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# BEST BOUNDS FOR THE APPROXIMATE UNITS FOR CERTAIN IDEALS OF $L^1(G)$ AND OF $A_p(G)$

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Abstract. We compute the best bound for the approximate units of the augmentation ideal of the group algebra $L^1(G)$ of a locally compact amenable group $G$. More generally such a calculation is performed for the kernel of the canonical map from $L^1(G)$ onto $L^1(G/H)$, $H$ being a closed amenable subgroup of $G$. Analogous results involving certain ideals of the Fourier algebra of an amenable group are also discussed.

## 1. Introduction

Let $T_{H,q}$ be the canonical map from $L^1(G)$ onto $L^1(G/H)$ where $H$ is a closed subgroup of a locally compact group $G$. In 1968, Reiter [15] proved that if $H$ is amenable the kernel of $T_{H,q}$ admits bounded approximate right units. He showed moreover that for $H = G$ this property characterizes the amenability of $H$. In 1978 the second author [3] obtained that this is also the case for a large class of subgroups of $G$ (including all lattices in $G$). But at the present time a full converse is still in doubt.

In *loco citato* Reiter more precisely proved that the amenability of $H$ implies the existence of approximate right units for $\ker T_{H,q}$ bounded by 2. One of the main results of this work is that 2 is the **best bound** if $H$ is infinite. For $H$ finite the best bound is $\dfrac{2(|H|-1)}{|H|}$. We also investigate the corresponding results for the Fourier algebra $A(G)$ of a locally compact **amenable** group $G$. The best bound for approximate units of the ideal $I(H)$ of all $u \in A(G)$ vanishing on a closed normal subgroup $H$ of $G$ is 2 if $G/H$ is infinite. It is $\dfrac{2(|G/H|-1)}{|G/H|}$ otherwise (for $H$ open in $G$ it is not necessary to assume the normality of $H$ in $G$!).

In section 2 we essentially develop the tools which permit estimates from above and from below for the bounds of approximate units. In section 3 we obtain new bounds for ideals in $L^1(G)$ of the form $T_H^{-1}(I)$, and section 4 is devoted to the corresponding results in the Figà-Talamanca Herz algebra $A_p(G)$ (recall that $A_2(G) = A(G)$). Our main results (Theorems 5, 10 and 11) are contained in sections 5 and 6.

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## 2. Bounded approximate units and projections

We collect in this rather technical section some results concerning an arbitrary normed algebra $A$. We will apply them later to $L^1(G)$ and $A_p(G)$.

The dual $A^*$ of $A$ carries a right $A$-module structure given by $(fa)(b) = f(ab)$ for $f \in A^*$ and $a, b \in A$. We denote by $\operatorname{Hom}_A A^*$ the Banach algebra of all bounded linear operators $T$ of $A^*$ with $T(fa) = (Tf)a$ for $f \in A^*$ and $a \in A$. For $A = L^1(G)$, $\operatorname{Hom}_A A^*$ is the algebra $\operatorname{Hom}_{L^1(G)} L^\infty(G)$ of all bounded operators $T$ of $L^\infty(G)$ with $T(f * \varphi) = f * T\varphi$ for $f \in L^1(G)$, $\varphi \in L^\infty(G)$ (to $\varphi \in L^\infty(G)$, we associate $F_\varphi$ the linear functional on $L^1(G)$ defined by $F_\varphi(f) = \int_G f(x)\overline{\varphi(x)}dx$ for $f \in L^1(G)$; the above right $L^1(G)$-module structure on $L^1(G)^*$ is given by $F_\varphi g = F_{g^* * \varphi}$ where $g^*(x) = \overline{g(x^{-1})}\Delta_G(x^{-1})$).

If $A = A_p(G)$, $A^*$ is the space $PM_p(G)$ of all $p$-pseudomeasures on $G$ and $\operatorname{Hom}_A A^*$ is the Banach algebra $\operatorname{Hom}_{A_p(G)} PM_p(G)$ of all bounded linear operators $\Phi$ of $PM_p(G)$ with $\Phi(uS) = u\Phi(S)$ for all $u \in A_p(G)$ and $S \in PM_p(G)$.

**Proposition 1.** *Let $I$ be a closed left ideal of $A$ having approximate right units bounded by $C \geq 0$. Then there exists a projection $P \in \operatorname{Hom}_A A^*$ from $A^*$ onto $I^\perp$ with $||\operatorname{Id} - P|| \leq C$.*

**Proposition 2.** *Let $C \geq 0$. Assume that $A$ admits two-sided approximate units bounded by $C$. Let $I$ be a closed left ideal of $A$. Assume the existence of $P \in \operatorname{Hom}_A A^*$ which is a projection from $A^*$ onto $I^\perp$. Then for $u \in A$, $v \in I$, $f \in A^*$ and $\varepsilon > 0$ there is $w \in I$ with $||w|| \leq C||\operatorname{Id} - P||$, $||v - vw|| < \varepsilon$ and $|f(uw) - f(u) + (Pf)(u)| < \varepsilon$.*

Both propositions are essentially known. The commutative case is due to Lust-Piquard [13, pp. 7 and 15] and the general case to Forrest [7, Proposition 6.4, p. 17]. However the estimates of $||\operatorname{Id} - P||$ (Prop. 1) and $||w||$ (Prop. 2) being probably new and *a fortiori* the condition involving $f$, $u$ and $P$, we present a complete proof of Proposition 2.

If $C = 0$, then $A = \{0\}$. Suppose $C > 0$ and $||\operatorname{Id} - P|| = 0$. We have $I = \{0\}$; it suffices to choose $w = 0$. We therefore suppose $C > 0$ and $||\operatorname{Id} - P|| > 0$. Let $F$ be a finite nonempty subset of $A^*$ and $\eta > 0$. We denote by $E(F, \eta)$ the set of all $w \in I$ with $||w|| \leq C||\operatorname{Id} - P||$, $|g(v) - g(vw)| < \eta$ for every $g \in F$ and $|f(uw) - f(u) + (Pf)(u)| < \varepsilon$.

