A NOTE ON INTERPOLATION IN THE HARDY SPACES OF THE UNIT DISC

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Abstract. In this note we formulate and solve a natural interpolation problem for the Hardy spaces in the unit disc in terms of maximal functions and weighted summable sequences.

1. Introduction

Let $D$ be the unit disc in the complex plane. For $0 < p < \infty$, $H^p(D)$ denotes the Hardy space of holomorphic functions in $D$ such that
\[ \|f\|_p^p = \sup_r \frac{1}{2\pi} \int_{-\pi}^{+\pi} |f(re^{i\theta})|^p \, d\theta < +\infty. \]

In this paper we are interested in the interpolating problem
\[ f(z_n) = w_n, \quad n = 1, 2, \ldots, \tag{1} \]
where $Z = \{z_n\}_{n=1}^\infty$ is a sequence in $D$ satisfying the Blaschke condition
\[ \sum_n (1 - |z_n|) < +\infty. \]

In [2] and [3], this problem has already been studied, proving that the restriction operator
\[ R: f \mapsto \{f(z_n)\}_{n=1}^\infty \]
maps $H^p$ onto $\{w_n: \sum_{n=1}^\infty |w_n|^p(1 - |z_n|) < +\infty\}$ if and only if $Z$ is uniformly separated, i.e.
\[ \inf_n \prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \bar{z}_k z_n} \right| \geq \delta > 0. \]

The starting point of this paper is the observation that the growth condition on the $\{w_n\}$,
\[ \sum_n (1 - |z_n|)|w_n|^p < +\infty, \tag{2} \]
is not necessary for a general Blaschke sequence, and in this sense the Shapiro-Shields result is somewhat unnatural. Here (Section 2) we first obtain elementary
necessary conditions on the \( \{w_n\}, \{z_n\} \) for the interpolation problem (1) to have a solution \( f \in H^p \). These conditions are expressed in terms of \( k \)-th order hyperbolic divided differences \( \Delta^k W \) of the sequence \( W = \{w_n\}_{n=1}^{\infty} \) and a corresponding maximal function \( W^*_k \). For \( k = 0 \) it is simply the statement that the maximal function

\[
W^*_0(e^{i\theta}) = \sup\{|w_n| : z_n \in C_t(\theta)\},
\]

where \( C_t(\theta) \) is the Stolz angle at \( e^{i\theta} \) of opening \( t \), must be in \( L^p(\mathbb{T}) \). This of course follows from the maximal characterization of \( H^p(\mathbb{D}) \). We also obtain necessary conditions of type (2) for a general Blaschke sequence \( Z \).

In Section 3 we pose and solve the corresponding interpolation problem, one for each \( k \). That is, if \( S^0_k(Z) = \{W = \{w_n\}_{n=1}^{\infty} : W^*_k \in L^p(\mathbb{T})\} \), we prove

**Theorem.** The restriction map \( R \) is onto from \( H^p \) to \( S^0_k(Z) \) if and only if \( Z \) is the union of \( k + 1 \) uniformly separated sequences.

As \( R \) always maps \( H^p(\mathbb{D}) \) into \( S^0_k(Z) \), for \( k = 0 \) this result might be called a “Shapiro-Shields theorem revisited”.

Finally, we mention that our study has close connections with [4], where a similar result is obtained for \( H^\infty \) (the first named author thanks Professor Nikolskii for pointing this out to him).

### 2. Necessary conditions

#### 2.1. We will denote by \( M_\alpha f \) the maximal function

\[
M_\alpha f(\theta) = \sup\{|f(z)|, z \in C_\alpha(\theta)\}
\]

corresponding to the angle \( \alpha \). For \( z, w \in \Delta \), we set

\[
\rho(z, w) = \frac{w - \overline{z}}{1 - \overline{z}w},
\]

so that \( |\rho(z, w)| \) is the pseudohyperbolic distance between \( z \) and \( w \).

