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THE NILPOTENCE HEIGHT OF P_t^s

KENNETH G. MONKS

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Abstract. The method of Walker and Wood is used to completely determine the nilpotence height of the elements P_t^s in the Steenrod algebra at the prime 2. In particular, it is shown that $(P_t^s)^{2\lfloor s/t\rfloor+2} = 0$ for all $s \geq 0$, $t \geq 1$. In addition, several interesting relations in A are developed in order to carry out the proof.

1. Introduction and main results

In [WW] Grant Walker and Reg Wood show that $Sq(2^s)^{2s+2} = 0$ for all $s \geq 0$ in the mod 2 Steenrod algebra, A, thereby settling a conjecture which dates back to 1975. Using their method, along with previous results of Andrew Gallant, Judith Silverman, and some new results, we prove the following theorem. Recall that P_t^s is the Milnor basis element $Sq(r_1, \ldots, r_t)$ with $r_t = 2^s$ and $r_i = 0$ for $i < t$.

Theorem 1.1. *For all $s \geq 0$, $t \geq 1$, $(P_t^s)^{2\lfloor s/t\rfloor+2} = 0$.*

This establishes a conjecture of the author [M1, Conjecture 3.1] and extends the result discussed above, which proves Theorem 1.1 for the case $t = 1$. It was shown in [M1] that $(P_t^s)^{2\lfloor s/t\rfloor+1} \neq 0$. Thus Theorem 1.1 completely determines the nilpotence height of the elements P_t^s. The problem of determining the nilpotence of elements of A has been of much recent interest and we refer the reader to [M1] and [WW] for more background on the general question.

In order to extend the proof of [WW] to arbitrary values of t we must derive appropriate generalizations of the propositions and lemmas used in their proof, many of which are interesting in their own right. These are discussed in detail in the following sections.

2. Relations in A and the antiautomorphism

In this section we prove some of the preliminary propositions needed for the proof of Theorem 1.1.

Much of the notation we will use follows [WW]. For $\theta \in A$ write $\widehat{\theta}$ for $\chi(\theta)$ where χ denotes the canonical antiautomorphism of A. Following [M2], let $Sq_t(r_1, \ldots, r_m)$ be the Milnor basis element $Sq(s_1, \ldots, s_{tm})$ where $s_{ti} = r_i$ and $s_j = 0$ if t does not divide j. In particular, $Sq_t(2^s) = P_t^s$ and $Sq_1(n) = Sq(n)$. If $R = (r_1, r_2, \ldots)$ is a sequence of nonnegative integers, we write $Sq_t\langle R\rangle$ for the corresponding Milnor

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basis element. The degree of $Sq_t\langle R\rangle$ is $|R|_t = \sum(2^{it}-1)r_i$, and the excess of $Sq_t\langle R\rangle$ is $\mathrm{ex}(R) = \sum r_i$.

For any positive integer n, let $\alpha_i(n)$ be the coefficient of 2^i in the binary expansion of n, i.e. $n = \sum_{i=0}^{\infty} \alpha_i(n)2^i$. We say that m and n are *disjoint* if $\alpha_i(m)+\alpha_i(n) \le 1$ for all i. This is equivalent to the condition that the binomial coefficient $\binom{m+n}{m}$ is odd [L], which is in turn equivalent to the condition that $m+n$ *dominates* m (written $m+n \succcurlyeq m$) in the notation of [S1]. If m and n are disjoint we will find it convenient to write $m \asymp n$. If either m or n is negative we will also say that m and n are not disjoint. We will also write $2^i \in n$ for $\alpha_i(n) = 1$ since the meaning will be clear from the context.

