CENTRAL UNITS OF INTEGRAL GROUP RINGS
OF NILPOTENT GROUPS

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Abstract. In this paper a finite set of generators is given for a subgroup of
finite index in the group of central units of the integral group ring of a finitely
generated nilpotent group.

In this paper we construct explicitly a finite set of generators for a subgroup of
finite index in the centre $Z(U(\mathbb{Z}G))$ of the unit group $U(\mathbb{Z}G)$ of the integral group
ring $\mathbb{Z}G$ of a finitely generated nilpotent group $G$. Ritter and Sehgal [4] did the
same for finite groups $G$, giving generators which are a little more complicated.
They also gave in [2] necessary and sufficient conditions for $Z(U(\mathbb{Z}G))$ to be trivial;
recall that the units $\pm G$ are called the trivial units. We first give a finite set of
generators for a subgroup of finite index in $Z(U(\mathbb{Z}G))$ when $G$ is a finite nilpotent
group. Next we consider an arbitrary finitely generated nilpotent group and prove
that a central unit of $\mathbb{Z}G$ is a product of a trivial unit and a unit of $\mathbb{Z}T$, where $T$
is the torsion subgroup of $G$. As an application we obtain that the central units
of $\mathbb{Z}G$ form a finitely generated group and we are able to give an explicit set of
generators for a subgroup of finite index.

1. Finite nilpotent groups

Throughout this section $G$ is a finite group. When $G$ is Abelian, it was shown in
[1] that the Bass cyclic units generate a subgroup of finite index in the unit group
$U(\mathbb{Z}G)$. Using a stronger version of this result, also proved by Bass in [1], we will construct
a finite set of generators from the Bass cyclic units when $G$ is finite nilpotent.

Our notation will follow that in [6]. The following lemma is proved in [1].

Lemma 1. The images of the Bass cyclic units of $\mathbb{Z}G$ under the natural homomorphism $j : U(\mathbb{Z}G) \to K_1(\mathbb{Z}G)$ generate a subgroup of finite index.

Let $L$ denote the kernel of this map $j$, and $B$ the subgroup of $U(\mathbb{Z}G)$ generated
by the Bass cyclic units. It follows that there exists an integer $m$ such that $z^m \in LB$
for all $z \in Z(U(\mathbb{Z}G))$, and so we can write $z^m = lb_1b_2 \cdots b_k$ for some $l \in L$ and
Bass cyclic units $b_i$. 

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Next, let $Z_i$ denote the $i$-th centre of $G$, and suppose from now on that $G$ is nilpotent of class $n$. For any $x \in G$ and Bass cyclic unit $b \in \mathbb{Z}(x)$, we define

$$b(1) = b$$

and for $2 \leq i \leq n$

$$b(i) = \prod_{g \in Z_i} b^g_{(i-1)},$$

where $\alpha^g = g^{-1} \alpha g$ for $\alpha \in \mathbb{Z}G$. Note that by induction $b(i)$ is central in $\mathbb{Z}(Z_i, x)$ and independent of the order of the conjugates in the product expression. In particular, $b(n) \in Z(\mathcal{U}(\mathbb{Z}G))$.

Recall again that if $z \in Z(\mathcal{U}(\mathbb{Z}G))$, then $z^m = lb_1b_2 \cdots b_k$ for some $l \in L$ and Bass cyclic units $b_i$. Since $K_1(\mathbb{Z}G)$ is Abelian, we can write

$$z^m|Z_2|Z_3| \cdots |Z_n| = (lb_1b_2 \cdots b_k)|Z_2|Z_3| \cdots |Z_n|$$

for some $l_1 \in L$ and for some $l_2 \in L$ and for some $l' \in L$. Since each $b_i(n)$ is in $Z(\mathcal{U}(\mathbb{Z}G))$, we conclude that $l' \in L \cap Z(\mathcal{U}(\mathbb{Z}G))$. But we shall show next that $L \cap Z(\mathcal{U}(\mathbb{Z}G))$ is trivial, so $l' \in \pm Z(G)$. The argument uses the same idea as in [3, Lemma 3.2].