The following well-known lemma is an obvious consequence of the Cauchy formula:

**Lemma 1.** Given \( 0 < \alpha < \beta < \pi \) there exists a constant \( C = C(\alpha, \beta) \) such that for all holomorphic \( f \) and all \( k \),

\[
\sup_{z \in C_\alpha(\theta)} (1 - |z|)^k |f^{(k)}(z)| \leq Ck! M_\alpha f(\theta).
\]

For a holomorphic function \( f \), we define

\[
\Delta^0 f(z) = f(z),
\]
\[
\Delta^1 f(z, w) = \frac{f(w) - f(z)}{\rho(z, w)}, \quad z, w \in \mathbb{D},
\]

and, inductively, for \( z_i \in \mathbb{D} \)

\[
(\Delta^k f)(z_1, \ldots, z_k, z_{k+1}) = \frac{(\Delta^{k-1} f)(z_1, \ldots, z_{k-1}, z_{k+1}) - (\Delta^{k-1} f)(z_1, \ldots, z_k, z_k)}{\rho(z_k, z_{k+1})}.
\]
Lemma 2. Given $0 < \alpha < \beta < \pi$, there exists a constant $C = C(\alpha, \beta)$ such that for any holomorphic function $f$ and $k \geq 1$, one has
\[
\sup_{z_1, \ldots, z_k \in C_\alpha(\theta)} \| (\Delta^k f)(z_1, \ldots, z_{k+1}) \| \leq C \sup_{t_1, \ldots, t_k \in C_\alpha(\theta)} \| (\Delta^{k-1} f)(t_1, \ldots, t_k) \|.
\]

Proof. First, let us consider the case $k = 1$. If $|\rho(z, w)| \geq \frac{1}{2}$, $|(\Delta^1 f)(z, w)| \leq 2(|f(z)| + |f(w)|)$, and if $|\rho(z, w)| < \frac{1}{2}$, $z, w \in C_\alpha(\theta)$, there exists an absolute constant $A$ such that $|(\Delta^1 f)(z, w)| \leq A \sup_{z \in C_\alpha(\theta)} (1 - |z|)|f'(z)|$. Hence
\[
\sup_{z_1, z_2 \in Z \cap C_\alpha(\theta)} \| (\Delta^1 f)(z_1, z_2) \| \leq 2M_\alpha(\theta) + A \sup_{z \in C_\alpha(\theta)} (1 - |z|)|f'(z)|
\]
and Lemma 1 finishes the proof.

For $k > 1$, fixed $z_1, \ldots, z_k$, consider $F_k(z) = (\Delta^{k-1} f)(z_1, \ldots, z_{k-1}, z)$ as a holomorphic function of $z$. Writing
\[
(\Delta^k f)(z_1, \ldots, z_{k+1}) = (\Delta^1 F_k)(z_k, z_{k+1})
\]
and applying the result for $k = 1$, one finishes the proof. \qed

The maximal characterization of $H^p(D)$ gives the following result.

Theorem 1. Let $f \in H^p$ and let $Z = \{z_n\}_{n=1}^\infty$ be a sequence of different points in $D$. Then, for $k \geq 0$
\[
\sup_{z_n \in Z \cap C_\alpha(\theta)} \| (\Delta^k f)(z_1, \ldots, z_{k+1}) \| \in L^p(T).
\]

This result immediately gives a set of necessary conditions for the problem (1). Denoting, as before, $W = \{w_n\}_{n=1}^\infty$, we introduce
\[
(\Delta^0 W)(w_n) = w_n, \quad (\Delta^1 W)(w_n, w_k) = \frac{w_k - w_n}{\rho(z_n, z_k)},
\]
\[
(\Delta^k W)(w_{n_1}, \ldots, w_{n_{k-1}}, w_{n_k}, w_{n_{k+1}}) = \frac{(\Delta^{k-1} W)(w_{n_1}, \ldots, w_{n_{k-1}}, w_{n_{k+1}}) - (\Delta^{k-1} W)(w_{n_1}, \ldots, w_{n_k})}{\rho(z_{n_k}, z_{n_{k+1}})}
\]
the maximal function
\[
W_k^\ast(e^{i\theta}) = \sup_{z_{n_1}, \ldots, z_{n_{k+1}} \in Z \cap C_\alpha(\theta)} \| (\Delta^k W)(z_1, \ldots, z_{k+1}) \|
\]
and the sequence spaces
\[
S_k^p(Z) = \{W: W_k^\ast \in L^p(T)\}
\]
with norm
\[
\|W\|_{p,0}^p = \|W_0^\ast\|_{L^p(T)}^p,
\]
\[
\|W\|_{p,k}^p = \|W_k^\ast\|_{L^p(T)}^p + \|W_{k-1}^\ast\|_{L^p(T)}^p.
\]

Then, $W \in S_k^p(Z)$ is a necessary condition for (1), for all $k$.\footnote{License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use}
2.2. Now we look for necessary conditions on $W = \{w_n\}_{n=1}^{\infty}$ for the problem (1) of the type of (2). The following lemma was proved in [1].