We first show that $E(F, \eta) \neq \emptyset$. There exists $u_1 \in A$ with $||u_1|| \leq C$ and

$$\begin{aligned}
||u - uu_1|| &< \frac{\varepsilon}{2(1 + ||(\operatorname{Id} - P)f||)}\,, \\
||v - vu_1|| &< \frac{\eta}{4(1 + \max_{g \in F} ||Pg||)(1 + \max_{g \in F} ||g||)}\,, \\
||v - u_1 v|| &< \frac{\eta}{4(1 + \max_{g \in F} ||Pg||)}\,.
\end{aligned}$$

For $g \in I^*$, we set $\gamma(g) = g_1(u_1) - (Pg_1)(u_1)$ where $g_1 \in A^*$ is such that the restriction of $g_1$ to $I$ is $g$. We have $\gamma \in I^{**}$ and $||\gamma|| \leq C||\operatorname{Id} - P||$. By the theorem of Goldstine, there is $w \in I$ with $||w|| \leq ||\gamma||$, $|(fu)(w) - \gamma(\operatorname{Res}_I(fu))| < \frac{\varepsilon}{2}$ and $|(gv)(w) - \gamma(\operatorname{Res}_I(gv))| < \frac{\eta}{4}$ for every $g \in F$. We obtain

$$|f(uw) - f(uu_1) + (Pf)(uu_1)| < \frac{\varepsilon}{2}$$

and therefore $|f(uw) - f(u) + (Pf)(u)| < \frac{\varepsilon}{2} + |(f - Pf)(uu_1 - u)| < \varepsilon$.

For $g \in F$, we have $|g(vw) - g(vu_1) + (Pg)(vu_1)| < \frac{\eta}{4}$. Taking into account that $u_1 v \in I$ and $(Pg)(u_1 v) = 0$, we have

$$\begin{aligned}
&|g(vw) - g(v)| \\
&\leq |g(vw) - g(vu_1) + (Pg)(vu_1)| + |g(vu_1) - g(v)| + |Pg(u_1 v) - (Pg)(vu_1)| \\
&< \frac{\eta}{4} + ||g||\,||vu_1 - v|| + ||Pg||\,||u_1 v - vu_1|| < \eta.
\end{aligned}$$

This proves that $w \in E(F, \eta)$.

Let $B$ be the set $\bigcup\{vE(F,\eta) \mid F$ nonempty finite subset of $A^*$ and $\eta > 0\}$. It is clear that $v$ lies in the $\sigma(A, A^*)$-closure of $B$ in $A$. Therefore $v$ lies in the norm closure in $A$ of the convex hull of $B$. Consequently we can find $m \in \mathbb{N}$, $F_1, \ldots, F_m$ finite nonempty subsets of $A^*$, $\eta_1, \ldots, \eta_m, c_1, \ldots, c_m > 0$, $w_1, \ldots, w_m \in I$ such that $c_1 + \ldots + c_m = 1$, $w_j \in E(F_j, \eta_j)$ $(1 \leq j \leq m)$ and $||v - \sum_{j=1}^{m} c_j v w_j|| < \varepsilon$. Consider $w = \sum_{j=1}^{m} c_j w_j$, indeed we obtain $w \in I$, $||w|| \leq C||\operatorname{Id} - P||$, $||v - vw|| < \varepsilon$ and $|f(uw) - f(u) + (Pf)(u)| < \varepsilon$.

We say that $C \geq 0$ is a bound of approximate right units of $A$ if for every $\varepsilon > 0$ and every $a \in A$ there is a $b \in A$ with $||a - ab|| < \varepsilon$ and $||b|| \leq C$. Let $\mathcal{C}$ be the set of all bounds of approximate right units. Then the infimum $D$ of $\mathcal{C}$ also is a bound of approximate right units. Let $\varepsilon > 0$ and $a \in A$; there is $C \in \mathcal{C}$ with $C < D + \eta$ where $0 < \eta < \frac{\varepsilon}{1+||a||}$; there is also $b \in A$ with $||b|| \leq C$ and $||a - ab|| < \eta$, we have $||a - ab_1|| < \varepsilon$ and $||b_1|| \leq D$ for $b_1 = \frac{Db}{D+\eta}$. We call $D$ the *best bound for the approximate right units of* $A$.

Assume that $A$ admits two-sided approximate units bounded by one. Let $I$ be a closed left ideal of $A$ having bounded approximate right units. Then the best bound for the approximate right units of $I$ is $\min\{||\operatorname{Id} - P|| \mid P \in \operatorname{Hom}_A A^*$, $P$ is a projection from $A^*$ onto $I^\perp\}$.

## 3. Bounds for approximate units of $T_H^{-1}(I)$

Let $q$ be a continuous strictly positive function on $G$ with

$$q(xh) = q(x)\Delta_H(h)\Delta_G(h^{-1}) \quad \text{for all } x \in G \text{ and } h \in H.$$

We choose measures $dx, dh, d_q\dot{x}$ on $G, H, G/H$ as in [14, p. 158]. For every $f \in L^1(G)$, we define $T_{H,q}f(\dot{x}) = \int_H \frac{f(xh)}{q(xh)} dh$ where $\dot{x} = xH = \omega(x)$. When we can choose $q = 1$ (this is the case if $H$ is normal in $G$) we will write $T_H$ instead of $T_{H,1}$.