In what follows we will make frequent use of the following trivial facts which we state without proof. Let $0 \le b < 2^t$.

$$2^l \in a \iff 2^{l+t} \in 2^t a + b, \tag{2.1}$$

$$2^l \in b \iff 2^l \in 2^t a + b \text{ and } l < t. \tag{2.2}$$

As in [M2] we let $\gamma_t(n) = \sum_{i=0}^{n-1} 2^{it}$ (take $\gamma_t(0) = 0$) for any integers $n \ge 0$ and $t \ge 1$. It follows that

$$\gamma_t(m+1) = 2^t\gamma_t(m) + 1. \tag{2.3}$$

We begin by proving a generalization of the Adem relation

$$Sq(2^{m+1}-1)Sq(2^m) = 0.$$

Proposition 2.1. *For any $t \ge 1$, $m \ge 0$, and $i < t$*

$$Sq_t(2^i\gamma_t(m+1))P_t^{mt+i} = 0.$$

Proof. By the Milnor product formula [Mil] $Sq_t(a)Sq_t(b) = \sum Sq_t(a+b-(2^t+1)j, j)$ where the sum is taken over all j such that $a-2^tj \asymp b-j$. Thus it suffices to show that $2^i\gamma_t(m+1) - 2^tj$ is not disjoint from $2^{mt+i}-j$ for any $0 \le j \le 2^i\gamma_t(m)$. We will prove something stronger, namely

Lemma 2.2. *For all $t \ge 1$, $m \ge 0$, $0 \le i < t$, and $0 \le j \le 2^i\gamma_t(m)$ there exists $k \ge 0$ such that*

$$2^{kt+i} \in 2^i\gamma_t(m+1) - 2^tj \quad \textit{and} \quad 2^{kt+i} \in 2^{mt+i} - j.$$

Proof. We will proceed by induction on m.

Base case. If $m = 0$, then $2^i\gamma_t(m+1) - 2^tj = 2^{mt+i} - j = 2^i$ so we can take k to be 0.

Inductive step. Assume that for all $t \ge 1$, $0 \le i < t$, and $0 \le j \le 2^i\gamma_t(m-1)$ there exists $k \ge 0$ such that $2^{kt+i} \in 2^i\gamma_t(m) - 2^tj$ and $2^{kt+i} \in 2^{(m-1)t+i} - j$.

Choose $t \ge 1$, $i < t$, and $0 \le j \le 2^i\gamma_t(m)$. To simplify the notation we let $X = 2^i\gamma_t(m+1) - 2^tj$ and $Y = 2^{mt+i} - j$.

If $j = 0$, then $X = 2^{mt+i} + 2^{(m-1)t+i} + \cdots + 2^i$ and $Y = 2^{mt+i}$ so that $2^{mt+i} \in X$ and $2^{mt+i} \in Y$. Thus in this case we can take k to be m.

Now suppose $j \ne 0$. We consider two cases.

Case 1. $2^i \notin j-1$. Using (2.3) to replace $\gamma_t(m+1)$ in X yields

$$\begin{aligned} X &= 2^i(2^t\gamma_t(m)+1) - 2^tj \\ &= 2^t(2^i\gamma_t(m) - j) + 2^i. \end{aligned}$$

Since $2^i < 2^t$, it follows from (2.2) that $2^i \in X$.

Since $2^i \notin j-1$, we can express $j-1 = 2^{i+1}q + r$ where $0 \le r < 2^i$. Substituting for j in Y yields

$$\begin{aligned} Y &= 2^{mt+i} - (2^{i+1}q + r + 1) \\ &= 2^{i+1}[2^{mt-1} - (q+1)] + 2^i + [2^i - (r+1)]. \end{aligned}$$

Since $r < 2^i$, it follows that the term $[2^i - (r+1)]$ is nonnegative and less than 2^i. Therefore $2^i \in Y$ by (2.1) and (2.2).

So $2^i \in X$ and $2^i \in Y$. Thus in this case we can take k to be 0.