**Lemma 3.** If $h \in H^\infty(D)$ and $\varepsilon > 0$, the measure

$$\frac{|h'(z)|^2}{|h(z)|^{4\varepsilon}} (1 - |z|) dV(z)$$

is a Carleson measure with constant $C \|h\|_\infty / \varepsilon^2$, that is, for all $f \in H^p(D)$

$$\int_D |f(z)|^p \frac{|h'(z)|^2}{|h(z)|^{4\varepsilon}} (1 - |z|) dV(z) \leq \frac{C}{\varepsilon^2} \|f\|_p \|h\|_\infty.$$ 

Let us apply this last inequality to $h = B$, the Blaschke product with zeros in $Z$. We use the notation

$$B_n(z) = \prod_{k \neq n} \frac{z - z_k}{|z_k| - z_k z}, \quad \mu_n = \inf_{k \neq n} |\rho(z_n, z_k)|,$$

i.e. $z_n$ is at hyperbolic distance $\mu_n$ from the other points in $Z$. We denote by $D_n$ the hyperbolic disk centered at $z_n$ of radius $\mu_n/2$. As these are disjoint,

$$\frac{C}{\varepsilon^2} \|f\|_p \geq \int_D |f(z)|^p \frac{|B'(z)|^2}{|B(z)|^{2\varepsilon}} (1 - |z|) dV(z) \geq \sum_n \int_{D_n} |f(z)|^p \frac{|B'(z)|^2}{|B(z)|^{2\varepsilon}} (1 - |z|) dV(z).$$

In $D_n, 1 - |z| \approx 1 - |z_n|$ and

$$|B(z)| = |B_n(z)| \left| \frac{z - z_n}{1 - z_n z} \right| \approx \frac{|B_n(z)|}{1 - |z_n|}.$$ 

Hence

$$\frac{C}{\varepsilon^2} \|f\|_p \geq \sum_n (1 - |z_n|)^{2\varepsilon} \int_{D_n} |f(z)|^p \frac{|B'(z)|^2}{|B_n(z)|^{2\varepsilon} |z - z_n|^{2\varepsilon}} dV(z).$$

We may think that $D_n$ is a euclidean disk centered at $z_n$ of radius $\mu_n(1 - |z_n|)$. Using polar coordinates in $D_n$, this last integral equals

$$\int_0^{\mu_n(1 - |z_n|)} \int_0^{2\pi} |f(z_n + re^{i\theta})|^p \frac{|B'(z_n + re^{i\theta})|^2}{|B_n(z_n + re^{i\theta})|^{2\varepsilon}} d\theta dr.$$

In $D_n, B_n$ does not vanish, hence by subharmonicity the integral in $\theta$ dominates

$$|f(z_n)|^p \frac{|B'(z_n)|^2}{|B_n(z_n)|^{2\varepsilon}} = |f(z_n)|^p \frac{|B_n(z_n)|^2}{(1 - |z_n|^2)^2}.$$

Thus we obtain

$$\frac{C}{\varepsilon^2} \|f\|_p \geq \sum_n (1 - |z_n|)(|B_n(z_n)|\mu_n)^\varepsilon |f(z_n)|^p.$$ 

We have therefore proved

**Theorem 2.** For a Blaschke sequence $\{z_n\}_{n=1}^{\infty}$, the measure

$$\sum_n (1 - |z_n|)(|B_n(z_n)|\mu_n)^\varepsilon \delta_{z_n}$$ 

is a Carleson measure with constant $C/\varepsilon, \varepsilon > 0$. 

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If \( \{z_n\}_{n=1}^{\infty} \) is a uniformly separated sequence, this result recaptures the well-known fact that
\[
\sum_n (1 - |z_n|) \delta_{z_n}
\]
is a Carleson measure.