**Theorem 3.** *Let $H$ be a closed normal amenable subgroup of $G$ and $I$ a closed left ideal of $L^1(G/H)$ having approximate right units bounded by $C \geq 0$. Then $T_H^{-1}(I)$ has approximate right units bounded by $C + 2$.*

The existence of bounded right units in $T_H^{-1}(I)$ is due to Reiter [16, p. 70]. He found for $T_H^{-1}(I)$ the bound $3C + 5$ (see pp. 31-33). Later, Doran and Wichman [6, pp. 43–44] obtained the bound $3C + 2$ with the same method.

For $F \in C^b_{\ell u}(G)^*$ ($C^b_{\ell u}(G)$ is the Banach space of all bounded left uniformly continuous functions on $G$) the relation

$$\langle f, \tau_G(F)t\rangle_{L^1(G),L^\infty(G)} = \int_G f(x)\overline{(\tau_G(F)t)(x)}dx = \overline{F(f^* * t)}$$

for $f \in L^1(G)$ and $t \in L^\infty(G)$ defines an element $\tau_G(F)$ of $\operatorname{Hom}_{L^1(G)} L^\infty(G)$. We recall that $\tau_G$ is a Banach algebra isomorphism from $C^b_{\ell u}(G)^*$ (with the Arens product) onto $\operatorname{Hom}_{L^1(G)} L^\infty(G)$; for $\Phi \in \operatorname{Hom}_{L^1(G)} L^\infty(G)$ and $t \in C^b_{\ell u}(G)$, we also have $\tau_G^{-1}(\Phi)(t) = \Phi(t)(e)$.

Let $M$ be a left-invariant mean on $C^b_{\ell u}(H)$. For $\varphi \in C^b_{\ell u}(G)$ and $x \in G$, we put $\gamma(\varphi)(\dot{x}) = M(\varphi_{x,H})$, where $\varphi_{x,H}(h) = \varphi(xh)$ for $h \in H$. It is straightforward to verify that $\alpha = \tau_G \circ \gamma^* \circ \tau_{G/H}^{-1}$ is a Banach algebra isometry from $\operatorname{Hom}_{L^1(G/H)} L^\infty(G/H)$ into $\operatorname{Hom}_{L^1(G)} L^\infty(G)$ (note the analogy with [1, Theorem 8, p. 501]). But in general $\alpha(\operatorname{Id}_{L^\infty(G/H)}) \neq \operatorname{Id}_{L^\infty(G)}$! According to Proposition 1 there is $P \in \operatorname{Hom}_{L^1(G/H)} L^\infty(G/H)$, a projection from $L^\infty(G/H)$ onto $I^\perp$, with $\|\operatorname{Id}_{L^\infty(G/H)} - P\| \leq C$. Let $t \in C^b_{\ell u}(G)$. For $f \in T_H^{-1}(I)$ we have

$$\langle f, \alpha(P)(t)\rangle_{L^1(G),L^\infty(G)} = \langle T_H f, P(\gamma(t))\rangle_{L^1(G/H),L^\infty(G/H)} = 0$$

and therefore $\alpha(P)(t) \in T_H^{-1}(I)^\perp$. From this we deduce $\alpha(P)(t) \in T_H^{-1}(I)^\perp$ for every $t \in L^\infty(G)$.

Let $t \in T_H^{-1}(I)^\perp$; for $f, g \in L^1(G)$ we have

$$\langle g * f, \alpha(P)t\rangle_{L^1(G),L^\infty(G)} = \langle T_H f, P(\gamma(g^* * t))\rangle_{L^1(G/H),L^\infty(G/H)} \ .$$

There is $u \in I^\perp \cap C^b_{\ell u}(G/H)$ with $u \circ \omega = g^* * t$, and therefore

$$\begin{aligned}\langle g * f, \alpha(P)t\rangle_{L^1(G),L^\infty(G)} &= \langle T_H f, Pu\rangle_{L^1(G/H),L^\infty(G/H)} \\ &= \langle T_H f, u\rangle_{L^1(G/H),L^\infty(G/H)} = \langle g * f, t\rangle_{L^1(G),L^\infty(G)} \ ,\end{aligned}$$

so we obtain $\alpha(P)t = t$. We have proved that $\alpha(P)$ is a projection from $L^\infty(G)$ onto $T_H^{-1}(I)^\perp$. The inequality

$$\|\operatorname{Id}_{L^\infty(G)} - \alpha(P)\| \leq \|\operatorname{Id}_{L^\infty(G)} - \alpha(\operatorname{Id}_{L^\infty(G/H)})\| + \|\operatorname{Id}_{L^\infty(G/H)} - P\|$$

permits us to conclude.

It is possible to avoid Propositions 1, 2 and the use of $\operatorname{Hom}_{L^1(G)} L^\infty(G)$. Nevertheless the following more direct proof gives perhaps less insight into the question.

Let $f \in T_H^{-1}(I)$ and $\varepsilon > 0$. There is $u \in L^1(G)$ with $\|f - f * u\|_1 < \dfrac{\varepsilon}{C+5}$ and $\|u\|_1 = 1$. By assumption there is $r \in I$ with $\|T_H f - T_H f * r\|_1 < \dfrac{\varepsilon}{C+5}$ and $\|r\|_1 \leq C$. Choose $s \in T_H^{-1}(I)$ with $T_H s = r$. Denote by $\mathcal{A}_H$ the convex hull of $\{A_h \mid h \in H\}$ where $(A_h\varphi)(x) = \varphi(xh)\Delta_G(h)$ for $\varphi \in \mathbb{C}^G$, $x \in G$ and $h \in H$. Using the amenability of $H$ we can find $A \in \mathcal{A}_H$ with $\|As\|_1 < \|r\|_1 + \eta$ where $\eta = \dfrac{\varepsilon}{(C+5)(1+\|f\|_1)}$ (see [14, p. 174]). There is also $B \in \mathcal{A}_H$ with

$$\|B(f * u - f * u * As)\|_1 < \frac{\varepsilon}{C+5} + \|T_H(f * u - f * u * As)\|_1 \ .$$

We have $u - Bu + u * BAs \in T_H^{-1}(I)$, $||f - f * (u - Bu + u * BAs)||_1 < \dfrac{\varepsilon(C+4)}{C+5}$ and $||u - Bu + u * BAs||_1 < 2 + C + \eta$. It suffices now to put

$$k = \frac{C+2}{C+2+\eta}(u - Bu + u * BAs)$$

to conclude $||k||_1 \leq C + 2$ and $||f - f * k||_1 < \varepsilon$.

