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FINITE AND ω-RESOLVABILITY

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Abstract. A topological space is *k-resolvable* ($2 \leq k \leq \omega$) if X has k disjoint dense subsets. In this paper, we prove that if X is *k-resolvable* for each positive integer k, then X is *ω-resolvable.*

Introduction

In [4], E. Hewitt defined a topological space to be *resolvable* (resp. *irresolvable*) if it is (resp. is not) the union of two disjoint dense subsets. Since then, more general notions related to resolvability have been studied by several topologists. For $0 \leq k \leq \omega$, a space X is *k-resolvable* if X has k disjoint dense subsets and X is *k-irresolvable* if X is not *k-resolvable.* For $k < \omega$, spaces which are k-resolvable but not $(k+1)$-resolvable have been constructed in [1], [3] and [5]. In this paper, we prove that if X is k-resolvable for each $k < \omega$, then X is ω-resolvable. The referee has communicated to the author that this theorem was proven by Eric K. Van Douwen but apparently never published.

Definition 1. A space X is *hereditarily irresolvable* if every nonempty open subset of X is irresolvable.

Lemma 2. *Suppose that $X = D_1 \cup \cdots \cup D_n$, where the sets $D_1, \cdots, D_n$ are pairwise disjoint and hereditarily irresolvable. Then X is $(n+1)$-irresolvable.*

Proof. First, we will prove the following assertion:

($*$) Suppose that $X = E_1 \cup \cdots \cup E_r$, where $E_1, \cdots, E_r$ are pairwise disjoint. Let U be a nonempty open subset of X; then there exists $i \in \{1, \ldots, r\}$ and there exists a nonempty open subset V of U such that $V \cap D_1 \subset E_i$.

In order to prove ($*$), we may assume that $D_1 \cap U \neq \emptyset$. From the "Baire Category Theorem" for finite sets, there exists $i \in \{1, \ldots, r\}$ such that $\mathrm{Int}_{(D_1 \cap U)}(\mathrm{Cl}_{(D_1 \cap U)}(D_1 \cap U \cap E_i)) \neq \emptyset$. Let V_1 be an open subset of X such that $V_1 \subset U$ and $V_1 \cap D_1 = \mathrm{Int}_{(D_1 \cap U)}(\mathrm{Cl}_{(D_1 \cap U)}(D_1 \cap U \cap E_i))$. Since $V_1 \cap D_1$ is irresolvable and $V_1 \cap D_1 \cap E_i$ is a dense subset of $V_1 \cap D_1$, then $V_1 \cap D_1 - E_i$ is not dense in $V_1 \cap D_1$. Therefore there exists an open subset V of V_1 such that $\emptyset \neq V \cap D_1 \subset E_i$. This ends the proof of the assertion ($*$).

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We are ready to prove the lemma. Suppose that $X = E_1 \cup \cdots \cup E_{n+1}$, where the sets $E_1, \ldots, E_{n+1}$ are pairwise disjoint and dense in X.

From $(*)$, there exists a nonempty open subset U_1 of X and there exists $i_1 \in \{1, \ldots, n+1\}$ such that $U_1 \cap D_1 \subset E_{i_1}$. Applying $(*)$ again, there exists a nonempty open subset U_2 of U_1 and there exists $i_2 \in \{1, \ldots, n+1\}$ such that $U_2 \cap D_2 \subset E_{i_2}$. Proceeding in this way, we have that there exists a sequence $U_1 \supset U_2 \supset \cdots \supset U_n$ of nonempty open subsets of X and there exists a sequence $\{i_1, \ldots, i_n\}$ of elements of $\{1, \ldots, n+1\}$ such that $U_k \cap D_k \subset E_{i_k}$ for every k.

Fix $i \in \{1, \ldots, n+1\} - \{i_1, \ldots, i_n\}$; then

$$U_n \cap E_i \subset ((U_1 \cap D_1) \cup \cdots \cup (U_n \cap D_n)) \cap E_i \subset (E_{i_1} \cup \ldots \cup E_{i_n}) \cap E_i = \emptyset.$$

This contradicts the density of E_i and completes the proof of the lemma.

