BEST BOUNDS FOR THE APPROXIMATE UNITS
FOR CERTAIN IDEALS OF $L^1(G)$ AND OF $A_p(G)$

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(Abstract. We compute the best bound for the approximate units of the augmentation ideal of the group algebra $L^1(G)$ of a locally compact amenable group $G$. More generally such a calculation is performed for the kernel of the canonical map from $L^1(G)$ onto $L^1(G/H)$, $H$ being a closed amenable subgroup of $G$. Analogous results involving certain ideals of the Fourier algebra of an amenable group are also discussed.

1. Introduction

Let $T_{H,q}$ be the canonical map from $L^1(G)$ onto $L^1(G/H)$ where $H$ is a closed subgroup of a locally compact group $G$. In 1968, Reiter [15] proved that if $H$ is amenable the kernel of $T_{H,q}$ admits bounded approximate right units. He showed moreover that for $H = G$ this property characterizes the amenability of $H$. In 1978 the second author [3] obtained that this is also the case for a large class of subgroups of $G$ (including all lattices in $G$). But at the present time a full converse is still in doubt.

In loco citato Reiter more precisely proved that the amenability of $H$ implies the existence of approximate right units for $ker T_{H,q}$ bounded by 2. One of the main results of this work is that 2 is the best bound if $H$ is infinite. For $H$ finite the best bound is $\frac{2(|H|-1)}{|H|}$. We also investigate the corresponding results for the Fourier algebra $A(G)$ of a locally compact amenable group $G$. The best bound for approximate units of the ideal $I(H)$ of all $a \in A(G)$ vanishing on a closed normal subgroup $H$ of $G$ is 2 if $G/H$ is infinite. It is $\frac{2(|G/H|-1)}{|G/H|}$ otherwise (for $H$ open in $G$ it is not necessary to assume the normality of $H$ in $G$!)

In section 2 we essentially develop the tools which permit estimates from above and from below for the bounds of approximate units. In section 3 we obtain new bounds for ideals in $L^1(G)$ of the form $T_{H,q}^{-1}(I)$, and section 4 is devoted to the corresponding results in the Figá-Talamanca Herz algebra $A_p(G)$ (recall that $A_2(G) = A(G)$). Our main results (Theorems 5, 10 and 11) are contained in sections 5 and 6.

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2. Bounded approximate units and projections

We collect in this rather technical section some results concerning an arbitrary normed algebra $A$. We will apply them later to $L^1(G)$ and $A_p(G)$.

The dual $A^*$ of $A$ carries a right $A$-module structure given by $(fa)(b) = f(ab)$ for $f \in A^*$ and $a, b \in A$. We denote by $\text{Hom}_A A^*$ the Banach algebra of all bounded linear operators $T$ of $A^*$ with $T(fa) = (Tf)a$ for $f \in A^*$ and $a \in A$. For $A = L^1(G)$, $\text{Hom}_A A^*$ is the algebra $\text{Hom}_{L^1(G)} L^\infty(G)$ of all bounded operators $T$ of $L^\infty(G)$ with $T(f) = T\varphi$ for $f \in L^1(G)$, $\varphi \in L^\infty(G)$ (to $\varphi \in L^\infty(G)$, we associate $T\varphi$ the linear functional on $L^1(G)$ defined by $F_\varphi(f) = \int f(x)\varphi(x)dx$ for $f \in L^1(G)$; the above right $L^1(G)$-module structure on $L^1(G)^*$ is given by $F_\varphi g = F_\varphi*$ where $F(x) = \frac{g(x)}{g(x-1)}\Delta_G(x)$).

If $A = A_p(G)$, $A^*$ is the space $PM_p(G)$ of all $p$-pseudomeasures on $G$ and $\text{Hom}_A A^*$ is the Banach algebra $\text{Hom}_{A_p(G)} P\text{M}_p(G)$ of all bounded linear operators $\Phi$ of $P\text{M}_p(G)$ with $\Phi(uS) = u\Phi(S)$ for all $u \in A_p(G)$ and $S \in P\text{M}_p(G)$.

**Proposition 1.** Let $I$ be a closed left ideal of $A$ having approximate right units bounded by $C > 0$. Then there exists a projection $P \in \text{Hom}_A A^*$ from $A^*$ onto $I^1$ with $\|\text{Id} - P\| \leq C$.

**Proposition 2.** Let $C \geq 0$. Assume that $A$ admits two-sided approximate units bounded by $C$. Let $I$ be a closed left ideal of $A$. Assume the existence of $P \in \text{Hom}_A A^*$ which is a projection from $A^*$ onto $I^1$. Then for $u \in A$, $v \in I$, $f \in A^*$ and $\varepsilon > 0$ there is $w \in I$ with $\|w\| \leq C\|\text{Id} - P\|$, $\|v - wv\| < \varepsilon$ and $\|f(uw) - f(u) + (Pf)(u)\| < \varepsilon$.

Both propositions are essentially known. The commutative case is due to Lust-Piquard [13, pp. 7 and 15] and the general case to Forrest [7, Proposition 6.4, p. 17]. However the estimates of $\|\text{Id} - P\|$ (Prop. 1) and $\|w\|$ (Prop. 2) being probably new and a fortiori the condition involving $f$, $u$ and $P$, we present a complete proof of Proposition 2.