## 4. Analogous result for the Figà-Talamanca Herz algebra $A_p(G)$

**Theorem 4.** *Let $G$ be an amenable locally compact group, $H$ a closed normal subgroup of $G$ and $I$ a closed ideal of $A_p(H)$. We assume that $I$ has approximate units bounded by $C$ ($C \geq 0$). Then the closed ideal $\{u \in A_p(G) \mid \operatorname{Res}_H u \in I\}$ has approximate units bounded by $C + 2$.*

For $p = 2$, the special case $I = \{0\}$, with the bound 3, was already obtained by Forrest [7, p. 6, Proposition 3.7]. More recently, Forrest [8, Proposition 3.4] treated also the corresponding result in $A_p(G)$ with a less explicit and certainly less precise constant.

Instead of using $C^b_{\ell u}(G)$, we consider $cv_p(G)$, the norm closure in the space $\mathcal{L}(L^p(G))$ of all bounded operators of all $p$-convolution operators with compact support. Let $\mathcal{P}$ be the map from $\mathcal{L}(L^p(G))$ into $\mathcal{L}(L^p(H))$ constructed in Theorem 3 of [2]. Let also $\sigma_G$ be the canonical Banach algebra isometry from $cv_p(G)^*$ onto $\operatorname{Hom}_{A_p(G)} PM_p(G)$ (see for example [1, p. 501]). The map $\lambda = \sigma_G \circ \mathcal{P}^* \circ \sigma_H^{-1}$ is a Banach algebra isometry from $\operatorname{Hom}_{A_p(H)} PM_p(H)$ into $\operatorname{Hom}_{A_p(G)} PM_p(G)$. In analogy with the $L^1$-case, we have in general $\lambda(\operatorname{Id}_{PM_p(H)}) \neq \operatorname{Id}_{PM_p(G)}$!

Let $i$ be the canonical map from $PM_p(H)$ into $PM_p(G)$ defined in [4, p. 76]. Then $i(I^\perp)$ coincides with $J^\perp$ where $J = \{u \in A_p(G) \mid \operatorname{Res}_H u \in I\}$. To verify this, consider $T \in J^\perp$, the support of $T$ lies in $H$. According to [12, p. 190, Théorème 5], there is $S \in PM_p(H)$ such that $i(S) = T$. We obtain $S \in I^\perp$. Conversely assume that $T = i(S)$ with $S \in I^\perp$. For $v \in J$ we have $\langle v, T\rangle_{A_p(G),PM_p(G)} = \langle \operatorname{Res}_H v, S\rangle_{A_p(H),PM_p(H)} = 0$ and thus $T \in J^\perp$.

For $\Phi \in \operatorname{Hom}_{A_p(H)} PM_p(H)$ we have $\lambda(\Phi) = i \circ \Phi \circ \mathcal{P}$. Take indeed $u \in A_p(G)$ and $T \in PM_p(G)$; then

$$\begin{aligned}\langle u, \lambda(\Phi)T\rangle_{A_p(G),PM_p(G)} &= \overline{\mathcal{P}^*(\sigma_H^{-1}(\Phi))(uT)} = \overline{\sigma_H^{-1}(\Phi)(\operatorname{Res}_H u\, \mathcal{P}(T))} \\ &= \langle \operatorname{Res}_H u, \Phi(\mathcal{P}(T))\rangle_{A_p(H),PM_p(H)} \\ &= \langle u, i(\Phi(\mathcal{P}(T)))\rangle_{A_p(G),PM_p(G)}\,.\end{aligned}$$

There exists $P \in \operatorname{Hom}_{A_p(H)} PM_p(H)$, a projection from $PM_p(H)$ onto $I^\perp$, with $||\operatorname{Id}_{PM_p(H)} - P|| \leq C$. The map $\lambda(P)$ is a projection from $PM_p(G)$ onto $J^\perp$. Let $T \in PM_p(G)$. For $w \in J$,

$$\langle w, \lambda(P)(T)\rangle_{A_p(G),PM_p(G)} = \langle \operatorname{Res}_H w, P(\mathcal{P}(T))\rangle_{A_p(H),PM_p(H)} = 0.$$

Therefore $\lambda(P)(T) \in J^\perp$. Let $T \in J^\perp$. There is $S \in I^\perp$ with $T = i(S)$. For

$w \in A_p(G)$ we have

$$\begin{aligned}\langle w, \lambda(P)(T)\rangle_{A_p(G),PM_p(G)} &= \langle \operatorname{Res}_H w, P(\mathcal{P}(i(S)))\rangle_{A_p(H),PM_p(H)}\\ &= \langle \operatorname{Res}_H w, P(S)\rangle_{A_p(H),PM_p(H)}\\ &= \langle \operatorname{Res}_H w, S\rangle_{A_p(H),PM_p(H)}\\ &= \langle w, i(S)\rangle_{A_p(G),PM_p(G)}\ .\end{aligned}$$

Finally, the inequality $||\operatorname{Id}_{PM_p(G)} - \lambda(P)|| \leq 2+C$ permits us as above to conclude.