Of course, Theorem 2 gives as a necessary condition on \( W = \{w_n\} \) for (1), namely
\[
\sum_n (1 - |z_n|)(|B_n(z_n)|\mu_n)^p|w_n|^p < +\infty, \quad \varepsilon > 0,
\]
a Shapiro-Shields type condition. We point out that (4) is already captured by the statement \( W \in S^p_0(Z) \). This follows from the fact that Carleson measures boundedly operate on (nonnecessarily holomorphic) functions having maximal function in \( L^p(T) \) (in this case the function equals \( w_n \) on \( z_n \) and 0 elsewhere).

Theorem 2 can be improved, in the sense that \( \varphi(t) = t^\varepsilon \) can be replaced by a function \( \varphi \) satisfying a Dini-type condition. For instance, multiplying both terms of (3) by \( \varepsilon^{-\beta} \) and integrating in \( \varepsilon \), obtains
\[
\sum_n (1 - |z_n|)(\log(|B_n(z_n)|\mu_n))^{-1-\beta}|f(z_n)|^p \leq C \varepsilon^{-\beta}, \quad \beta > 0,
\]
which can be integrated again, and so on. This leads to improvements of (4), all of them included in the statement \( W \in S^p_0(Z) \). In fact, it is an interesting question to obtain conditions like (4) from \( W \in S^p_0(Z) \) using only the geometry of the sequence \( Z \).

### 3. Sufficient conditions

Let \( Z = \{z_n\} \) be a Blaschke sequence. In section 2.1 it has been shown that the restriction operator
\[
R: f \to \{f(z_n)\}_{n=1}^{\infty}
\]
maps \( H^p \) into \( S^p_k(Z) \), \( k = 0, 1, 2, \ldots \).

**Theorem 3.** Let \( Z = \{z_n\} \) be a Blaschke sequence and \( k \geq 0 \). The restriction operator \( R \) maps \( H^p \) onto \( S^p_k(Z) \) if and only if \( Z \) is the union of \( k + 1 \) uniformly separated sequences.

**Proof.** Assume \( R \) is onto. Consider \( W = \{w_n\} \), \( w_n = \delta_{n,m} \), i.e. \( w_n = 0 \) if \( n \neq m \) and \( w_m = 1 \). An easy inductive argument shows
\[
W_k(\varepsilon^\beta) \leq 2^k \frac{1}{\rho(z_m, z_{m_1}) \cdots \rho(z_m, z_{m_k})}, \quad z_m \in C_\alpha(\theta),
\]
and hence
\[
\|W\|_{p,k} \leq 2^k (1 - |z_m|)^{1/p} \frac{1}{\rho(z_m, z_{m_1}) \cdots \rho(z_m, z_{m_k})},
\]
where \( \{z_m : j = 1, \ldots, k\} \) are the \( k \) points in \( \{z_n\} \) closest in the pseudohyperbolic distance to \( z_m \). Now, since \( R \) is onto, by the open mapping theorem there exists \( f_m \in H^p \), \( f_m(z_n) = w_n, \|f_m\|_p \leq C\|W\|_{p,k} \) where \( C \) is a constant independent of \( m \).
Hence, \( f_m = B_m \cdot g_m \) and
\[
|B_m(z_m)|^{-1} = |g_m(z_m)| \leq C_1 \frac{\|g_m\|_p}{(1 - |z_m|)^{1/p}} \leq \frac{C_1 C 2^k}{|\rho(z_m, z_{m_1}) \cdots \rho(z_m, z_{m_k})|}.
\]
So,
\[(5) |B_m(z_m)| \geq A|\rho(z_m, z_{m_1}) \cdots \rho(z_m, z_{m_k})|.
\]
We will show that (5) implies that \( Z \) is the union of \( k + 1 \) uniformly separated sequences. By Zorn’s lemma, there exists a maximal subset \( Z_1 \) of \( Z \) such that if \( z_r, z_j \in Z_1 \) one has \( |\rho(z_r, z_j)| > 2^{-1}A \). Do the same for \( Z \) replaced by \( Z \setminus Z_1 \) and repeat the process to obtain \( Z_1, \ldots, Z_{k+1} \). By (5) these sequences are uniformly separated. Now let us show
\[
Z = \bigcup_{j=1}^{k+1} Z_j.
\]
If this were not true, there exists \( z_m \in Z \setminus \bigcup_{j=1}^{k+1} Z_j \). By the maximality of each \( Z_j \), there exists \( z_{m,j} \in Z_j \) such that \( |\rho(z_m, z_{m,j})| < 2^{-1}A \). Hence, there exist \( k + 1 \) points in \( Z \) at pseudohyperbolic distance from \( z_m \) less than \( 2^{-1}A \). This contradicts (5).