Case 2. $2^i \in j-1$. In this case we let

$$j - 1 = 2^t q + r \qquad \text{where } 0 \le r < 2^t. \tag{2.4}$$

Indeed, since $2^i \in j-1$, we must have $2^i \le r < 2^t$ by (2.2). Replacing j via (2.4) and $\gamma_t(m)$ via (2.3) in the assumption that $j \le 2^i\gamma_t(m)$ and solving for $q+1$ shows

$$q + 1 \le 2^i\gamma_t(m-1) + \frac{2^t + 2^i - (r+1)}{2^t}.$$

But $\frac{2^t+2^i-(r+1)}{2^t} < 1$ since $r \ge 2^i$. Thus $q+1 \le 2^i\gamma_t(m-1)$ and so by the inductive hypothesis there exists $k \ge 0$ such that $2^{kt+i} \in 2^i\gamma_t(m) - 2^t(q+1)$ and $2^{kt+i} \in 2^{(m-1)t+i} - (q+1)$. Let k be any such value. We will now show that $2^{(k+1)t+i} \in X$ and $2^{(k+1)t+i} \in Y$, which will complete the induction and hence the proof.

To see $2^{(k+1)t+i} \in X$, we use (2.3) and (2.4) to obtain

$$\begin{aligned} X &= 2^i(2^t\gamma_t(m) + 1) - 2^t(2^t q + r + 1) \\ &= 2^t\left([2^i\gamma_t(m) - 2^t(q+1)] + [(2^t-1) - r]\right) + 2^i. \qquad (2.5) \end{aligned}$$

Now $2^i \in j-1 \implies 2^i \in r \implies 2^i \notin [(2^t-1)-r]$. Thus since $(2^t-1)-r < 2^t$, it follows that $[(2^t-1)-r] + 2^i < 2^t$ also. Writing $2^i\gamma_t(m) - 2^t(q+1)$ in the form $2^t[2^i\gamma_t(m-1) - (q+1)] + 2^i$, we see that $2^{kt+i} \in 2^i\gamma_t(m) - 2^t(q+1)$ implies $2^{kt+i} \in 2^i\gamma_t(m) - 2^t(q+1) + [(2^t-1)-r]$, which in turn implies $2^{(k+1)t+i} \in X$ by (2.1), (2.2), and (2.5).

Finally, to see $2^{(k+1)t+i} \in Y$, we use (2.4) to obtain

$$\begin{aligned} Y &= 2^{mt+i} - (2^t q + r + 1) \\ &= 2^t[2^{(m-1)t+i} - (q+1)] + [2^t - (r+1)]. \end{aligned}$$

But $[2^t - (r+1)] < 2^t$. Thus by (2.1), $2^{kt+i} \in 2^{(m-1)t+i} - (q+1)$ implies $2^{(k+1)t+i} \in Y$. □□

The following result of Andrew Gallant [G, Corollary 1a] describes the product of an arbitrary element $Sq_t(u)$ with $\chi Sq_t(v)$:

(2.6)

$$Sq_t(u)\widehat{Sq_t(v)} = \sum Sq_t\langle R\rangle : |R|_t = (2^t-1)(u+v);\ |R|_t + \mathrm{ex}(R) \not\geq 2^t u.$$

This generalizes a formula of Don Davis [D]. Using Davis' method one can derive the analogous formula for $\widehat{Sq_t(u)}Sq_t(v)$,

$$\widehat{Sq_t(u)}Sq_t(v) = \sum Sq_t\langle R\rangle : |R|_t = (2^t-1)(u+v);\ \mathrm{ex}(R) \not\geq v. \tag{2.7}$$

Using these formulae one can prove the following.

Proposition 2.3. *Let a, b, and t be positive integers with $a \geq t$. Then*

$$\widehat{P_t^a} Sq_t(2^a(2^b-1)) = Sq_t(2^{a-t}(2^{b+t}-1))\widehat{P_t^{a-t}}.$$