It is also possible to write another more direct proof:

Let $u \in J$ and $\varepsilon > 0$. There is $v \in A_p(G)$ with $||v||_{A_p(G)} = 1$ and $||u - uv||_{A_p(G)} < \dfrac{\varepsilon}{4}$. By assumption, there exists $w \in I$ with $||\operatorname{Res}_H(uv) - w \operatorname{Res}_H(uv)||_{A_p(H)} < \dfrac{\varepsilon}{4}$ and $||w||_{A_p(H)} \leq C$. Using [9, p. 92, Theorem 1b] there is $a \in A_p(G)$ with $\operatorname{Res}_H a = w$ and $||a||_{A_p(G)} < ||w||_{A_p(H)} + \eta$ where $0 < \eta < \min\{1, \dfrac{\varepsilon}{4(C+3)(1+||u||_{A_p(G)})}\}$. By [5, p. 102, Proposition 10] there is $b \in A_p(G/H)$ with $b(\dot{e}) = 1$, $||b||_{A_p(G/H)} < 1+\eta$ and $||b\circ\omega(uv-uva)||_{A_p(G)} < \dfrac{\varepsilon}{4} + ||\operatorname{Res}_H(uv-uva)||_{A_p(H)}$. Let $d = v - b\circ\omega\, v + b\circ\omega\, va$. We have $d \in J$; from

$$||u - ud||_{A_p(G)} \leq ||u - uv||_{A_p(G)} + ||b\circ\omega(uv - uva)||_{A_p(G)},$$

we deduce $||u-ud||_{A_p(G)} < \dfrac{3\varepsilon}{4}$. Moreover we have $||d||_{A_p(G)} < C+2+2\eta+\eta C+\eta^2$. Consider $f = \dfrac{(C+2)d}{C+2+2\eta+\eta C+\eta^2}$. We have $f \in J$ and $||f||_{A_p(G)} \leq C+2$. From $||u - uf||_{A_p(G)} \leq ||u - ud||_{A_p(G)} + ||ud - uf||_{A_p(G)}$ and

$$||ud - uf||_{A_p(G)} \leq (2\eta + \eta C + \eta^2)||u||_{A_p(G)}$$

we obtain $||ud - uf||_{A_p(G)} < \dfrac{\varepsilon}{4}$ and finally $||u - uf||_{A_p(G)} < \varepsilon$.

## 5. Best bound for the approximate units of the ideal $\ker T_{H,q}$

**Theorem 5.** *Let $H$ be a closed amenable subgroup of $G$. The best bound for the right approximate units of $\ker T_{H,q}$ is 2 if $H$ is infinite. It is $\dfrac{2(|H|-1)}{|H|}$ otherwise.*

This theorem is a consequence of the following two propositions.

**Proposition 6.** *Let $P \in \operatorname{Hom}_{L^1(G)} L^\infty(G)$ which is a projection from $L^\infty(G)$ onto $\ker T_{H,q}{}^\perp$, $H$ being a closed noncompact subgroup of $G$. Then we have*

1) *$P(f) = 0$ for all $f \in C_0(G)$ (the set of all continuous functions on $G$ vanishing at infinity).*
2) *$||\operatorname{Id} - P|| \geq 2$.*

The existence of $P$ is, for $H$ amenable, a consequence of Proposition 1.

To prove 1), observe first that $P(t) \in C^b_{\ell u}(G)$ for $t \in C^b_{\ell u}(G)$. For every $r \in C_{00}(G)$ (i.e. $r$ is a continuous function with compact support on $G$), $t \in C^b_{\ell u}(G)$, $h \in H$, we have $\langle r - r_{h^{-1}}\Delta_G(h^{-1}), P(t)\rangle_{L^1(G),L^\infty(G)} = 0$ (for $a, x \in G$, $\varphi \in \mathbb{C}^G$,

${}_a\varphi(x) = \varphi(ax)$, $\varphi_a(x) = \varphi(xa)$). This implies $P(t)_h = P(t)$ and consequently $P({}_ht)(e) = P(t)(e)$. Now let $f \in C_{00}(G)$. There is a sequence $(h_n)_{n=1}^{\infty} \subset H$ such that $\operatorname{supp}_{h_n} f \cap \operatorname{supp}_{h_m} f = \emptyset$ for $n \neq m$. For $N \in \mathbb{N}$ we have

$$\left| P\left( \sum_{k=1}^{N} {}_{h_k} f \right)(e) \right| = |N(Pf)(e)| \leq ||P||\, ||f||_\infty \ ,$$

which implies $P(f)(e) = 0$. Consequently, for every $x \in G$, $P({}_xf)(e) = 0$ and therefore $P(f) = 0$.

To prove 2), it suffices to choose $f \in C_{00}(G)$ with $0 \leq f \leq 1_G$ and $f(e) = 1$. The function $g = 2f - 1_G$ satisfies the following properties: $g \in C^b_{\ell u}(G)$, $||g||_\infty = 1$ and $(\operatorname{Id} - P)g = 2f$. We finally obtain $||\operatorname{Id} - P|| \geq 2$.

**Proposition 7.** *Let $H$ be a compact subgroup of $G$ and $P \in \operatorname{Hom}_{L^1(G)} L^\infty(G)$ a projection from $L^\infty(G)$ onto $\ker {T_H}^\perp$. If $H$ is infinite, then $||\operatorname{Id} - P|| \geq 2$ and $||\operatorname{Id} - P|| \geq \dfrac{2(|H|-1)}{|H|}$ otherwise.*

*Preliminary remark.* If $H$ is a finite, $P(t) = \dfrac{1}{|H|} \displaystyle\sum_{h \in H} t_h$ (for $t \in L^\infty(G)$) defines $P \in \operatorname{Hom}_{L^1(G)} L^\infty(G)$, which is a projection from $L^\infty(G)$ onto $\ker {T_H}^\perp$. We have $||\operatorname{Id} - P|| \leq \dfrac{2(|H|-1)}{|H|}$.

*Proof.* 1) We have $(Pf)(e) = \displaystyle\int_H f(h)dh$ for all $f \in C_0(G)$ such that ${}_hf = f_h$ for every $h \in H$.