For every primitive central idempotent $e$ in the rational group algebra $QG$, the simple ring $QGe$ has a reduced norm which we denote by $nr_e$. Further, denote

$$m_e = \sqrt{[QGe : Z(QGe)]}$$

and let

$$r = \prod_e m_e.$$

Now let $l' \in L \cap Z(\mathcal{U}(\mathbb{Z}G))$. By definition of $K_1(\mathbb{Z}G)$ this means that a suitable matrix

$$\begin{bmatrix}
    l' \\
    1 \\
    \cdots \\
    1
\end{bmatrix}$$

is a product of commutators. Therefore $l'e$ has reduced norm one. Since $l'e$ is also central, we obtain

$$(l'e)^{m_e} = nr(l'e)e = e.$$  

Hence

$$l'e = 1.$$

So $l'$ is a torsion central unit, and therefore is trivial [7, Corollary 1.7, page 4].

Since $Z(\mathcal{U}(\mathbb{Z}G))$ is finitely generated (see, e.g., [2]), $(\mathcal{U}(\mathbb{Z}G))|Z_2|Z_3| \cdots |Z_n|$ is of finite index. But we have just seen that the latter subgroup is contained in the subgroup generated by $\pm Z(G)$ and $\{b_i(n) : b \text{ a Bass cyclic unit} \}$. We have proved
Proposition 2. Let \( G \) be a finite nilpotent group of class \( n \). Then
\[
\langle b_{i(a)} \mid b \text{ a Bass cyclic} \rangle
\]
is of finite index in \( Z(\mathcal{U}(\mathbb{Z}G)) \).

Remark. Note that our method for constructing generators for a subgroup of finite index in \( Z(\mathcal{U}(\mathbb{Z}G)) \) can be adapted for some other classes of finite groups \( G \). For example, if \( G = D_{2n} = \langle a, b \mid x^n = 1, y^2 = 1, yx = x^{n-1}y \rangle \), the dihedral group of order \( 2n \), then the only nontrivial Bass cyclic units \( b \) of \( \mathbb{Z}D_{2n} \) belong to \( \mathbb{Z}(x) \). It follows that \( bb^y = b^yb \) is central. Our proof now remains valid and yields that \( \langle bb^y \mid b \text{ a Bass cyclic in } \mathbb{Z}(x) \rangle \) is of finite index in \( Z(\mathcal{U}(\mathbb{Z}D_{2n})) \).

2. FINITELY GENERATED NILPOTENT GROUPS

We will now consider central units of an integral group ring of an arbitrary finitely generated nilpotent group \( G \). The torsion subgroup of \( G \) is denoted \( T \). First we show that central units of \( \mathbb{Z}G \) have the following decomposition.

Proposition 3. Let \( G \) be a finitely generated nilpotent group. Every \( u \in Z(\mathcal{U}(\mathbb{Z}G)) \) can be written as \( u = rg, r \in \mathbb{Z}T, g \in G \).

Proof. Let \( F = G/T \). Since \( T \) is finite and \( F \) acts on the set of primitive central idempotents of \( \mathbb{Q}T \) by conjugation, by adding the idempotents in an orbit of this action we obtain
\[
\mathbb{Q}T = \bigoplus (\mathbb{Q}T)e_i = \bigoplus R_i,
\]
where \( e_i \) are primitive central idempotents of \( \mathbb{Q}G \). Then \( \mathbb{Q}G \) is the crossed product
\[
(\ast) \quad \mathbb{Q}G = \mathbb{Q}T \ast F = \left( \bigoplus R_i \right) \ast F = \bigoplus R_i \ast F.
\]
Decompose \( u \) as a sum of elements in (\( \ast \)):
\[
u = \bigoplus \left( \sum_{j=1}^{n} u_jf_j \right), \quad 0 \neq u_j \in R_i, f_j \in G, \text{ for each } j.
\]
We assume that we have put together the \( u_j \)'s with the same \( f_jT \in G/T \), namely for \( k \neq j, f_kT \neq f_jT \).