Proof. By (2.6) and (2.7) it suffices to show that for any sequence R in degree $|R|_t = 2^{a+b}(2^t-1)$, $\mathrm{ex}(R) \succcurlyeq 2^a(2^b-1) \iff 2^{a+b}(2^t-1) + \mathrm{ex}(R) \succcurlyeq 2^a(2^{b+t}-1)$. Notice that $\mathrm{ex}(R) \leq 2^{a+b}$ and that equality holds only when R is the sequence (2^{a+b}). But in this case it is easy to check that neither $\mathrm{ex}(R) \succcurlyeq 2^a(2^b-1)$ nor $2^{a+b}(2^t-1) + \mathrm{ex}(R) \succcurlyeq 2^a(2^{b+t}-1)$. Thus assume $\mathrm{ex}(R) < 2^{a+b}$. Then

$$\begin{aligned}
\mathrm{ex}(R) \succcurlyeq 2^a(2^b-1) \quad &\iff \quad \mathrm{ex}(R) - 2^a(2^b-1) \asymp 2^a(2^b-1) \\
&\overset{\text{by (2.2)}}{\iff} \quad \mathrm{ex}(R) - 2^a(2^b-1) \asymp 2^{a+b}(2^t-1) + 2^a(2^b-1) \\
&\iff \quad 2^{a+b}(2^t-1) + \mathrm{ex}(R) - 2^a(2^{b+t}-1) \asymp 2^a(2^{b+t}-1) \\
&\iff \quad 2^{a+b}(2^t-1) + \mathrm{ex}(R) \succcurlyeq 2^a(2^{b+t}-1),
\end{aligned}$$

which completes the proof. □

Let n, k, and t be positive integers with $n = k + mt$ for some $m \geq 0$. Extending the notation of Arnon [Ar] we define

$$(X_t)_k^n = Sq_t(2^n)Sq_t(2^{n-t}) \cdots Sq_t(2^k) = P_t^n P_t^{n-t} P_t^{n-2t} \cdots P_t^k.$$

Applying Proposition 2.3 repeatedly yields the following corollary.

Corollary 2.4. *Let $m \geq 1$, $c \geq 0$, and $a = c + mt$. Then*

$$(\widehat{X_t})_{c+t}^a Sq_t(2^a(2^b-1)) = Sq_t(2^c(2^{b+mt}-1))(\widehat{X_t})_c^{a-t}.$$

Finally, using (2.6) we can easily prove a generalization of Straffin's formula [St].

Proposition 2.5. *For any $s \geq 0$, $t \geq 1$,*

$$\widehat{P_t^s} = P_t^s + P_t^{s-1}\widehat{P_t^{s-1}}.$$

Proof. By (2.6), $P_t^{s-1}\widehat{P_t^{s-1}} = \sum Sq_t\langle R\rangle : |R|_t = 2^s(2^t-1);\ 2^s(2^t-1) + \mathrm{ex}(R) \succcurlyeq 2^t 2^{s-1}$. Since $\widehat{Sq_t(n)}$ is the sum of all $Sq_t\langle R\rangle$ in degree $n(2^t-1)$ (cf. [M2] or [G]), $\widehat{P_t^s} = \sum Sq_t\langle R\rangle : |R|_t = 2^s(2^t-1)$. Thus it suffices to show that $2^s(2^t-1)+\mathrm{ex}(R) \succcurlyeq 2^{s+t-1}$ if and only if $R \neq (2^s)$.

If $R = (2^s)$, then $2^s(2^t-1) + \mathrm{ex}(R) = 2^{s+t}$, which does not dominate 2^{s+t-1}. If $R \neq (2^s)$, then $\mathrm{ex}(R) < 2^s$ so that $2^s(2^t-1) + \mathrm{ex}(R) = 2^{s+t-1} + 2^{s+t-2} + \cdots + 2^s + \mathrm{ex}(R)$ clearly dominates 2^{s+t-1}. □

3. The Kristensen operations

Kristensen [K] developed a family of linear operations indexed by the admissible monomials that can be used to derive relations in A from other relations. Following Kristensen, we define the family of operators $\kappa_\theta : A \to A$ indexed by the Milnor basis elements θ by

$$\phi(x) = \sum \kappa_\theta(x) \otimes \theta \tag{3.1}$$

where $\phi : A \to A \otimes A$ is the diagonal map and $\phi(x)$ is expressed in the Milnor basis (i.e. the basis for $A \otimes A$ is taken to be tensor products of Milnor basis elements). The operations κ_θ have many nice properties and we refer the reader to [Gra] for details. We will refer to the action of κ_θ on A as 'stripping by θ' (cf. [WW]).