Using the continuity of the map $h \mapsto f_h$ from $H$ into $C_0(G)$, we obtain the existence and the unicity of $g \in C_0(G)$ such that $L(g) = \displaystyle\int_H L(f_h)dh$ for every $L \in C_0(G)^*$. It follows that $g(x) = \displaystyle\int_H f(xh)dh$ for every $x \in G$ and consequently $P(g) = g$. We also have

$$(Pg)(e) = \int_H P(f_h)(e)dh = \int_H P({}_hf)(e)dh = \int_H (Pf)(h)dh = \int_H P(f)(e)dh.$$

We therefore conclude that $g(e) = (Pf)(e)$.

2) Assume that $H$ is infinite. Let $\varepsilon > 0$. There is an open neighbourhood $U$ of $e$ in $G$ such that $m_H(H \cap U) < \dfrac{\varepsilon}{2}$ where $m_H$ is the normalized Haar measure of $H$. There is also a compact neighbourhood $V$ of $e$ in $G$ with $V = V^{-1}$, $V^2 \subset U$ and $hV = Vh$ for every $h \in H$. Consider then $\varphi = \dfrac{1_V * 1_V}{m_G(V)}$. We have ${}_h\varphi = \varphi_h$ for every $h \in H$, $2\varphi - 1_G \in C^b_{\ell u}(G)$, $(2\varphi - 1_G)(e) = 1$, $||2\varphi - 1_G||_\infty = 1$ and $(\operatorname{Id} - P)(2\varphi - 1_G)(e) = 2 - 2P(\varphi)(e)$. From

$$P(\varphi)(e) = \int_H \varphi(h)dh \leq m_H(U \cap H) < \frac{\varepsilon}{2},$$

we deduce $||\operatorname{Id} - P|| > 2 - \varepsilon$.

3) In the finite case it suffices to consider an open neighbourhood $U$ of $e$ in $G$ with $U \cap H = \{e\}$. We choose then $V$ and $\varphi$ as in 2). We obtain

$$(\mathrm{Id} - P)(2\varphi - 1_G)(e) = 2 - 2P(\varphi)(e) = 2 - \frac{2}{|H|}\sum_{h\in H}\varphi(h) = 2 - \frac{2}{|H|}$$

and therefore $||\,\mathrm{Id} - P|| \geq \dfrac{2(|H|-1)}{|H|}$.

It would be interesting (for $H$ amenable) to obtain a description of the set of all projections $P$ from $L^\infty(G)$ onto $\ker T_H{}^\perp$ with $P \in \mathrm{Hom}_{L^1(G)} L^\infty(G)$. For $H$ compact and *normal* in $G$ we can show that this set consists of a unique element given by $P(t)(x) = \displaystyle\int_H t(xh)dh$ for $t \in C^b_{lu}(G)$.

Let $f \in L^1(G)$, $t \in L^\infty(G)$ and $\varepsilon > 0$. By Proposition 2 there is $g \in \ker T_H$ with $||g||_1 \leq ||\,\mathrm{Id} - P||$, $||f - T_H f \circ \omega - (f - T_H f \circ \omega) * g|| < \dfrac{\varepsilon}{2(1 + ||t||_\infty)}$ and $|\langle f, P(t)\rangle_{L^1(G),L^\infty(G)} - \langle f, t\rangle_{L^1(G),L^\infty(G)} + \langle f * g, t\rangle_{L^1(G),L^\infty(G)}| < \dfrac{\varepsilon}{2}$. The subgroup $H$ being normal in $G$, for every $x \in G$ we have

$$(T_H f \circ \omega) * g(x) = \int_{G/H} (T_H f)(\omega(x)\dot{y}^{-1})\Delta_G(y^{-1})\Big(\int_H g(yh)dh\Big)d\dot{y}$$

and therefore $(T_H f \circ \omega) * g = 0$. Taking into account that

$$\begin{aligned}
&|\langle f, P(t)\rangle_{L^1(G),L^\infty(G)} - \langle T_H f \circ \omega, t\rangle_{L^1(G),L^\infty(G)}| \\
&\quad \leq |\langle f, P(t)\rangle_{L^1(G),L^\infty(G)} - \langle f, t\rangle_{L^1(G),L^\infty(G)} + \langle f * g, t\rangle_{L^1(G),L^\infty(G)}| \\
&\qquad + |\langle f - T_H f \circ \omega - (f - T_H f \circ \omega) * g, t\rangle_{L^1(G),L^\infty(G)}|
\end{aligned}$$

we obtain that $|\langle f, P(t)\rangle_{L^1(G),L^\infty(G)} - \langle T_H f \circ \omega, t\rangle_{L^1(G),L^\infty(G)}| < \varepsilon$. In other words $\langle f, P(t)\rangle_{L^1(G),L^\infty(G)} = \langle T_H f \circ \omega, t\rangle_{L^1(G),L^\infty(G)}$. This implies for $t \in C^b_{lu}(G)$ and $x \in G$ that $P(t)(x) = \displaystyle\int_H t(xh)dh$.

## 6. Best bound for the approximate units of certain ideals of the Fourier algebra

For an arbitrary subset $F$ of $G$ we denote by $I(F)$ the closed ideal of $A_p(G)$ consisting of those functions vanishing on $F$. Motivated by the assertion 1) of Proposition 6 we first prove the following result.

