1, \zeta_{i+1,i+2} = 3);$$

for the “trivial” arrays the probabilities like [1] and [13] concern two-block-factors.

We shall need the following auxiliary result.

Lemma.

$$\left[\overset{s}{\circ} \longleftarrow \overset{t}{\circ} \right]^2 \leq [st], \quad s, t \in S.$$

Proof. From Theorem 3.1 of [10] we deduce that the array ζ , without its diagonal, can be represented in the form

$$(\zeta_{i,j}; i, j \geq 1, i \neq j) = (g(\beta_i, \beta_j, \gamma_{i,j}); i, j \geq 1, i \neq j)$$

with β_i a $\gamma_{i,j}$ as before but $\gamma_{i,j} = \gamma_{j,i}$. The equality is taken almost sure. Let $i, j, k \geq 1$ be distinct and $\alpha_{i,j}^s, s \in S$, denote the indicator of the event $(\zeta_{i,j} = s)$. Then by using the Cauchy inequality twice we get

$$\begin{aligned} \left[\begin{array}{c} s \\ \text{---} \\ t \end{array} \right] &= \mathbb{E}(\mathbb{E}(\alpha_{i,j}^s \alpha_{j,i}^t | \beta_j)) \\ &\leq \mathbb{E}(\mathbb{E}(\alpha_{i,j}^s | \beta_j)^{1/2} \mathbb{E}(\alpha_{j,i}^t | \beta_j)^{1/2}) \leq \mathbb{E}(\mathbb{E}(\alpha_{i,j}^s | \beta_j) \mathbb{E}(\alpha_{j,i}^t | \beta_j))^{1/2}. \end{aligned}$$

Since $\mathbb{E}(\alpha_{j,i}^t | \beta_j) = \mathbb{E}(\alpha_{j,k}^t | \beta_j)$ a.s. and $\alpha_{i,j}^s$ is conditionally independent of $\alpha_{j,k}^t$ given β_j (note that $\alpha_{i,j}^s$ is a function of β_i, β_j and $\gamma_{i,j}$), we can conclude

$$\left[\begin{array}{c} s \\ \text{---} \\ t \end{array} \right]^2 \leq \mathbb{E}(\mathbb{E}(\alpha_{i,j}^s \alpha_{j,k}^t | \beta_j)) = [st]. \quad \square$$

The following result can be formulated equivalently for the two-block-factors as well.

Theorem. *The inequality*

$$[13] + [24]^{1/2} + [32]^{1/2} + [41] + [51] + [52] + [55] \geq \frac{1}{4} [5]^2 [11][12][23]$$

is valid for every jointly exchangeable and dissociated array ζ .

Proof. We start with the calculation

$$\left[\begin{array}{c} 1 \\ \text{---} \\ 4 \end{array} \right] \left[\begin{array}{c} 1 \\ \text{---} \\ 4 \end{array} \right] = \sum_{s \in S} \left[\begin{array}{c} 1 \quad s \quad 1 \\ \text{---} \\ 4 \quad 4 \end{array} \right] \leq [41] + [24] + [13] + [41] + [51]$$

using the dissociation property. The whole proof consists exclusively of tricks of this kind. The next one is

$$[5][12] = \sum_{s \in S} \left[\begin{array}{c} 1 \quad 2 \\ \text{---} \\ 5 \quad s \end{array} \right] \leq [51] + [52] + [32] + \left[\begin{array}{c} 1 \\ \text{---} \\ 4 \end{array} \right] + [55]$$

which, together with the foregoing inequality, gives

$$[5][12] \left[\begin{array}{c} 1 \\ \text{---} \\ 4 \end{array} \right] \leq 2([41] + [51]) + [13] + [24] + [32] + [52] + [55].$$

Further the inequality

$$\begin{aligned} [23] \left[\begin{array}{c} 3 \\ \text{---} \\ 1 \quad 1 \end{array} \right] &= \sum_{s \in S} \left[\begin{array}{c} 3 \quad 1 \\ \text{---} \\ 1 \quad s \quad 1 \\ 2 \quad 3 \end{array} \right] \\ &\leq [13] + [32] + [13] + \left[\begin{array}{c} 1 \\ \text{---} \\ 4 \end{array} \right] + \left[\begin{array}{c} 2 \quad 5 \end{array} \right] \end{aligned}$$

provides the estimation

$$[5][12][23] \left[\begin{array}{c} \circ \\ \downarrow 3 \\ \circ \xrightarrow{1} \circ \xrightarrow{1} \circ \end{array} \right] \leq 3[13] + 2([32] + [41] + [51]) \\ + [24] + [52] + [55] + \left[\begin{array}{c} \circ \xrightarrow{2} \circ \xrightarrow{5} \circ \end{array} \right].$$

Similarly from

$$[5][11] = \sum_{s \in S} \left[\begin{array}{c} \circ \xrightarrow{1} \circ \xrightarrow{1} \circ \\ \downarrow s \\ \circ \xrightarrow{5} \circ \end{array} \right] \leq [51] + [52] + \left[\begin{array}{c} \circ \\ \downarrow 3 \\ \circ \xrightarrow{1} \circ \xrightarrow{1} \circ \end{array} \right] + [41] + [55]$$

we obtain

$$[5]^2[11][12][23] \leq 3([13] + [41] + [51]) + 2([32] + [52] + [55]) \\ + [24] + \left[\begin{array}{c} \circ \xrightarrow{2} \circ \xrightarrow{5} \circ \end{array} \right].$$

Finally, the casting

$$\left[\begin{array}{c} \circ \xrightarrow{2} \circ \xrightarrow{5} \circ \end{array} \right] = \sum_{s \in S} \left[\begin{array}{c} \circ \xrightarrow{2} \circ \xrightarrow{5} \circ \\ \downarrow s \\ \circ \end{array} \right] \leq [51] + [52] + \left[\begin{array}{c} \circ \xrightarrow{2} \circ \\ \downarrow 3 \\ \circ \end{array} \right] + \left[\begin{array}{c} \circ \xrightarrow{2} \circ \\ \downarrow 4 \\ \circ \end{array} \right] + [55]$$

together with the Lemma lead to the assertion of the Theorem. \square

The inequality of our Theorem improves Theorem 4 of [1] which claims

$$[13] + [24] + [32] + [41] + [51] + [52] + [55] = 0 \Rightarrow [11][12][14][23][53][54] = 0,$$

for probabilities in the two-block-factors of i.i.d. sequences.

Consequence. *The homogeneous Markov sequence on S with the uniform initial distribution and with the transition matrix*

$$\frac{1-\varepsilon}{10} \begin{pmatrix} 4 & 2 & 0 & 1 & 3 \\ 2 & 4 & 1 & 0 & 3 \\ 4 & 0 & 1 & 3 & 2 \\ 0 & 4 & 3 & 1 & 2 \\ 0 & 0 & 5 & 5 & 0 \end{pmatrix} + \frac{\varepsilon}{5} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad 0 \leq \varepsilon \leq 1,$$

is not a two-block-factor of an i.i.d. sequence if ε is close to zero. Trivially, it is one-dependent (the square of the transition matrix has constant entries) and all cylinders have positive probabilities for positive ε . \square

The Markov sequence with $\varepsilon = 0$ was found in [1]. Knowing that this sequence is not a two-block-factor of an i.i.d. sequence the Consequence follows also from the closedness of $\mathcal{P}_{2bf}(S)$.

Let us mention that in the perturbed sequences of Burton, Goulet and Meester [5] (p. 2163) cylinders have positive probabilities, too.

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