If $C = 0$, then $A = \{0\}$. Suppose $C > 0$ and $\|\text{Id} - P\| = 0$. We have $I = \{0\}$; it suffices to choose $w = 0$. We therefore suppose $C > 0$ and $\|\text{Id} - P\| > 0$. Let $F$ be a finite nonempty subset of $I^*$ and $\eta > 0$. We denote by $E(F, \eta)$ the set of all $w \in I$ with $\|w\| \leq C\|\text{Id} - P\|$, $\|g(v) - g(wv)\| < \eta$ for every $g \in F$ and $\|f(uw) - f(u) + (Pf)(u)\| < \varepsilon$.

We first show that $E(F, \eta) \neq \emptyset$. There exists $u_1 \in A$ with $\|u_1\| \leq C$ and

\[
\|u - uu_1\| < \frac{\varepsilon}{2(1 + \|\text{Id} - P\|f)} \eta,
\]

\[
\|v - vu_1\| < \frac{\varepsilon}{4(1 + \max_{g \in F} \|P_g\|)(1 + \max_{g \in F} \|g\|)} \eta,
\]

\[
\|v - u_1v\| < \frac{\varepsilon}{4(1 + \max_{g \in F} \|P_g\|)} \eta.
\]

For $g \in I^*$, we set $\gamma(g) = g_1(u_1) - (P_g)(u_1)$ where $g_1 \in A^*$ is such that the restriction of $g_1$ to $I$ is $g$. We have $\gamma \in I^{**}$ and $\|\gamma\| \leq C\|\text{Id} - P\|$. By the theorem of Goldstine, there is $w \in I$ with $\|w\| \leq C\|\gamma\|$, $\|f(u)(w) - \gamma(\text{Res}_I fu)| < \frac{\varepsilon}{4}$ and $\|g(u)(w) - \gamma(\text{Res}_I gw)| < \frac{\varepsilon}{4}$ for every $g \in F$. We obtain

\[
\|f(uw) - f(uu_1) + (Pf)(uu_1)\| < \frac{\varepsilon}{2}.
\]
and therefore \(|f(uw) - f(u) + ( Pf)(u)| < \frac{\varepsilon}{2} + |(f - Pf)(uu_1 - u)| < \varepsilon.\)

For \(g \in G\), we have \(|g(vw) - g(vu_1) + (Pg)(vu_1)| < \frac{\varepsilon}{2}\). Taking into account that \(u_1 v \in I\) and \((Pg)(u_1 v) = 0\), we have

\[
|g(vw) - g(v)| \\
\leq |g(vw) - g(vu_1) + (Pg)(vu_1)| + |g(vu_1) - g(v)| + |Pg(u_1 v) - (Pg)(vu_1)| \\
\leq \frac{\eta}{4} + \|g\| \|vu_1 - v\| + \|Pg\| \|u_1 v - vu_1\| < \eta.
\]

This proves that \(w \in E(F, \eta)\).

Let \(B\) be the set \(\bigcup\{v E(F, \eta) \mid F \text{ nonempty finite subset of } A^* \text{ and } \eta > 0\}\). It is clear that \(v\) lies in the \(\sigma(A, A^*)\)-closure of \(B\) in \(A\). Therefore \(v\) lies in the norm closure in \(A\) of the convex hull of \(B\). Consequently we can find \(m \in \mathbb{N}, F_1, \ldots, F_m\) finite nonempty subsets of \(A^*, \eta_1, \ldots, \eta_m, c_1, \ldots, c_m > 0, w_1, \ldots, w_m \in I\) such that \(c_1 + \ldots + c_m = 1, w_j \in E(F_j, \eta_j)\) \((1 \leq j \leq m)\) and \(\|v - \sum_{j=1}^{m} c_j w_j\| < \varepsilon\).

Consider \(w = \sum_{j=1}^{m} c_j w_j\), indeed we obtain \(w \in I, \|w\| \leq C\|\text{Id} - P\| \|v - vw\| < \varepsilon\)
and \(|f(uw) - f(u) + ( Pf)(u)| < \varepsilon.\)

We say that \(C \geq 0\) is a bound of approximate right units of \(A\) if for every \(\varepsilon > 0\) and every \(a \in A\) there is a \(b \in A\) with \(||a - ab|| < \varepsilon\) and \(||b|| \leq C\). Let \(C\) be the set of all bounds of approximate right units. Then the infimum \(D\) of \(C\) also is a bound of approximate right units. Let \(\varepsilon > 0\) and \(a \in A;\) there is \(C \in C\) with \(C < D + \varepsilon\) where \(0 < \eta \leq \frac{\varepsilon}{\sum \|a\|}\); there is also \(b \in A\) with \(||b|| \leq C\) and \(||a - ab|| < \eta\), we have \(||a - ab_1|| < \varepsilon\) and \(||b_1|| \leq D\) for \(b_1 = \frac{b}{\sum \|a\||}\). We call \(D\) the best bound for the approximate right units of \(A\).

Assume that \(A\) admits two-sided approximate units bounded by one. Let \(I\) be a closed left ideal of \(A\) having bounded approximate right units. Then the best bound for the approximate right units of \(I\) is \(\min\{||\text{Id} - P|| \mid P \in \text{Hom}_A A^*\}, P\) is a projection from \(A^*\) onto \(I^+\).

