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NONNORMAL SPACES $C_p(X)$ WITH COUNTABLE EXTENT

WINFRIED JUST, OL'GA V. SIPACHEVA, AND PAUL J. SZEPTYCKI

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Abstract. Examples of spaces X are constructed for which $C_p(X)$ is not normal but all closed discrete subsets are countable. A monolithic example is constructed in ZFC and a separable first countable example is constructed using $\diamondsuit$.

1. Introduction

Reznichenko [R] has shown that if $e(C_p(X)) > \aleph_0$, then $C_p(X)$ is not normal. The question arises whether every space of the form $C_p(X)$ with countable extent is normal. For Lindelöf Σ-spaces X, $e(C_p(X)) = \aleph_0$ does imply $C_p(X)$ is Lindelöf (and therefore, normal) (D.P.Baturov). In fact, Baturov proved that $e(Y) = l(Y)$ for every $Y \subseteq C_p(X)$ with X a Lindelöf Σ-space [Ba]. Therefore Reznichenko's theorem implies that for the class of Lindelöf Σ-spaces, countable extent, normality and the Lindelöf property all coincide. However, if X is the one-point Lindelöfication of ω_1 with the discrete topology, then $C_p(X)$ is normal (and, therefore, $e(C_p(X)) = \aleph_0$), but it is not Lindelöf. It is of general interest to specify classes of spaces for which countable extent and normality or the Lindelöf property are well correlated. For example, it would be interesting to know whether the countable version of the Baturov Theorem remains true for countably compact or pseudocompact spaces and whether there exists an example of a Lindelöf space X for which $C_p(X)$ is not normal and has countable extent. (Both questions are due to Reznichenko).

In this note we construct two examples of spaces X for which $C_p(X)$ is not normal and the extent of $C_p(X)$ is countable. One example is constructed in ZFC but is monolithic and of character ω_1, and the other, while both separable and first countable, is constructed from the combinatorial principle $\diamondsuit$. These examples answer a question of Reznichenko (problem 57 in [A2]).

For the background material on function spaces see [A1], and for the set theoretic concepts such as $\diamondsuit$ and almost disjoint families we refer the reader to [K]. For a good introduction to elementary submodel techniques we recommend [D].

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2. A ZFC EXAMPLE

Let τ be a cardinal number. A space X is called τ-monolithic if the net weight of the closure of any subset of X with cardinality at most τ does not exceed τ. A space is monolithic if it is τ-monolithic for all τ. In other words, a space is monolithic iff the net weight of any its subspace does not exceed the density of the subspace. Thus, for spaces having uncountable net weight the property of being monolithic is opposite in a certain sense to the property of being separable.

Example 2.1. There is a monolithic space X such that $C_p(X)$ is nonnormal and $e(C_p(X)) \leq \aleph_0$.

Let ω_1 be endowed with the discrete topology, and let $L(\omega_1) = \omega_1 \cup \{*\}$ be its one point Lindelöfication. Neighborhoods of points $\alpha \in \omega_1$ are isolated while neighborhoods of $*$ are cocountable. We let Y denote the space $L(\omega_1) \times (\omega+1)$ and let $X = L(\omega_1) \times (\omega+1) \setminus \{(*,\omega)\}$ be given the subspace topology. Clearly, the closure of any countable subset of X is countable, and therefore X is monolithic.

Claim 2.2. $C_p(Y)$ is normal.

Proof. Note that $C_p(Y)$ can be embedded naturally into $(C_p(\omega+1))^{L(\omega_1)}$: to each $f \in C_p(Y)$ there corresponds $\{f_x\}_{x \in L(\omega_1)}$, where f_x is the restriction of f to $\{(x,n) : n < \omega\}$ for each $x \in L(\omega_1)$. Let $\mathbf{0}_*$ denote the function on $\{(*,n) : n < \omega\}$ identically equal to zero, and let

$$Z = \{f \in C_p(Y) : f_* = \mathbf{0}_*\}.$$

Then to each f from Z there corresponds $\{f_x\}_{x \in L(\omega_1)}$ such that f_x is identically equal to zero for all but countably many x and for $x = *$ in particular. Thus,