We claim that \( n = 1 \). Let us denote by \( - \) the projection of \( \mathbb{Q}G \) onto \( R_i \ast F \). Then since \( u \) is central we have \( \mathbb{Q}T \pi = \pi \mathbb{Q}T \), which implies \( \mathbb{Q}T u_jf_j = u_jf_j \mathbb{Q}T \) for all \( j \). It follows that \( u_j \) is not a zero divisor provided \( R_i \) has only one simple (artinian) component, and so \( u_j \) is a unit. The only time \( u_j \) can be a nonunit is when it has some zero components in the simple components of \( R_i \). However, by the construction of \( R_i \), these latter components can be moved to any other place by conjugating suitably. But they must stay put due to the facts that \( F \) is ordered and \( \pi \) is central. It follows that \( u_j \) is a unit for all \( j \). Hence, working in \( R_i \ast F \) and using again that \( F \) is ordered, it follows by a classical argument that \( \pi = \sum_j u_jf_j \) is simply equal to \( u_nf_n \) as claimed.

Changing notation, we write
\[
u = \bigoplus \alpha f, \quad \alpha \in R_i, f \in G.
\]
Let $k = |\text{Aut}(T)|$, so $f^k$ commutes with $T$ for $f \in G$. Hence
\[ u^k = \bigoplus (\alpha f)^k = \bigoplus \beta f^k, \quad \beta \in R_t \]
(note that the number of summands in $u^k$ is the same as the number of summands in $u$, because each $\alpha$ is a unit in $R_t$), and thus
\[ u^k = (u^k)^t = \bigoplus (\beta f^k)^t = \bigoplus \beta f^k, \quad t \in T. \]
The last step follows from the fact that conjugation will preserve the order on the $fT$'s in the ordered group $F$. Since $(f^k)^t = tf^k$, we can choose $k$ large enough so that all the $f^k$ commute with each other and with $T$. Thus we may assume that
\[ u^k = \bigoplus \beta f^k. \]
Again, we put together all $\beta$ with the same $f^kT$. In other words, we assume that $u^k = \bigoplus \beta f^k$ with all $f^kT$ different. Note that these new values of $\beta$ all lie in $ZT$. Furthermore, we now obtain for each $t \in T$,
\[ u^k = (u^k)^t = \bigoplus (\beta f^k)^t, \]
and thus $\beta^t = \beta$. So the ring $R$ generated by all the $\beta$ is commutative. Again, if necessary, replacing $k$ by a high enough power, we may assume that the group $A$ generated by all the $f^k$ in the summation of $u^k$ is a torsion-free Abelian group, and thus a free Abelian group. Consequently
\[ u^k \in RA, \]
the commutative group ring of $A$ over $R$. Let $N = \text{Rad}(R)$ be the set of nilpotent elements of $R$. Now $ZT$ has only trivial idempotents [6, Theorem 2.20, page 25]. Hence since $R \subseteq ZT$ and since idempotents of $R/N$ can be lifted to $R$, it follows that $R/N$ also has only trivial idempotents. Therefore [6, Lemma 3.3, page 55] together with an inductive argument tells us that $(R/N)A$ has only trivial units. It follows that
\[ u^k = \beta f^k + \text{nilpotent elements}. \]
But as each $\beta$ is a sum of units in various $R_t$, it follows that the last term must be zero. Hence $u^k = \beta f^k$, and thus all $fT$'s in the original decomposition of $u = \bigoplus \alpha f$ were in the same coset of $T$. Thus $u = rf$ as required. \hfill $\Box$

We give two important consequences of the last result. We say that $Z(\mathcal{U}(ZG))$ is trivial if it contains only trivial units.