### 3. Bounds for approximate units of \(T_H^{-1}(I)\)

Let \(q\) be a continuous strictly positive function on \(G\) with

\[q(xh) = q(x)\Delta_H(h)\Delta_G(h^{-1}) \quad \text{for all } x \in G \text{ and } h \in H.\]

We choose measures \(dx, dh, d_q \hat{x}\) on \(G, H, G/H\) as in \([14, \text{p. } 158]\). For every \(f \in L^1(G)\), we define \(T_{H,q}f(\hat{x}) = \int_H \frac{f(xh)}{q(xh)}dh\) where \(\hat{x} = xH = \omega(x)\). When we can choose \(q = 1\) (this is the case if \(H\) is normal in \(G\)) we will write \(T_H\) instead of \(T_{H,1}\).

**Theorem 3.** Let \(H\) be a closed normal amenable subgroup of \(G\) and \(I\) a closed left ideal of \(L^1(G/H)\) having approximate right units bounded by \(C \geq 0\). Then \(T_{H}^{-1}(I)\) has approximate right units bounded by \(C + 2\).

The existence of bounded right units in \(T_{H}^{-1}(I)\) is due to Reiter \([16, \text{p. } 70]\). He found for \(T_{H}^{-1}(I)\) the bound \(3C + 5\) (see pp. 31-33). Later, Doran and Wichman \([6, \text{pp. } 43-44]\) obtained the bound \(3C + 2\) with the same method.
For $F \in C^h_{\text{sa}}(G)^*$ ($C^0_{\text{sa}}(G)$ is the Banach space of all bounded left uniformly continuous functions on $G$) the relation
\[ \langle f, \tau_G(F) t \rangle_{L^1(G), L^\infty(G)} = \int_G f(x) \overline{\tau_G(F)(t)(x)} \, dx = F(f^* t) \]
for $f \in L^1(G)$ and $t \in L^\infty(G)$ defines an element $\tau_G(F)$ of $\text{Hom}_{L^1(G)} L^\infty(G)$. We recall that $\tau_G$ is a Banach algebra isomorphism from $C^0_{\text{sa}}(G)^*$ (with the Arens product) onto $\text{Hom}_{L^1(G)} L^\infty(G)$; for $\Phi \in \text{Hom}_{L^1(G)} L^\infty(G)$ and $t \in C^0_{\text{sa}}(G)$, we also have $\tau^{-1}_G(\Phi)(t) = \Phi(t)e$.

Let $M$ be a left-invariant mean on $C^0_{\text{sa}}(H)$. For $\varphi \in C^0_{\text{sa}}(G)$ and $x \in G$, we put $\gamma(\varphi)(x) = M(\varphi \cdot H)$, where $\varphi \cdot H(h) = \varphi(\gamma x h)$ for $h \in H$. It is straightforward to verify that $\alpha = \gamma \circ \gamma^{-1}_G$ is a Banach algebra isometry from $\text{Hom}_{L^1(G/H)} L^\infty(G/H)$ into $\text{Hom}_{L^1(G)} L^\infty(G)$ (note the analogy with [1, Theorem 8, p. 50]). But in general $\alpha(Id_{L^\infty(G/H)}) \neq Id_{L^\infty(G)}$. According to Proposition 1 there is $P \in \text{Hom}_{L^1(G/H)} L^\infty(G/H)$, a projection from $L^\infty(G/H)$ onto $I^\perp$, with $\|Id_{L^\infty(G/H)} - P\| \leq C$. Let $t \in C^0_{\text{sa}}(G)$. For $f \in T_H^{-1}(I)$ we have
\[ \langle f, \alpha(P)(t) \rangle_{L^1(G), L^\infty(G)} = (T_H f, P(\gamma(t)))_{L^1(G/H), L^\infty(G/H)} = 0 \]
and therefore $\alpha(P)(t) \in T_H^{-1}(I)^\perp$. From this we deduce $\alpha(P)(t) \in T_H^{-1}(I)^\perp$ for every $t \in L^\infty(G)$.

Let $t \in T_H^{-1}(I)^\perp$; for $f, g \in L^1(G)$ we have
\[ \langle g \ast f, \alpha(P)(t) \rangle_{L^1(G), L^\infty(G)} = (T_H f, P(\gamma(g^* \ast t)))_{L^1(G/H), L^\infty(G/H)} . \]
There is $u \in I^\perp \cap C^0_{\text{sa}}(G/H)$ with $u \circ \omega = g^* \ast t$, and therefore
\[ \langle g \ast f, \alpha(P)(t) \rangle_{L^1(G), L^\infty(G)} = (T_H f, Pu)_{L^1(G/H), L^\infty(G/H)} = (T_H f, u)_{L^1(G/H), L^\infty(G/H)} = (g \ast f, t)_{L^1(G), L^\infty(G)} , \]
so we obtain $\alpha(P)t = t$. We have proved that $\alpha(P)$ is a projection from $L^\infty(G)$ onto $T^{-1}_H(I)^\perp$. The inequality
\[ \|Id_{L^\infty(G)} - \alpha(P)\| \leq \|Id_{L^\infty(G)} - \alpha(Id_{L^\infty(G/H)})\| + \|Id_{L^\infty(G/H)} - P\| \]
permits us to conclude.