It follows immediately from (3.1) that stripping a Milnor basis element by $\theta = Sq(t_1, t_2, \dots)$ is given by

$$\kappa_\theta(Sq(r_1, r_2, \dots)) = Sq(r_1 - t_1, r_2 - t_2, \dots, r_i - t_i, \dots) \tag{3.2}$$

where the right-hand side is taken to be 0 if $r_i < t_i$ for any i. In particular we see that stripping P_t^s by P_t^s yields $Sq(0) = 1$.

Recall that A_n is the subalgebra of A generated by $Sq(2^k)$ with $k \leq n$. A_n has a graded vector space basis consisting of the Milnor basis elements $Sq(r_1, \dots, r_{n+1})$ having $r_i < 2^{n+2-i}$ for all i. Thus $P_t^s \in A_{s+t-1} \setminus A_{s+t-2}$ and the result of stripping any element of A_{s+t-2} by P_t^s is zero by (3.2) and the fact that the Kristensen operations are linear. In particular, since $P_t^{s-1} \in A_{s+t-2}$, stripping $\widehat{P_t^s}$ by P_t^s yields 1 by Proposition 2.5.

The Steenrod algebra is a Hopf algebra with dual A^*. A^* is isomorphic to the graded polynomial algebra $\mathbb{Z}_2[\xi_1, \xi_2, \dots]$ on generators ξ_i in dimension $2^i - 1$. If $R = (r_1, \dots, r_m)$ we will write ξ^R to mean the monomial $\xi_1^{r_1} \cdots \xi_m^{r_m}$. The basis of monomials ξ^R in A^* is dual to the Milnor basis for A. As is common we will write $\langle x, y \rangle$ for the evaluation of $y \in A^*$ on $x \in A$. The algebra homomorphism $\mu^* : A^* \to A^* \otimes A^*$ defined by

$$\mu^*(\xi_k) = \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j \tag{3.3}$$

is the dual of the product map in A. All of this is described in [Mil].

It can be shown (cf. [Gra, Proposition 28.13]) that for any $x, y \in A$ and any Milnor basis element $Sq\langle R \rangle$

$$\kappa_{Sq\langle R \rangle}(xy) = \sum_{I,J} \lambda_R^{I,J} \kappa_{Sq\langle I \rangle}(x) \kappa_{Sq\langle J \rangle}(y) \tag{3.4}$$

where $\lambda_R^{I,J} = \langle Sq\langle I \rangle Sq\langle J \rangle, \xi^R \rangle$. In particular, it follows from (3.3) that

$$\kappa_t^s(xy) = \sum_{j=0}^{t} \kappa_{t-j}^{j+s}(x) \kappa_j^s(y) \tag{3.5}$$

where we abbreviate $\kappa_{P_u^v}$ as κ_u^v and take κ_0^u to be the identity map.

For a fixed t let B_t be the vector subspace of A with basis the set of all $Sq_t(R)$. One can easily verify [G], either directly or by using the results in [AD], that B_t is a Hopf subalgebra of A. Note that $B_1 = A$ and that stripping an element of B_t by a Milnor basis element not in B_t must yield zero by (3.2).

Therefore if x and y are elements of B_t, then stripping by P_t^s is a differential, i.e.

$$\kappa_t^s(xy) = \kappa_t^s(x) y + x \kappa_t^s(y). \tag{3.6}$$

This follows from (3.5) since all of the other terms in the sum correspond to stripping by Milnor basis elements which are not in B_t and therefore give zero.