To prove the converse, consider first the case \( k = 0 \), that is, \( Z = \{z_n\} \) a uniformly separated sequence and \( W = \{w_n\} \in S_p^p(Z) \), i.e. \( W_0^s(e^{ih}) = \sup\{|w_n| : z_n \in C_i(\theta)| \) in \( L^p(T) \). Since Carleson measures boundedly operate on functions having maximal function in \( L^p(T) \), (2) is satisfied and the Shapiro-Shields result gives \( f \in H^p(D), f(z_n) = w_n, n = 1, 2, \ldots \). However, using that \( W \in S_p(Z) \) we can give a more elementary proof.

By normal families, the result will be proved if we show that there exists \( C > 0 \) such that for any \( N \), there is \( f_N \in H^p(D) \), satisfying \( f_N(z_n) = w_n, n = 1, \ldots, N \), and \( \|f_N\|_p \leq C \).

Take \( \delta > 0 \) such that \( D_n = \{z : \rho(z, z_n) \leq 2\delta\} \) are pairwise disjoint. Let \( H = H_N \) be a \( C^\infty \) in \( D, H(z) = w_n \) if \( \rho(z, z_n) \leq \delta, H = 0 \) or \( D \setminus \bigcup_{n=1}^N D_n \) and \( |H(z)| \leq |w_n| \) for \( z \in D_n \). It is clear that \( \|M_{\beta}(H)\|_p \leq \|W\|_{p,0} \) for some \( \beta < \alpha \). Let \( B \) be the Blaschke product with zero set \( Z \). We look for solutions of (1) of the form \( H = BG \), where
\[(6) \quad \partial(G) = B^{-1} \partial(H), \quad \|G\|_{L^p(\gamma)} \leq C
\]
and \( C \) is a constant independent on \( N \).

Since \( Z = \{z_n\} \) is uniformly separated, one has \( |B(z)| \geq C \inf_n |\rho(z, z_n)| \). Hence,
\[
|B(z)^{-1} \partial(H(z))| dm(z) \leq C(\delta) \sum_n |w_n| (1 - |z_n|)^{-1} dm_{D_n} \leq C(\delta) |H(z)| \sum_n (1 - |z_n|)^{-1} dm_{D_n}.
\]
Observe that \( \mu = \sum_n (1 - |z_n|)^{-1} dm_{D_n} \) is a Carleson measure. Now, the function
\[
G(z) = \frac{1}{\pi} \int_\mathbb{D} \frac{1 - |\xi|^2}{(\xi - z)(1 - \xi z)} B(\xi)^{-1} \partial H(\xi) dm(\xi)
\]
satisfies \( \partial G = B^{-1} \partial H \). We estimate \( \|G\|_p \) by duality.
Let $A \in L^q(T), p^{-1} + q^{-1} = 1$ and denote by $P[A](\xi)$ the Poisson integral of $A$ at the point $\xi$. One has
\[
\left| \int_{\mathbb{D}} G(e^{i\theta})A(e^{i\theta}) \, d\theta \right| \leq \int_{\mathbb{D}} |P[A](\xi)||B(\xi)|^{-1} |\partial H(\xi)\, d\mu(\xi)
\leq C(\delta) \int_{\mathbb{D}} |P[|A|](\xi)||H(\xi)| \, d\mu(\xi) \leq C(\delta)C_1 \|A\|_{L^q(T)},
\]
where $C_1$ is independent on $N$, because $P[|A|](\xi) \cdot H(\xi)$ has maximal function in $L^q(T)$, so the function $G$ satisfies (6) and this finishes the proof for $k = 0$. \hfill \Box

Assume the proof is completed for $k$ and let us show it for $k+1$, that is, assume $Z$ is the union of $k+1$ uniformly separated sequences. One can split the sequence $Z = Z_1 \cup Z_2$, where $Z_1 = \{a_n\}$ is the union of $k$ uniformly separated sequences and $Z_2 = \{z_n\}$ is uniformly separated.