It is possible to avoid Propositions 1, 2 and the use of $\text{Hom}_{L^1(G)} L^\infty(G)$. Nevertheless the following more direct proof gives perhaps less insight into the question.

Let $f \in T_H^{-1}(I)$ and $\varepsilon > 0$. There is $u \in L^1(G)$ with $\|f - f \ast u\|_1 < \frac{\varepsilon}{C + 5}$ and $\|u\|_1 = 1$. By assumption there is $r \in I$ with $\|T_H f - T_H f \ast r\|_1 < \frac{\varepsilon}{C + 5}$ and $\|r\|_1 \leq C$. Choose $s \in T_H^{-1}(I)$ with $T_H s = r$. Denote by $A_H$ the convex hull of \{h | h \in H\} where $\langle A_h \varphi \rangle(x) = \varphi(xh)\Delta_G(h)$ for $\varphi \in C^0, x \in G$ and $h \in H$. Using the amenability of $H$ we can find $A \in A_H$ with $\|A \|_1 < \|r\|_1 + \eta$ where $\eta = \frac{\varepsilon}{(C + 5)(1 + \|f\|_1)}$ (see [14, p. 174]). There is also $B \in A_H$ with
\[ \|B(f \ast u - f \ast u \ast As)\|_1 < \frac{\varepsilon}{C + 5} + \|T_H (f \ast u - f \ast u \ast As)\|_1 . \]
We have $u - Bu + u * BAs \in T_H^{-1}(I)$, $||f - f * (u - Bu + u * BAs)||_1 < \frac{\varepsilon (C + 4)}{C + 5}$ and $||u - Bu + u * BAs||_1 < 2 + C + \eta$. It suffices now to put

$$k = \frac{C + 2}{C + 2 + \eta} (u - Bu + u * BAs)$$

to conclude $||k||_1 \leq C + 2$ and $||f - f * k||_1 < \varepsilon$.

4. Analogous result for the Figà-Talamanca Herz algebra $A_p(G)$

**Theorem 4.** Let $G$ be an amenable locally compact group, $H$ a closed normal subgroup of $G$ and $I$ a closed ideal of $A_p(H)$. We assume that $I$ has approximate units bounded by $C$ ($C \geq 0$). Then the closed ideal $\{u \in A_p(G) \mid \text{Res}_H u \in I\}$ has approximate units bounded by $C + 2$.

For $p = 2$, the special case $I = \{0\}$, with the bound $3$, was already obtained by Forrest [7, p. 6, Proposition 3.7]. More recently, Forrest [8, Proposition 3.4] treated also the corresponding result in $A_p(G)$ with a less explicit and certainly less precise constant.

Instead of using $C_{0u}^p(G)$, we consider $cv_p(G)$, the norm closure in the space $L(L^p(G))$ of all bounded operators of all $p$-convolution operators with compact support. Let $P$ be the map from $L(L^p(G))$ into $L(L^p(H))$ constructed in Theorem 3 of [2]. Let also $\sigma_G$ be the canonical Banach algebra isometry from $cv_p(G)$ onto $\text{Hom}_{A_p(G)}(PM_p(H))$ (see for example [1, p. 501]). The map $\lambda = \sigma_G \circ P^* \circ \sigma_H^{-1}$ is a Banach algebra isometry from $\text{Hom}_{A_p(H)}(PM_p(H))$ into $\text{Hom}_{A_p(G)}(PM_p(G))$. In analogy with the $L^1$-case, we have in general $\lambda(Id_{PM_p(H)}) \neq Id_{PM_p(G)}$.

Let $i$ be the canonical map from $PM_p(H)$ into $PM_p(G)$ defined in [4, p. 76]. Then $i(I^\perp)$ coincides with $J^\perp$ where $J = \{u \in A_p(G) \mid \text{Res}_H u \in I\}$. To verify this, consider $T \in J^\perp$, the support of $T$ lies in $H$. According to [12, p. 190, Théorème 5], there is $S \in PM_p(H)$ such that $i(S) = T$. We obtain $S \in I^\perp$.

Conversely assume that $T = i(S)$ with $S \in I^\perp$. For $v \in J$ we have $(v, T)_{A_p(G),PM_p(G)} = (\text{Res}_H v, S)_{A_p(H),PM_p(H)} = 0$ and thus $T \in J^\perp$.

For $\Phi \in \text{Hom}_{A_p(H)}(PM_p(H))$ we have $\lambda(\Phi) = i \circ \Phi \circ P$. Take indeed $u \in A_p(G)$ and $T \in PM_p(G)$; then

$$\langle u, \lambda(\Phi)T \rangle_{A_p(G),PM_p(G)} = \langle P^*(\sigma_H^{-1}(\Phi))(uT), \sigma_H^{-1}(\Phi)(\text{Res}_H u, P(T)) \rangle_{A_p(H),PM_p(H)} = \langle (\text{Res}_H u, \Phi(P(T)))_{A_p(H),PM_p(H)} \rangle_{A_p(G),PM_p(G)}.$$

There exists $P \in \text{Hom}_{A_p(H)}(PM_p(H))$, a projection from $PM_p(H)$ onto $I^\perp$, with $||\text{Id}_{PM_p(H)} - P|| \leq C$. The map $\lambda(P)$ is a projection from $PM_p(G)$ onto $J^\perp$. Let $T \in PM_p(G)$. For $w \in J$,

$$\langle w, \lambda(P)(T) \rangle_{A_p(G),PM_p(G)} = \langle (\text{Res}_H w, P(P(T)))_{A_p(H),PM_p(H)} \rangle_{A_p(G),PM_p(G)} = 0.$$