Iterating (3.6) we see that for any $x_1, \dots, x_n \in B_t$

$$\kappa_t^s(x_1 x_2 \cdots x_n) = \sum_{i=1}^{n} x_1 \cdots x_{i-1} \kappa_t^s(x_i) x_{i+1} \cdots x_n. \tag{3.7}$$

Using these facts we can now obtain the following technical lemma which is analogous to [WW, Lemma 1.4] and whose proof is entirely analogous to the proof of [WW, Lemma 1.4]. Namely, the second relation follows from the first by stripping the first by P^s_t using (3.7), and then multiplying the result by P^s_t.

Lemma 3.1. *Let $\theta \in B_t$ be an element which gives zero when stripped by P^s_t. Then*

$$\theta(\widehat{P^s_t})(P^s_t)^{2k+1} = 0 \qquad \Longrightarrow \qquad \theta(P^s_t)^{2k+2} = 0.$$

4. The doubling isomorphism

Let E be the exterior subalgebra of A generated by $\{\, P^0_t \,|\, t \geq 1 \}$. Recall that there is an algebra isomorphism $D : A \to A//E$ given by $D(Sq(s_1, s_2, \dots)) = [Sq(2s_1, 2s_2, \dots)]$ where $[x]$ denotes the equivalence class in $A//E$ of $x \in A$. In particular, since $D(P^{s-1}_t) = [P^s_t]$, we have $(P^s_t)^n = 0 \implies [P^s_t]^n = [0] \implies D\left((P^{s-1}_t)^n\right) = [0] \implies (P^{s-1}_t)^n = 0$ (since D is an isomorphism). Thus, by iterating we see that $(P^s_t)^n = 0$ implies $(P^i_t)^n = 0$ for all $i \leq s$. This proves the following.

Lemma 4.1. *For $t > 1$, if Theorem 1.1 holds for all $s \equiv -1 \mod t$, then it holds for all s.*

5. Proof of the main result

Having developed the necessary tools, the proof of Theorem 1.1 now follows by directly imitating the proof of [WW] for the case $t = 1$. Accordingly we proceed by proving the following two equations by induction on k for $1 \leq k \leq m$:

$$(\widehat{X_t})^{mt-1}_{kt-1}(P^{mt-1}_t)^{2k-1} = 0, \tag{5.1}$$

$$(\widehat{X_t})^{(m-1)t-1}_{kt-1}(P^{mt-1}_t)^{2k} = 0. \tag{5.2}$$

To begin the induction we note that $(\widehat{X_t})^{mt-1}_{t-1} = Sq_t(2^{t-1}\gamma_t(m))$ by [S2, Theorem 1.1]. Thus equation (5.1) for $k = 1$ follows from Proposition 2.1.

Equation (5.1) for k is equivalent to the relation

$$(\widehat{X_t})^{(m-1)t-1}_{kt-1}\widehat{P^{mt-1}_t}(P^{mt-1}_t)^{2k-1} = 0,$$

which implies equation (5.2) for k by Lemma 3.1 (interpret $(\widehat{X_t})^{(m-1)t-1}_{mt-1}$ as 1).

Finally, by Corollary 2.4 with $a = mt - 1$, $b = 1$, and $c = kt - 1$ we have

$$\begin{aligned}(\widehat{X_t})^{mt-1}_{(k+1)t-1}(P^{mt-1}_t)^{2k+1} &= (\widehat{X_t})^{mt-1}_{(k+1)t-1}P^{mt-1}_t(P^{mt-1}_t)^{2k} \\ &= Sq_t(z)(\widehat{X_t})^{(m-1)t-1}_{kt-1}(P^{mt-1}_t)^{2k}\end{aligned}$$

for some integer z so that equation (5.2) for k implies equation (5.1) for $k+1$. Thus by induction on k we see that equations (5.1) and (5.2) both hold for $1 \leq k \leq m$. Equation (5.2) for $k = m$ is $(P^{mt-1}_t)^{2m} = 0$, which proves the theorem for all $s \equiv -1 \mod t$. So by Lemma 4.1 this proves Theorem 1.1. □

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Department of Mathematics, University of Scranton, Scranton, Pennsylvania 18510

E-mail address: monks@uofs.edu