**Corollary 4.** Let $G$ be a finitely generated nilpotent group. If $Z(\mathcal{U}(ZT))$ is trivial, then $Z(\mathcal{U}(ZG))$ is trivial.

**Proof.** Let $u \in Z(\mathcal{U}(ZG))$ be nontrivial. Then the support of $u$ contains two different elements, say $x$ and $y$. Since finitely generated nilpotent groups are residually finite, there exists a finite factor $G/N = \overline{G}$ so that $\overline{x} \neq \overline{y}$ in $\overline{G}$ (see [5, page 149]). Hence $\overline{x}$ has in its support at least two different elements, and thus $\overline{x}$ is of infinite order ([7, Corollary 1.7, page 4]). By Proposition 3 we write $u = rg$, $r \in ZT$, $g \in G$. Since $u$ is central, $r$ commutes with $g$. It then follows easily that $\overline{r}$, and hence also $r$, is of infinite order. Moreover, there exists a positive integer $n$ such that $(g^n, T) = 1$. Consequently it follows from $u^n = r^ng^n$ that $r^k$ commutes with $T$. Thus $r^k$ is a nontrivial unit of $Z(\mathcal{U}(ZT))$. \hfill $\Box$
Corollary 5. Let $G$ be a finitely generated nilpotent group. Then $Z(\mathcal{U}(\mathbb{Z}G))$ is finitely generated. Furthermore, $(Z(\mathcal{U}(\mathbb{Z}G))\cap Z(\mathcal{U}(\mathbb{Z}T)))Z(G)$ is of finite index in $Z(\mathcal{U}(\mathbb{Z}G))$.

Proof. Let $S = Z(\mathcal{U}(\mathbb{Z}G)) \cap Z(\mathcal{U}(\mathbb{Z}T))$. First we show that $Z(\mathcal{U}(\mathbb{Z}G))/SZ(G)$ is a torsion group of bounded exponent. Indeed, let $u \in Z(\mathcal{U}(\mathbb{Z}G))$. Because of Proposition 3 write $u = rg$, with $r \in \mathcal{U}(\mathbb{Z}T)$ and $g \in G$. Considering the natural epimorphism $\mathbb{Z}G \to \mathbb{Z}(G/T)$ and using the fact that $Z(\mathcal{U}(\mathbb{Z}(G/T)))$ is trivial because $G/T$ is ordered, it follows that $gT \in Z(G/T)$. Hence $(g^k, T) = 1$ and $g^l \in Z(G)$ for $k = [\text{Aut}(T)]$ and $l = k[T]$. Now since $u$ is central, $r$ and $g$ commute. Therefore

$$u^l = r^l g^l$$

Consequently $u^l \in SZ(G)$, and the claim follows.

As a subgroup of the finitely generated group $Z(\mathcal{U}(\mathbb{Z}T))$, the group $S$ is itself finitely generated. Hence so is $SZ(G)$. Since the torsion subgroup of $Z(\mathcal{U}(\mathbb{Z}G))$ is finite (see for example [6, page 46]), the above claim now easily yields that $Z(\mathcal{U}(\mathbb{Z}G))$ is indeed finitely generated.

We will now construct finitely many generators for the central units of any finitely generated nilpotent group.