Therefore $\lambda(P)(T) \in J^\perp$. Let $T \in J^\perp$. There is $S \in I^\perp$ with $T = i(S)$. For
Let \( w \in A_p(G) \) we have

\[
\langle w, \lambda(P)(T) \rangle_{A_p(G), PM_p(G)} = (\text{Res}_H w, P(P(i(S))))_{A_p(H), PM_p(H)}
\]

\[
= (\text{Res}_H w, P(S))_{A_p(H), PM_p(H)}
\]

\[
= (\text{Res}_H w, S)_{A_p(H), PM_p(H)}
\]

\[
= \langle w, i(S) \rangle_{A_p(G), PM_p(G)}.
\]

Finally, the inequality \( || \text{Id}_{PM_p(G)} - \lambda(P)|| \leq 2 + C \) permits us as above to conclude.

It is also possible to write another more direct proof:

Let \( u \in J \) and \( \varepsilon > 0 \). There is \( v \in A_p(G) \) with \( ||v||_{A_p(G)} = 1 \) and \( ||u - uv||_{A_p(G)} < \frac{\varepsilon}{4} \). By assumption, there exists \( w \in I \) with \( ||\text{Res}_H (uv) - w \text{Res}_H (uv)||_{A_p(H)} < \frac{\varepsilon}{4} \) and \( ||w||_{A_p(H)} \leq C \). Using [9, p. 92, Theorem 1b] there is \( a \in A_p(G) \) with \( \text{Res}_H a = w \) and \( ||u||_{A_p(G)} < ||w||_{A_p(H)} + \eta \) where \( 0 < \eta < \min\{1, \frac{C}{4(1 + ||w||_{A_p(H)})}\} \).

By [5, p. 102, Proposition 10] there is \( b \in A_p(G/H) \) with \( b(\ell) = 1 \), \( ||b||_{A_p(G/H)} < 1 + \eta \) and \( ||b \circ \omega (uv - uv)|_{A_p(G)} < \frac{\varepsilon}{4} + ||\text{Res}_H (uv - uv)|_{A_p(H)} \). Let \( d = v - b \circ \omega v + b \circ \omega va \). We have \( d \in J \); from

\[
||u - ud||_{A_p(G)} \leq ||u - uv||_{A_p(G)} + ||b \circ \omega (uv - uv)|_{A_p(G)},
\]

we deduce \( ||u - ud||_{A_p(G)} < \frac{3\varepsilon}{4} \). Moreover we have \( ||d||_{A_p(G)} < C + 2 + 2\eta + \eta C + \eta^2 \).

Consider \( f = \frac{C + 2 + 2\eta + \eta C + \eta^2}{C + 2 + 2\eta + \eta C + \eta^2} \). We have \( f \in J \) and \( ||f||_{A_p(G)} \leq C + 2 \). From

\[
||u - uf||_{A_p(G)} \leq ||u - ud||_{A_p(G)} + ||ud - uf||_{A_p(G)}
\]

and

\[
||ud - uf||_{A_p(G)} \leq (2\eta + \eta C + \eta^2)||u||_{A_p(G)}
\]

we obtain \( ||ud - uf||_{A_p(G)} < \frac{\varepsilon}{4} \) and finally \( ||u - uf||_{A_p(G)} < \varepsilon \).

5. **Best bound for the approximate units of the ideal \( \ker T_{H,q} \)**

**Theorem 5.** Let \( H \) be a closed amenable subgroup of \( G \). The best bound for the right approximate units of \( \ker T_{H,q} \) is \( 2 \) if \( H \) is infinite. It is \( \frac{2(|H| - 1)}{|H|} \) otherwise.

This theorem is a consequence of the following two propositions.

**Proposition 6.** Let \( P \in \text{Hom}_{L^1(G)} L^\infty(G) \) which is a projection from \( L^\infty(G) \) onto \( \ker T_{H,q} \), \( H \) being a closed noncompact subgroup of \( G \). Then we have

1) \( P(f) = 0 \) for all \( f \in C_0(G) \) (the set of all continuous functions on \( G \) vanishing at infinity).

2) \( ||\text{Id} - P|| \geq 2 \).

The existence of \( P \) is, for \( H \) amenable, a consequence of Proposition 1.

To prove 1), observe first that \( P(t) \in C^0_{b,a}(G) \) for \( t \in C^0_{b,a}(G) \). For every \( r \in C^0_{b,a}(G) \) (i.e. \( r \) is a continuous function with compact support on \( G \), \( t \in C^0_{b,a}(G) \), \( h \in H \), we have \( \langle r - r_{h^{-1}} \Delta_G (h^{-1}), P(t) \rangle_{L^1(G), L^\infty(G)} = 0 \) for \( a, x \in G, \varphi \in C^0 \),
We therefore conclude that $P(h) = P(t)$ and consequently $P(h_t(c)) = P(t)(c)$. Now let $f \in C_0(G)$. There is a sequence $(h_n)_{n=1}^\infty \subseteq H$ such that $\supp h_n \cap \supp h_m = \emptyset$ for $n \neq m$. For $N \in \mathbb{N}$ we have

$$|P\left(\sum_{k=1}^N h_k f\right)(e)| = |N(Pf)(e)| \leq ||P|| ||f||_\infty ,$$

which implies $P(f)(e) = 0$. Consequently, for every $x \in G$, $P(x f)(e) = 0$ and therefore $P(f) = 0$.