**Proposition 8.** *Let $G$ be amenable and let $H$ be a closed normal nonopen subgroup of $G$. Let $P \in \mathrm{Hom}_{A_p(G)} PM_p(G)$ be a projection from $PM_p(G)$ onto $I(H)^\perp$. Then $P(T) = 0$ for every $T \in PF_p(G)$.*

$PF_p(G)$ is the norm closure of $L^1(G)$ in $PM_p(G)$. For $G$ abelian $PF_2(G)$ is, via the Fourier transform, isomorphic to $C_0(\hat{G})$. To $P$ there corresponds $\hat{P} \in \mathrm{Hom}_{L^1(\hat{G})} L^\infty(\hat{G})$, a projection from $L^\infty(\hat{G})$ onto $(\ker T_{H^\perp})^\perp$ where $H^\perp$ is the set of all continous characters of $G$ equal to 1 on $H$. Moreover $H$ is nonopen if and only if $H^\perp$ is noncompact.

The existence of $P$ (in Proposition 8) is a consequence of Proposition 1 and Theorem 4.

Let $f \in C_{00}(G)$, $K = \operatorname{supp} f^*$, $u \in A_p(G)$, $\varepsilon > 0$ and $U$ be an open neighbourhood of $H \cap K$ in $G$ such that

$$m_G(U) < \frac{\varepsilon}{4(||f^*||_\infty + 1)(||u||_{A_p(G)} + 1)(||\operatorname{Id} - P|| + 1)}\,.$$

It is possible to choose $v \in I(H) \cap C_{00}(G)$ with $v = 1$ on $K_1 = K - U$. By Proposition 2 there is $w \in I(H)$ with $||w||_{A_p(G)} \leq ||\operatorname{Id} - P||$,

$$||vw - v||_{A_p(G)} < \frac{\varepsilon}{4(||f||_1 + 1)(||u||_{A_p(G)} + 1)}$$

and

$$\begin{aligned}&|\langle uw, \lambda_G^p(f)\rangle_{A_p(G),PM_p(G)}\\ &\quad - \langle u, \lambda_G^p(f)\rangle_{A_p(G),PM_p(G)} + \langle u, P(\lambda_G^p(f))\rangle_{A_p(G),PM_p(G)}| < \frac{\varepsilon}{2}\end{aligned}$$

(for a bounded measure $\mu$, $\lambda_G^p(\mu)$ is the convolution operator defined by $\lambda_G^p(\mu)(\varphi)(x) = \int_G \varphi(xy)\Delta_G(y)^{1/p} d\mu(y)$). We obtain the estimate

$$|\langle u, P(\lambda_G^p(f))\rangle_{A_p(G),PM_p(G)}| < \frac{\varepsilon}{2} + |\langle u - uw, \lambda_G^p(f)\rangle_{A_p(G),PM_p(G)}|\,.$$

Taking into account that

$$\langle u - uw, \lambda_G^p(f)\rangle_{A_p(G),PM_p(G)} = \int_G (u(x) - u(x)w(x)) f^*(x) dx\,,$$

we can write

$$\begin{aligned}&|\langle u - uw, \lambda_G^p(f)\rangle_{A_p(G),PM_p(G)}|\\ &\quad \leq \int_{K_1} |u(x)v(x) - v(x)u(x)w(x)|\,|f^*(x)|\,dx\\ &\qquad + \int_{K\cap U} |u(x) - u(x)w(x)|\,|f^*(x)|\,dx\,.\end{aligned}$$

We estimate $\int_{K_1} |u(x)v(x) - v(x)u(x)w(x)|\,|f^*(x)|\,dx$ by $||u||_{A_p(G)}||v - vw||_{A_p(G)}||f||_1$ and $\int_{K\cap U} |u(x) - u(x)w(x)|\,|f^*(x)|\,dx$ by $m_G(K \cap U)(||u||_\infty + ||u||_\infty||w||_\infty)||f^*||_\infty$. We obtain therefore that $|\langle u - uw, \lambda_G^p(f)\rangle_{A_p(G),PM_p(G)}| < \frac{\varepsilon}{2}$ and finally

$$|\langle u, P(\lambda_G^p(f))\rangle_{A_p(G),PM_p(G)}| < \varepsilon.$$

We also need an analog of the assertion 2) of Proposition 6:

**Proposition 9.** *There is $T \in PF_2(G)$ with $|||T|||_2 = 1$ and $|||\operatorname{Id}_{L^2(G)} - 2T|||_2 = 1$.*

Let $f \in C_{00}(G)$ such that $f \neq 0$ and $f = f^*$. Consider the $C^*$-algebra $\overline{\lambda_G^2(M^1(G))}$ and denote by $A$ and $B$ the $C^*$-subalgebras generated by $\lambda_G^2(f)$, respectively $\lambda_G^2(f)$ and $\operatorname{Id}_{L^2(G)}$. Of course $A$ and $B$ are abelian and $B$ is unital. Let $\Omega(A) \subset \Omega(B)$ be the spectra of $A$ and $B$. The space $\Omega(A)$ is locally compact (and nonempty), $\Omega(B)$ is compact (it is actually the one-point compactification of $\Omega(A)$). It follows that the Gelfand transformation $\mathcal{F} : B \longrightarrow C_0(\Omega(B)) = C(\Omega(B))$ is an isometric isomorphism. Moreover the restriction of $\mathcal{F}$ to $A$ is an isomorphism onto $C_0(\Omega(A))$. Choose now $\varphi \in C(\Omega(B))$ such that $0 \leq \varphi \leq 1$, $||\varphi||_\infty = 1$ and $\operatorname{supp}\varphi \subset \Omega(A)$. Let $T = \mathcal{F}^{-1}(\varphi) \in A \subset PF_2(G)$ and $|||T|||_2 = ||\varphi||_\infty = 1 = |||\operatorname{Id} - 2T|||_2$.

**Theorem 10.** *Suppose that $G$ is amenable, and let $H$ be a closed normal* **nonopen** *subgroup of $G$. The best bound for approximate units of $I(H)$ (in $A_2(G)$) is 2.*

Let $P \in \operatorname{Hom}_{A(G)} PM(G)$ be a projection from $PM(G)$ onto $I(H)^\perp$. There is $T \in PF(G)$ with $|||T|||_2 = 1$ and $|||\operatorname{Id}_{L^2(G)} - T|||_2 = 1$. We have

$$(\operatorname{Id} - P)(\operatorname{Id}_{L^2(G)} - T) = -2T.$$

This implies $||\operatorname{Id} - P|| \geq 2$.