Let $n$ be the nilpotency class of $T$ and $h$ the Hirsch number of $G/T$. Let $k = [\text{Aut}(T)]$. Further let $x_1, \cdots, x_h$ be elements of $G$ such that for each $1 \leq i \leq h$ the group $G_i = \langle T, x_1, \cdots, x_i \rangle$ is normal in $G$ and $G_i/G_{i-1} \cong \mathbb{Z}$, where $G_0 = T$. For any generator $b_{(n)}$ described in Proposition 2 define

$$b_{(n)}^{(0)} = b_{(n)}$$

and for $1 \leq i \leq h$

$$b_{(n)}^{(i)} = \prod_{0 \leq j < k} \left(b_{(n)}^{(i-1)}\right)^{x^j_i}.$$ 

Since each $b_{(n)}^{(i)}$ is in $Z(\mathcal{U}(\mathbb{Z}T))$, the order of the conjugates in the product expression for $b_{(n)}^{(i)}$ is unimportant. It follows by induction that $b_{(n)}^{(i)}$ is in $Z(\mathcal{U}(\mathbb{Z}G_i))$. In particular, $b_{(n)}^{(h)} \in Z(\mathcal{U}(\mathbb{Z}G))$.

Theorem 6. Let $G$ be a finitely generated nilpotent group. Suppose $n$ is the nilpotency class of $T$ and $h$ is the Hirsch number of $G/T$. Then

$$\langle b_{(n)}^{(h)} \mid b \text{ a Bass cyclic of } \mathbb{Z}T \rangle Z(G)$$

is of finite index in $Z(\mathcal{U}(\mathbb{Z}G))$.

Proof. Because of Corollary 5 the group $SZ(G)$ with $S = Z(\mathcal{U}(\mathbb{Z}G)) \cap Z(\mathcal{U}(\mathbb{Z}T))$ is of finite index in $Z(\mathcal{U}(\mathbb{Z}G))$. Let $\alpha_1, \cdots, \alpha_p$ be a finite set of generators for $S$. By Proposition 2 there exists a positive integer $m$ such that all $\alpha_1^m, \cdots, \alpha_p^m$ are in $\langle b_{(n)} \mid b \text{ a Bass cyclic in } \mathbb{Z}T \rangle$. For simplicity, write $\alpha = \alpha_1^m$. Then

$$\alpha = \prod b_{(n)}^{(i)},$$

where the product runs over a finite number of Bass cyclic units of $\mathbb{Z}T$. Since $\alpha$ is in $Z(\mathcal{U}(\mathbb{Z}G))$, and using the notation introduced above, we obtain

$$\alpha^k = \alpha x_1^i \cdots x_h^{k-i}.$$
As each $b_{(n)}$ is central in $ZT$, this implies
\[ \alpha^k = \prod b_{(n)}^{(1)}. \]
Continuing this process one obtains that
\[ \alpha^{kh} = \prod b_{(n)}^{(h)}. \]
Since the group generated by $\alpha_{mkh}, \cdots, \alpha_{pkh}$ is of finite index in $S$, the result follows.

Note that Corollary 4 can now also be obtained as an easy consequence of Theorem 6.

We now give an example showing that the converse of Corollary 4 does not hold.

**Example.** Let $G = \langle a, x \mid a^x = a^3, a^8 = 1 \rangle$. Clearly $G$ is a nilpotent group with $T = \langle a \rangle$, a cyclic group of order 8. From Higman’s result (see [6]) it follows that $Z(U(ZT))$, modulo the trivial units, is a free Abelian group of rank 1. We now show that $Z(U(ZG))$ contains only trivial units. For this suppose $u$ is a nontrivial central unit in $ZG$. By Proposition 3, we can write $u = rx^i$ for some integer $i$ and $r \in U(ZT)$. We know from the above that $r$ is of infinite order, and since $r$ commutes with $x$, it must be in $Z(U(ZG))$.

Because the only Bass cyclic unit, up to inverses, in $ZT$ is
\[ b = (1 + a + a^2)^4 - 10\hat{a}, \quad \hat{a} = 1 + a + \cdots + a^7, \]
Proposition 2 yields that
\[ r^k = b^{l}, \]
for some nonzero integers $k, l$. Observe, however, that $b^5 = b^{-3}$. Since $b^k = r^k$ is central in $ZG$, we obtain $b^l = b^{-3}$, contradicting the fact that $b$ is of infinite order.

**References**

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