To prove 2), it suffices to choose $f \in C_0(G)$ with $0 \leq f \leq 1_G$ and $f(e) = 1$. The function $g = 2f - 1_G$ satisfies the following properties: $g \in C^0_t(G), ||g||_\infty = 1$ and $(\Id - P)g = 2f$. We finally obtain $||\Id - P|| \geq 2$.

**Proposition 7.** Let $H$ be a compact subgroup of $G$ and $P \in \text{Hom}_{L^1(G)}L^\infty(G)$ a projection from $L^\infty(G)$ onto $\ker T_H^\perp$. If $H$ is infinite, then $||\Id - P|| \geq 2$ and $||\Id - P|| \geq \frac{2(2[H] - 1)}{|H|}$ otherwise.

**Preliminary remark.** If $H$ is a finite, $P(t) = \frac{1}{|H|} \sum_{h \in H} t_h$ (for $t \in L^\infty(G)$) defines $P \in \text{Hom}_{L^1(G)}L^\infty(G)$, which is a projection from $L^\infty(G)$ onto $\ker T_H^\perp$. We have $||\Id - P|| \leq \frac{2(2[H] - 1)}{|H|}$.

**Proof.** 1) We have $(Pf)(e) = \int_H f(h) dh$ for all $f \in C_0(G)$ such that $h f = f_h$ for every $h \in H$.

Using the continuity of the map $h \mapsto f_h$ from $H$ into $C_0(G)$, we obtain the existence and the unicity of $g \in C_0(G)$ such that $L(g) = \int_H L(f_h) dh$ for every $L \in C_0(G)^\ast$. It follows that $g(x) = \int_H f(xh) dh$ for every $x \in G$ and consequently $P(g) = g$. We also have

$$P(g)(e) = \int_H P(f_h)(e) dh = \int_H P(h_f)(e) dh = \int_H (Pf)(h) dh = \int_H Pf(e) dh .$$

We therefore conclude that $g(e) = (Pf)(e)$.

2) Assume that $H$ is infinite. Let $\varepsilon > 0$. There is an open neighbourhood $U$ of $e$ in $G$ such that $m_H(H \cap U) < \frac{\varepsilon}{2}$ where $m_H$ is the normalized Haar measure of $H$. There is also a compact neighbourhood $V$ of $e$ in $G$ with $V = V^{-1}, V^2 \subseteq U$ and $h V = V h$ for every $h \in H$. Consider then $\varphi = \frac{1_V * 1_V}{m_G(V)}$. We have $h \varphi = \varphi_h$ for every $h \in H, 2\varphi - 1_G \in C^0_t(G), (2\varphi - 1_G)(e) = 1, ||2\varphi - 1_G||_\infty = 1$ and $(\Id - P)(2\varphi - 1_G)(e) = 2 - 2P(\varphi)(e)$. From

$$P(\varphi)(e) = \int_H \varphi(h) dh \leq m_H(U \cap H) < \frac{\varepsilon}{2} ,$$

we deduce $||\Id - P|| > 2 - \varepsilon$. 
3) In the finite case it suffices to consider an open neighbourhood $U$ of $e$ in $G$ with $U \cap H = \{e\}$. We choose then $V$ and $\varphi$ as in 2). We obtain

$$ (\text{Id} - P)(2\varphi - 1_G)(e) = 2 - 2\varphi(e) = 2 - \frac{2}{|H|} \sum_{h \in H} \varphi(h) = 2 - \frac{2}{|H|} $$

and therefore $|\text{Id} - P| \geq \frac{2(|H| - 1)}{|H|}$.

It would be interesting (for $H$ amenable) to obtain a description of the set of all projections $P$ from $L^\infty(G)$ onto $\ker T_H^\perp$ with $P \in \text{Hom}_{L^1(G)} L^\infty(G)$. For $H$ compact and normal in $G$ we can show that this set consists of a unique element given by $P(t)(x) = \int_H t(xh)dh$ for $t \in C_0^b(G)$.

Let $f \in L^1(G)$, $t \in L^\infty(G)$ and $\varepsilon > 0$. By Proposition 2 there is $g \in \ker T_H$ with $|g|_1 \leq |\text{Id} - P|_1$. \(|f - T_H f \circ \omega - (f - T_H f \circ \omega) \ast g| < \frac{\varepsilon}{2(1 + ||t||_\infty)}$$ and$$ |\langle f, P(t) \rangle_{L^1(G), L^\infty(G)} - \langle f, t \rangle_{L^1(G), L^\infty(G)} + \langle f \ast g, t \rangle_{L^1(G), L^\infty(G)}| < \frac{\varepsilon}{2}.$$ 