Lemma 3. [Compare with Fact 3.1 in [5]] *If X is irresolvable, then there exists a nonempty, hereditarily irresolvable open subset of X.*

Proof. Let $U = \bigcup\{V\colon\ V$ is a resolvable open subset of $X\}$. From the main theorem in [2], U is resolvable. Then $\mathrm{Cl}(U)$ is resolvable and $X \neq \mathrm{Cl}(U)$. Hence $W = X - \mathrm{Cl}(U)$ is a nonempty, hereditarily irresolvable open subset of X.

Lemma 4. *Let X be a topological space. Then there exists an open subset W of X such that $X - \mathrm{Cl}(W)$ is ω-resolvable and W contains a hereditarily irresolvable dense subset. (W could be equal to X or to the empty set.)*

Proof. Let $\mathcal{U} = \{U \subset X\colon\ U$ is a nonempty open subset of X and U contains a hereditarily irresolvable dense subset$\}$.

If $\mathcal{U} \neq \emptyset$, let $\mathcal{A}$ be a maximal family of nonempty, pairwise disjoint elements of $\mathcal{U}$. $W = \bigcup\{U\colon\ U \in \mathcal{A}\}$. In the case that $\mathcal{U} = \emptyset$, define $W = \emptyset$.

For each $U \in \mathcal{A}$, let D_U be a hereditarily irresolvable dense subset of U. Let $D_0 = \bigcup\{D_U\colon\ U \in \mathcal{A}\}$. Clearly D_0 is a hereditarily irresolvable dense subset of W.

Now, we will show that every dense subset of $X - \mathrm{Cl}(W)$ is resolvable. For this, let D be a dense subset of $X - \mathrm{Cl}(W)$. Assume, on the contrary, that D is irresolvable. From Lemma 3, there exists a nonempty, hereditarily irresolvable open subset U_1 of D. Let U_0 be an open subset of X such that $U_1 = U_0 \cap D$ and $U_0 \subset X - \mathrm{Cl}(W)$. Then U_1 is dense in U_0, so $U_0 \in \mathcal{U}$. This contradicts the maximality of $\mathcal{A}$ and completes the proof of the assertion.

Finally, we will prove that $X - \mathrm{Cl}(W)$ is ω-resolvable. The assertion implies that there exist disjoint dense subsets D_1 and E_1 of $X - \mathrm{Cl}(W)$. Applying it again, there exist disjoint dense subsets D_2 and E_2 of E_1 such that $E_1 = D_2 \cup E_2$. Proceeding in this way, sequences $\{D_n\}_n$ and $\{E_n\}_n$ can be constructed such that, for each n, D_{n+1} and E_{n+1} are disjoint dense subsets of E_n such that $E_n = D_{n+1} \cup E_{n+1}$. Then

$$X - \mathrm{Cl}(W) = (D_1 \cup ((X - \mathrm{Cl}(W)) - \bigcup\{D_n\colon\ n \geq 2\})) \cup D_2 \cup D_3 \cup \cdots .$$

The sets in this union are dense pairwise disjoint subsets of $X - \mathrm{Cl}(W)$. Hence $X - \mathrm{Cl}(W)$ is ω-resolvable.

Theorem 5. *If X is n-resolvable for each n, then X is ω-resolvable.*

Proof. From Lemma 4, there exists an open subset W_1 in X and there exists a hereditarily irresolvable dense subset D_1 of W_1 such that $X - \mathrm{Cl}(W_1)$ is ω-resolvable.

Let $X - \mathrm{Cl}(W_1) = E_1^{(1)} \cup E_2^{(1)} \cup \cdots$, where $E_1^{(1)}, E_2^{(1)}, \ldots$ are pairwise disjoint dense subsets of $X - \mathrm{Cl}(W_1)$. Let $X_0 = X$.

Let $X_1 = W_1 - D_1$. Applying again Lemma 4, there exists an open subset W_2 of X_1 such that W_2 contains a hereditarily irresolvable dense subset D_2 and $X_1 - \mathrm{Cl}_{X_1}(W_2)$ is ω-resolvable.