*Remark.* We are unable to prove the corresponding result in $A_p(G)$ for $p \neq 2$!

**Theorem 11.** *Let $H$ be an open (not necessarily normal) subgroup of an amenable group $G$. The best bound for approximate units of $I(H)$ (in $A_2(G)$) is 2 if $G/H$ is infinite, and is $\dfrac{2(|G/H| - 1)}{|G/H|}$ otherwise.*

Let $MA(G)$ be the Banach algebra of all pointwise multipliers of $A(G)$ with the multiplier norm. We have $1_H \in MA(G)$. Clearly $P_0(T) = 1_H T$ defines a map which belongs to $\operatorname{Hom}_{A(G)} PM(G)$ and projects $PM(G)$ onto $I(H)^\perp$. From the decomposition $T = 1_H T + 1_{G-H} T$ it follows that any $P \in \operatorname{Hom}_{A(G)} PM(G)$ which projects $PM(G)$ onto $I(H)^\perp$ coincides with $P_0$. Therefore the best bound for the approximate units of $I(H)$ is precisely $||\operatorname{Id} - P_0||$, i.e. $||1_{G-H}||_{MA(G)}$.

Now $G$ being amenable $MA(G)$ coincides (isometrically) with the intricate Banach algebra $B_2(G)$ introduced by C. Herz [10]. We recall the necessary notions. Let $X$ be a nonempty set (with the discrete topology). Every $k \in C_{00}(X \times X)$ is the kernel of a bounded operator of $\ell^2(X)$. The corresponding norm is denoted $|||k|||_2$. $V_2(X)$ is the space of all $\varphi \in \mathbb{C}^{X \times X}$ for which there is $C > 0$ with $|||\varphi k|||_2 \leq C|||k|||_2$ for every $k \in C_{00}(X \times X)$. The smallest possible $C$ is $||\varphi||_{V_2(X)}$. By definition $B_2(G)$ is the set of all $\varphi \in C(G)$ for which $M_G\varphi \in V_2(G_d)$ where $M_G\varphi(x, y) = \varphi(y^{-1}x)$ and $||\varphi||_{B_2(G)} = ||M_G\varphi||_{V_2(G_d)}$.

Moreover, by an important result of C. Herz [11, Theorem 5],

$$||1_{G-H}||_{B_2(G)} = ||1_{G/H \times G/H - \Delta(G/H)}||_{V_2((G/H)_d)}$$

where $\Delta(G/H)$ is the diagonal in $G/H \times G/H$. Now $G/H$ carries a structure of abelian group. Let $L_d$ be this group with the discrete topology. We have $||1_{G-H}||_{B_2(G)} = ||1_{L_d - \{e\}}||_{B_2(L_d)}$. From above $||1_{L_d - \{e\}}||_{B_2(L_d)}$ is the best bound for approximate units of $\ker T_{\widehat{L_d}}$. We conclude then with the help of Theorem 5.

## References


1. J. Delaporte and A. Derighetti, *On ideals of $A_p$ with bounded approximate units and certain conditional expectations*, J. London Math. Soc. **47** (1993), 497–507. MR **94k:**43004
2. ———, *p-Pseudomeasures and closed subgroups*, Mh. Math. **119** (1995), 37–47.
3. A. Derighetti, *Some remarks on $L^1(G)$*, Math. Z. **164** (1978), 189–194. MR **80f:**43009
4. ———, *Relations entre les convoluteurs d'un groupe localement compact et ceux d'un sous-groupe fermé*, Bull. Sci. Math. **106** (1982), 69–84. MR **83j:**43008
5. ———, *Quelques observations concernant les ensembles de Ditkin d'un groupe localement compact*, Mh. Math. **101** (1986), 95–113. MR **88c:**43006
6. R. S. Doran and J. Wichman, *Approximate Identities and Factorization in Banach Modules*, Lecture Notes in Math. 768, Springer-Verlag, Berlin, 1979. MR **83e:**46044
7. B. Forrest, *Amenability and bounded approximate identities in ideals of $A(G)$*, Illinois J. Math. **34** (1990), 1–25.
8. ———, *Amenability and the structure of the algebras $A_p(G)$*, Trans. Amer. Math. Soc. **343** (1994), 233–243. MR **94g:**43001
9. C. S. Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier **23** (1973), no. 3, 91–123. MR **50:**7956
10. ———, *Une généralisation de la notion de transformée de Fourier-Stieltjes*, Ann. Inst. Fourier **24** (1974), no. 3, 145–157. MR **54:**13466
11. ———, *Asymmetry of norms of convolution operators II: nilpotent Lie groups*, Symposia Mathematica **22** (1977), 223–230. MR **58:**6932
12. N. Lohoué, *Estimations $L^p$ de coefficients de représentations et opérateurs de convolution*, Adv. in Math. **38** (1980), 178–221. MR **82m:**43004
13. F. Lust-Piquard, *Propriétés harmoniques et géométriques des sous-espaces invariants par translation de $L^\infty(G)$*, Thèse, Université de Paris-Sud, 1978. MR **58:**23346
14. H. Reiter, *Classical harmonic analysis and locally compact groups*, Oxford University Press, Oxford, 1968. MR **46:**5933
15. ———, *Sur certains idéaux de $L^1(G)$*, C. R. Acad. Sc. Paris **267** (1968), 882–885. MR **39:**6025
16. ———, *$L^1$-Algebras and Segal Algebras*, Lecture Notes in Math. 231, Springer-Verlag, Berlin, 1971. MR **55:**13158


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