The subgroup $H$ being normal in $G$, for every $x \in G$ we have

$$ (T_H f \circ \omega) \ast g(x) = \int_{G/H} (T_H f)(\omega(x)g^{-1}) \Delta_G(g^{-1}) \left( \int_H g(yh)dh \right) dy $$

and therefore $(T_H f \circ \omega) \ast g = 0$. Taking into account that

$$ |\langle f, P(t) \rangle_{L^1(G), L^\infty(G)} - \langle T_H f \circ \omega, t \rangle_{L^1(G), L^\infty(G)}|$$

$$ \leq |\langle f, P(t) \rangle_{L^1(G), L^\infty(G)} - (f, t)_{L^1(G), L^\infty(G)} + (f \ast g, t)_{L^1(G), L^\infty(G)}|$$

$$ + |\langle f - T_H f \circ \omega - (f - T_H f \circ \omega) \ast g, t \rangle_{L^1(G), L^\infty(G)}|$$

we obtain that $|\langle f, P(t) \rangle_{L^1(G), L^\infty(G)} - \langle T_H f \circ \omega, t \rangle_{L^1(G), L^\infty(G)}| < \varepsilon$. In other words $\langle f, P(t) \rangle_{L^1(G), L^\infty(G)} = \langle T_H f \circ \omega, t \rangle_{L^1(G), L^\infty(G)}$. This implies for $t \in C_0^b(G)$ and $x \in G$ that $P(t)(x) = \int_H t(xh)dh$.

6. Best bound for the approximate units of certain ideals of the Fourier algebra

For an arbitrary subset $F$ of $G$ we denote by $I(F)$ the closed ideal of $A_p(G)$ consisting of those functions vanishing on $F$. Motivated by the assertion 1) of Proposition 6 we first prove the following result.

**Proposition 8.** Let $G$ be amenable and let $H$ be a closed normal nonopen subgroup of $G$. Let $P \in \text{Hom}_{A_p(G)} \text{PM}_p(G)$ be a projection from $\text{PM}_p(G)$ onto $I(H)^\perp$. Then $P(T) = 0$ for every $T \in \text{PF}_p(G)$.

$\text{PF}_p(G)$ is the norm closure of $L^1(G)$ in $\text{PM}_p(G)$. For $G$ abelian $\text{PF}_2(G)$ is, via the Fourier transform, isomorphic to $C_0(G)$. To $P$ there corresponds $P \in \text{Hom}_{L^1(G)} L^\infty(G)$, a projection from $L^\infty(G)$ onto $\ker T_H^\perp$ where $H^\perp$ is the set of all continuous characters of $G$ equal to 1 on $H$. Moreover $H$ is nonopen if and only if $H^\perp$ is noncompact.
The existence of $P$ (in Proposition 8) is a consequence of Proposition 1 and Theorem 4.

Let $f \in C_00(G)$, $K = \text{supp } f^*$, $u \in A_p(G)$, $\varepsilon > 0$ and $U$ be an open neighbourhood of $H \cap K$ in $G$ such that

$$m_G(U) < \frac{\varepsilon}{4(||f^*||_\infty + 1)(||u||_{A_p(G)} + 1)||1|| + 1).$$

It is possible to choose $v \in I(H) \cap C_00(G)$ with $v = 1$ on $K_1 = K - U$. By Proposition 2 there is $w \in I(H)$ with $||w||_{A_p(G)} \leq ||1 - P||$,

$$||uw - v||_{A_p(G)} < \frac{\varepsilon}{4(||f||_1 + 1)(||u||_{A_p(G)} + 1)},$$

and

$$\langle uw, \lambda_G^p(f) \rangle_{A_p(G), PM_p(G)} - \langle u, \lambda_G^p(f) \rangle_{A_p(G), PM_p(G)} + \langle u, P(\lambda_G^p(f)) \rangle_{A_p(G), PM_p(G)} < \frac{\varepsilon}{2}$$

(for a bounded measure $\mu$, $\lambda_G^p(\mu)$ is the convolution operator defined by $\lambda_G^p(\mu)(\varphi)(x) = \int_G \varphi(xy) \Delta_G(y)^{1/p} d\mu(y)$). We obtain the estimate

$$||u, P(\lambda_G^p(f)) \rangle_{A_p(G), PM_p(G)} < \frac{\varepsilon}{2} + ||uw, \lambda_G^p(f) \rangle_{A_p(G), PM_p(G)}.$$

Taking into account that

$$\langle u - uw, \lambda_G^p(f) \rangle_{A_p(G), PM_p(G)} = \int_G (u(x) - u(x)w(x)) f^*(x) dx,$$

we can write

$$||u - uw, \lambda_G^p(f) \rangle_{A_p(G), PM_p(G)}$$

$$\leq \int_{K_1} |u(x)v(x) - v(x)u(x)w(x)||f^*(x)|| dx$$

$$+ \int_{K \cap U} |u(x) - u(x)w(x)||f^*(x)|| dx.$$

We estimate

$$\int_{K_1} |u(x)v(x) - v(x)u(x)w(x)||f^*(x)|| dx$$

by $||u||_{A_p(G)} ||v - uv||_{A_p(G)} ||f^*||_t$ and

$$\int_{K \cap U} |u(x) - u(x)w(x)||f^*(x)|| dx$$

by $m_G(K \cap U)(||u||_\infty + ||u||_\infty ||w||_\infty ||f^*||_\infty$.

