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FIXED POINTS IN BOUNDARIES OF NEGATIVELY CURVED GROUPS

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An important feature of the natural action of a negatively curved (= word hyperbolic) group on its boundary is that the fixed points of hyperbolic (= infinite order) elements form a dense subset of the boundary. Gromov states this in his influential treatise *Hyperbolic Groups* [4, 8.2 D] and his explanation involves Stalling's theorem on groups with infinitely many ends. This short note provides an elementary proof of this fact. The proof relies on a simple enhancement of the Pumping Lemma from the theory of finite state automata and on the fact that the geodesic words with respect to a fixed generating set for a negatively curved group form a regular language. For background material on negatively curved groups see [1] and on automata and regular languages on groups see [2]. For a different proof see [5].

Definition. Let L be a language over the finite alphabet A. An infinite sequence $\{a_i\}$ of letters from A is an *L-ray* provided, for every integer $n > 0$, there exists an integer $N \geq n$ such that $a_1 \cdots a_N$ is a word in the language L.

The following lemma, which we term an *Ultra-Pumping Lemma* (UPL), is our aforementioned simple enhancement of the Pumping Lemma.

Ultra-Pumping Lemma. *Let $\{a_i\}$ be an L-ray where L is a regular language over the finite alphabet A. There exists an integer $M \geq 0$ such that, for each integer $n > 0$, there exists an integer $N \geq n$ such that $(a_1 \cdots a_M)(a_{M+1} \cdots a_{M+N})^r \in L$ for every integer $r \geq 0$.*

Proof. Let W be a finite state automaton accepting L. The L-ray $\{a_i\}$ describes an infinite path in W and, as there are only finitely many (accept) states in W and infinitely many initial segments of $\{a_i\}$ in L, this path visits some accept state s_∞ infinitely often. Let M be the step at which this path first visits s_∞ and choose $N \geq n$ such that $a_{M+1} \cdots a_{M+N}$ describes a loop in W based at s_∞. □

We use the term *δ-negatively curved* in the sense of Gromov's δ-hyperbolic condition [4, p. 89]. A group G is δ-negatively curved with respect to a finite generating set $A = A^{-1}$ if the corresponding Cayley graph $\Gamma_A(G)$ with the usual word metric has a δ-hyperbolic inner product based at the identity; see [1, Definition 1.1, p. 8]. A group G is *negatively curved* (or *word hyperbolic*) if it is δ-negatively curved with

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respect to some finite generating set for some $\delta \geq 0$. For an infinite order element g of the negatively curved group G, let $g^{\pm\infty}$ denote the two fixed points of the action of g on $G \cup \partial G$, where ∂G denotes the boundary of G. Recall that $g^{\pm\infty}$ are elements of ∂G and $\lim g^i = g^{+\infty}$ and $\lim g^{-i} = g^{-\infty}$.

Theorem [4, 8.2D]. *Let G be a negatively curved group with boundary ∂G. The set*

$$\{g^{\pm\infty} : g \in G,\ \text{order}(g) = \infty\}$$

is a dense subset of ∂G.

Proof. Recall that ∂G is metrizable [1] and as such sequences suffice to verify denseness. Let G be δ-negatively curved with respect to the finite generating set $A = A^{-1}$ and let $x \in \partial G$. Choose an infinite geodesic path $a_1 a_2 \cdots$ based at 1 in the Cayley graph $\Gamma_A(G)$ such that $\lim \overline{a_1 \cdots a_i} = x$. Here, $a_1 \cdots a_i$ denotes a word over A and $\overline{a_1 \cdots a_i}$ denotes the element of G obtained by multiplying. The sequence $\{a_i\}$ is an L-ray, where L is the language of geodesic words of G with respect to A. Recall that L is a regular language [2] and apply the UPL to obtain an integer M and a strictly increasing sequence $0 < N(1) < N(2) < \ldots$ of integers such that for all integers $i > 0$ and $r \geq 0$,

$$(a_1 \cdots a_M)(a_{M+1} \cdots a_{M+N(i)})^r \in L.$$

Let $a = a_1 \cdots a_M$, $b_i = a_{M+1} \cdots a_{M+N(i)}$, and $g_i = \overline{a}\overline{b}_i\overline{a}^{-1}$. Our claim is that the sequence $\{g_i^{+\infty}\}$ converges to x. The point is that since $ab_i^r \in L$ for all integers $r > 0$, the path $ab_ib_i\cdots$, where b_i is repeated infinitely, is a geodesic path from 1 to $g_i^{+\infty}$ that shares the initial segment ab_i of length $M + N(i)$ with the geodesic path $a_1a_2\cdots$ from 1 to x. Since $N(i) \to \infty$ as $i \to \infty$, this forces the convergence of $\{g_i^{+\infty}\}$ to x. To be precise, the Gromov inner product (extended to the boundary [1]) applied to x and $g_i^{+\infty}$ yields

$$(x.g_i^{+\infty}) \geq M + N(i) - 2\delta;$$

see [1, Lemma 4.6(4), p. 51]. Hence $(x.g_i^{+\infty}) \to \infty$ as $i \to \infty$ and therefore $\{g_i^{+\infty}\}$ converges to x as desired. □

Observe that the proof of the theorem yields the following interesting fact: an infinite geodesic path $a_1a_2\cdots$ based at 1 may be approximated in $\Gamma_A(G) \cup \partial G$ by traveling along an initial segment $(a_1 \cdots a_M)$ and then infinitely often repeating large segments of the path. The UPL and the proof of the theorem may be applied to verify that an infinite quasi-geodesic path may be approximated by traveling on an initial segment of the path and then repeating a subpath, chosen arbitrarily large, infinitely often.

As an application of the theorem, we give a quick proof of a known result.

Corollary. *If H is an infinite normal subgroup of the negatively curved group G, then the limit set $\Lambda(H) = \partial G \cap \mathrm{Cl}_{G\cup\partial G}(H)$ is equal to ∂G.*

Proof. Let $x \in \partial G$ and let U be an open neighborhood of x in ∂G. Let h be an infinite order element of H [3] and choose an infinite order element g of G such that $g^{+\infty} \in U$. Since $g^{+\infty}$ is an attractive fixed point of the action of g on $G \cup \partial G$ (induced by left multiplication), there is an integer $N > 0$ such that $g^N \cdot h^{+\infty} \in U$. Since $\lim(g^N h^i g^{-N}) = g^N \cdot h^{+\infty}$ and H is normal, $g^N \cdot h^{+\infty} \in \Lambda(H)$. Since U is an arbitrary neighborhood of x, we conclude that $x \in \Lambda(H)$. □

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