Let $W \in S^p_{k+1}(Z)$. The previous splitting for $Z$ gives $W = W_1 \cup W_2$, $W_1 = \{s_n\}$, $W_2 = \{w_n\}$. Applying the result for $k = 0$, one gets $f_2 \in H^p(\mathbb{D}), f_2(z_n) = w_n$, $n = 1, 2, \ldots$. Let $B_2$ be the Blaschke product with zero sequence $Z_2$. Now we look for a function $f \in H^p(\mathbb{D})$ such that
\[
f(a_n) = \frac{s_n - f_2(a_n)}{B_2(a_n)}, \quad n = 1, 2, \ldots,
\]
because $f_2 + B_2 f$ will interpolate $W$ at the points $Z$. By induction, (7) is solvable if and only if
\[
\{(s_n - f_2(a_n))B_2(a_n)^{-1}\} \in S^p_k(Z_1).
\]
Let $z_{k(n)}$ be the closest point, in the pseudohyperbolic metric, in $Z_2$ to $a_n$. Then,
\[
(s_n - f_2(a_n))B_2(a_n)^{-1} = \frac{s_n - w_{k(n)} + \rho(a_n, z_{k(n)})}{\rho(a_n, z_{k(n)})} \frac{\rho(a_n, z_{k(n)})}{B_2(a_n)} + \frac{f_2(z_{k(n)}) - f_2(a_n)}{\rho(a_n, z_{k(n)})} = \frac{B_2(z_{k(n)})}{B_2(a_n)}.
\]
Now, since $W \in S^p_{k+1}(Z)$ and $f_2 \in H^p(\mathbb{D})$, one has
\[
\left\{ \frac{s_n - w_{k(n)}}{\rho(a_n, z_{k(n)})} \right\} \in S^p_k(Z_1), \quad \left\{ \frac{f_2(z_{k(n)}) - f_2(a_n)}{\rho(a_n, z_{k(n)})} \right\} \in S^p_k(Z_1).
\]
Hence in order to finish the proof it is sufficient to show the following two auxiliary results.

**Lemma 4.** Let $Z$ be a Blaschke sequence, $W = \{w_n\}$ and $A = \{a_n\}$ two sequences of complex numbers and denote by $WA$ the sequence $\{w_n a_n\}$. Then for $k \geq 0$,
\[
(\Delta^k(WA))(w_{n_1}, \ldots, w_{n_k}, a_{n_{k+1}})
= \sum_{j=0}^{k} (\Delta^j W)(w_{n_1}, \ldots, w_{n_j}) \cdot (\Delta^{k-j}A)(a_{n_{j+1}}, \ldots, a_{n_{k+1}}).
\]

**Lemma 5.** Let $Z = \{z_n\}$ be a uniformly separated sequence, $B$ the Blaschke product with zero set $Z$ and $0 < \delta$ such that the discs $D_n = \{z : |\rho(z, z_n)| \leq \delta\}$ are pairwise disjoint. Consider $\Omega = \bigcup_n D_n$ and $\varphi : \Omega \rightarrow \mathbb{C}, \varphi(a) = B_{b(a)}(a)^{-1}$ where $b(a) = z_n$ if $a \in D_n$. Let $A = \{a_n\} \in \Omega$ and $\varphi(A) = \{\varphi(a_n)\}$. Then $\varphi(A) \in S^p_k(A)$, for any $k \geq 0$. \hfill \Box
Lemma 4 follows from a simple inductive argument. The case $k = 0$ of Lemma 5 follows from the fact that $Z$ is a uniformly separated sequence. For $k > 0$, one shows by induction that

$$z \to \Delta^n(a_{n_1}, \ldots, a_{n_m}, z)$$

is a bounded analytic function in $\Omega$.

Finally, concerning the necessary condition (4), since it is captured from the fact $W \in S^p_0(Z)$, Theorem 3 shows

$$R(H^p(D)) = \{W : W \text{ satisfies (4)}\}$$

if and only if $Z$ is a uniformly separated sequence.

References


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