We obtain therefore that $||u - uw, \lambda_G^p(f) \rangle_{A_p(G), PM_p(G)} < \frac{\varepsilon}{2}$ and finally

$$||\langle u, P(\lambda_G^p(f)) \rangle_{A_p(G), PM_p(G)} < \varepsilon.$$

We also need an analog of the assertion 2) of Proposition 6:
Proposition 9. There is $T \in PF_2(G)$ with $|||T|||_2 = 1$ and $|||\text{Id}_{L^2(G)} - 2T|||_2 = 1$.

Let $f \in C_0(G)$ such that $f \neq 0$ and $f = f^*$. Consider the $C^*$-algebra $\lambda^2_G(M^1(G))$ and denote by $A$ and $B$ the $C^*$-subalgebras generated by $\lambda^2_G(f)$, respectively $\lambda^2_G(f)$ and $\text{Id}_{L^2(G)}$. Of course $A$ and $B$ are abelian and $B$ is unital. Let $\Omega(A) \subset \Omega(B)$ be the spectra of $A$ and $B$. The space $\Omega(A)$ is locally compact (and nonempty). $\Omega(B)$ is compact (it is actually the one-point compactification of $\Omega(A)$). It follows that the Gelfand transformation $\mathcal{F} : B \longrightarrow C_0(\Omega(B)) = C(\Omega(B))$ is an isometric isomorphism. Moreover the restriction of $\mathcal{F}$ to $A$ is an isomorphism onto $C_0(\Omega(A))$. Choose now $\varphi \in C(\Omega(B))$ such that $0 \leq \varphi \leq 1$, $||\varphi||_\infty = 1$ and supp $\varphi \subset \Omega(A)$. Let $T = \mathcal{F}^{-1}(\varphi) \in A \subset PF_2(G)$ and $|||T|||_2 = ||\varphi||_\infty = 1 = |||\text{Id} - 2T|||_2$.

Theorem 10. Suppose that $G$ is amenable, and let $H$ be a closed normal nonopen subgroup of $G$. The best bound for approximate units of $I(H)$ (in $A_2(G)$) is $2$.

Let $P \in \text{Hom}_{A(G)} PM(G)$ be a projection from $PM(G)$ onto $I(H)^\perp$. There is $T \in PF(G)$ with $|||T|||_2 = 1$ and $|||\text{Id}_{L^2(G)} - T|||_2 = 1$. We have

$$(\text{Id} - P)(\text{Id}_{L^2(G)} - T) = -2T.$$ 

This implies $||\text{Id} - P|| \geq 2$.

Remark. We are unable to prove the corresponding result in $A_p(G)$ for $p \neq 2$!

Theorem 11. Let $H$ be an open (not necessarily normal) subgroup of an amenable group $G$. The best bound for approximate units of $I(H)$ (in $A_2(G)$) is $2$ if $G/H$ is infinite, and is $\frac{2[G/H] - 1}{|G/H|}$ otherwise.

Let $MA(G)$ be the Banach algebra of all pointwise multipliers of $A(G)$ with the multiplier norm. We have $1_H \in MA(G)$. Clearly $P_0(T) = 1_RT$ defines a map which belongs to $\text{Hom}_{A(G)} PM(G)$ and projects $PM(G)$ onto $I(H)^\perp$. From the decomposition $T = 1_RT + 1_{G\setminus H}T$ it follows that any $P \in \text{Hom}_{A(G)} PM(G)$ which projects $PM(G)$ onto $I(H)^\perp$ coincides with $P_0$. Therefore the best bound for the approximate units of $I(H)$ is precisely $||\text{Id} - P_0||$, i.e. $||1_{G\setminus H}||_{MA(G)}$.

Now $G$ being amenable $MA(G)$ coincides (isometrically) with the intricate Banach algebra $B_2(G)$ introduced by C. Herz [10]. We recall the necessary notions. Let $X$ be a nonempty set (with the discrete topology). Every $k \in C_00(X \times X)$ is the kernel of a bounded operator of $C(X)$. The corresponding norm is denoted $|||k|||_2$. $V_2(X)$ is the space of all $\varphi \in C^{X \times X}$ for which there is $C > 0$ with $|||\varphi k|||_2 \leq C|||k|||_2$ for every $k \in C_00(X \times X)$. The smallest possible $C$ is $||\varphi||_{V_2(X)}$. By definition $B_2(G)$ is the set of all $\varphi \in C(G)$ for which $M_G \varphi \in V_2(G_d)$ where $M_G \varphi(x,y) = \varphi(y^{-1}x)$ and $|||\varphi|||_{B_2(G)} = |||M_G \varphi|||_{V_2(G_d)}$.

Moreover, by an important result of C. Herz [11, Theorem 5],

$$|||1_{G\setminus H}|||_{B_2(G)} = |||1_{G/H \times G/H \setminus \Delta(G/H)}|||_{V_2(G/H_d)}$$

where $\Delta(G/H)$ is the diagonal in $G/H \times G/H$. Now $G/H$ carries a structure of abelian group. Let $L_d$ be this group with the discrete topology. We have $|||1_{G\setminus H}|||_{B_2(G)} = |||1_{L_d\setminus \{e\}}|||_{B_2(L_d)}$. From above $|||1_{L_d\setminus \{e\}}|||_{B_2(L_d)}$ is the best bound for approximate units of $ker T_{L_d}^\perp$. We conclude then with the help of Theorem 5.
BEST BOUNDS FOR APPROXIMATE UNITS

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