Proceeding in this way, it is possible to find sequences $\{X_n\}_n$, $\{W_n\}_n$, and $\{D_n\}_n$ such that $X_n = W_n - D_n$, W_{n+1} is an open subset of X_n, D_n is a hereditarily irresolvable dense subset of W_n and $X_n - \mathrm{Cl}_{X_n}(W_{n+1})$ is ω-resolvable. For each n, let $X_n - \mathrm{Cl}_{X_n}(W_{n+1}) = E_{n+1}^{(n+1)} \cup E_{n+2}^{(n+1)} \cup \cdots$, where $E_{n+1}^{(n+1)}$, $E_{n+2}^{(n+1)}, \ldots$ are pairwise disjoint dense subsets of $X_n - \mathrm{Cl}_{X_n}(W_{n+1})$.

Define $F_1^* = E_1^{(1)} \cup D_1$. For each $n \geq 2$, define $F_n = E_n^{(1)} \cup \cdots \cup E_n^{(n)} \cup D_n$. And define $F_1 = F_1^* \cup (X - \bigcup\{F_n\colon\ n \geq 2\})$. Clearly, $X = \bigcup\{F_n\colon\ n \geq 1\}$.

If $n < m$, $E_k^{(m)} \subset X_{m-1} \subset X_n \subset W_n$ and $E_j^{(n)} \subset X_{n-1} - W_n$. Hence the elements of the family $\{E_k^{(n)}\colon\ n \in \mathbb{N} \text{ and } k \geq n\}$ are pairwise disjoint. If $n < m$, $D_m \subset W_m \subset X_{m-1} \subset X_n \subset W_n - D_n$, so $D_1, D_2, \ldots$ are pairwise disjoint. If $m \geq n$, $D_m \subset W_m \subset W_n$ and $E_k^{(n)} \subset X_{n-1} - W_n$, then $E_k^{(n)} \cap D_m = \varnothing$, if $m < n$, $E_k^{(n)} \subset X_{n-1} \subset X_m = W_m - D_m$, so $E_k^{(n)} \cap D_m = \varnothing$. Therefore, $F_1, F_2, \ldots$ are pairwise disjoint.

We will prove that X_n is dense in W_n. Suppose, on the contrary, that there exists an open subset V of X such that $\varnothing \neq V \cap W_n \subset W_n - X_n$. For each $k = 1, \ldots, n$, let Z_k be an open subset of X such that $W_k = Z_k \cap X_{k-1}$. Define $U_0 = V \cap Z_1 \cap \cdots \cap Z_n$. We assert that $U_0 \subset D_1 \cup \cdots \cup D_n$. If there exists a point $p \in U_0 - (D_1 \cup \cdots \cup D_n)$, then $p \in Z_1 = Z_1 \cap X_0 = W_1$, so $p \in X_1 \cap Z_2 = W_2$. Proceeding in this way, we obtain that $p \in V \cap W_n \subset W_n - X_n$. This contradiction proves that $U_0 = (U_0 \cap D_1) \cup \cdots \cup (U_0 \cap D_n)$. Since $\varnothing \neq V \cap W_n \subset W_{n-1} \subset \cdots \subset W_1$, then U_0 is nonempty. For each $k = 1, \ldots, n, V \cap W_n \subset U_0 \cap W_k$. Then $U_0 \cap W_k$ is a nonempty subset of W_k; thus $U_0 \cap D_k$ is a nonempty open subset of D_k. Since D_k is hereditarily irresolvable, then $U_0 \cap D_k$ is hereditarily irresolvable. From Lemma 2 we have that U_0 is not $(n+1)$-resolvable. This implies that X is not $(n+1)$-resolvable. This contradiction proves that X_n is dense in W_n.

Finally, we will prove that F_n is dense in X. For this, let U be a nonempty subset of X. If $U \subset X - \mathrm{Cl}(W_1)$, then $\varnothing \neq U \cap E_n^{(1)} \subset U \cap F_n$. Then, we may assume that $U \cap W_1 \neq \varnothing$, then $U \cap X_1 \neq \varnothing$. If $U \cap X_1 \subset X_1 - \mathrm{Cl}_{X_1}(W_2)$. Then $\varnothing \neq U \cap X_1 \cap E_n^{(2)} \subset U \cap F_n$. Then we may assume that $U \cap W_2 \neq \varnothing$, so $U \cap X_2 \neq \varnothing$. Proceeding in this way we have that $U \cap F_n \neq \varnothing$ or $U \cap W_n \neq \varnothing$. In the second case, $\varnothing \neq U \cap W_n \cap D_n \subset U \cap F_n$. This proves that F_n is dense in X and completes the proof of the theorem.

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