$$\begin{aligned} Z &= \big\{\{f_x\}_{x \in L(\omega_1)} \in C_p(\omega+1)^{L(\omega_1)} : f_* \equiv 0 \& |\{x : f_x \not\equiv 0\}| \leq \aleph_0\big\} \\ &= \big\{\{f_\alpha\}_{\alpha<\omega_1} \in C_p(\omega+1)^{\omega_1} : |\{\alpha : f_\alpha \not\equiv 0\}| \leq \aleph_0\big\} \times \{\mathbf{0}_*\} \\ &\cong \big\{\{f_\alpha\}_{\alpha<\omega_1} \in C_p(\omega+1)^{\omega_1} : |\{\alpha : f_\alpha \not\equiv 0\}| \leq \aleph_0\big\} \\ &= \Sigma(\mathbf{0}) \subset C_p(\omega+1)^{\omega_1}, \end{aligned}$$

where $\mathbf{0}$ is the element of $C_p(\omega+1)^{\omega_1}$ all the coordinates of which are zero and $\Sigma(\mathbf{0})$ is the usual Σ-product of ω_1 copies of $C_p(\omega+1)$ about $\mathbf{0}$. To complete the proof, it suffices to note that the space $C_p(\omega+1) \times \Sigma(\mathbf{0})$ is a Σ-product of metrizable spaces and therefore normal ([G], [Ru]), and that $C_p(Y)$ is homeomorphic to $C_p(\omega+1) \times \Sigma(\mathbf{0})$: a homeomorphism can be defined by $f \mapsto (f - (f_*)^{\omega_1}, f_*)$.

It is easy to see that X is C-embedded in Y. Hence the projection map $\pi : C_p(Y) \to C_p(X)$ is onto. By [R], $C_p(Y)$ has countable extent; hence its continuous image $C_p(X)$ does too. Clearly the closure of any countable subset of X is countable and therefore X is monolithic.

Lemma 2.3. *$C_p(X)$ is not normal.*

Proof. Let Z be the space ω_1 with the discrete topology. Z is embedded as a closed subset of X, so by a theorem of Uspenskii (I.6.2 in [A1]) it suffices to prove that $C_p(X|Z)$ is not normal. It is easy to see that $C_p(X|Z)$ is just the collection of real valued functions from ω_1 that are eventually constant. The fact that this space is not normal follows from the standard proof that ω^{ω_1} is not normal. Indeed, if we let

$$K_0 = \{f \in \omega^{\omega_1} : \exists \alpha < \omega_1 \ f \restriction \alpha \textit{ is } 1\text{-}1 \textit{ and } f(\omega_1 \setminus \alpha) = \{0\}\}$$

and

$$K_1 = \{f \in \omega^{\omega_1} : \exists \alpha < \omega_1 \; f \restriction \alpha \textit{ is } \text{1-1} \textit{ and } f(\omega_1 \setminus \alpha) = \{1\}\},$$

then K_0 and K_1 are disjoint closed sets that can't be separated (see 2.7 in [P]).

3. A separable example from $\diamondsuit$

Example 3.1. Assuming $\diamondsuit$, there is a first countable separable space X such that $C_p(X)$ is not normal and $e(C_p(X)) = \aleph_0$.

Question 3.2. *Can one construct a space as in 3.1 in ZFC alone?*

Definition 3.3. Let $\mathcal{F} \subseteq P(\omega)$. We call $\mathcal{F}$ an *ω_1-p-ultrafilter* if $\mathcal{F}$ satisfies the following conditions:

- **(a)** $\mathcal{F}$ is an ultrafilter.
- **(b)** There exists a sequence $(p_\xi : \xi < \omega_1)$ of infinite subsets of ω such that $\forall \xi < \eta < \omega_1$:
  - **(b1)** $p_\eta \subseteq^* p_\xi$,
  - **(b2)** $|p_\xi \setminus p_\eta| = \aleph_0$, and
  - **(b3)** $\forall a \in \mathcal{F} \; \exists \xi \; p_\xi \subseteq^* a$.

The existence of an ω_1-p-ultrafilter follows from CH, and is also consistent with the negation of CH, but it is not a theorem of ZFC.

Let $\mathcal{F}$ be an ω_1-p-ultrafilter and let $(p_\xi : \xi < \omega_1)$ be a sequence that witnesses (b) of the definition. For every ξ fix an infinite set $a_\xi \subseteq p_\xi \setminus p_{\xi+1}$. Note that the sets a_ξ are pairwise almost disjoint (i.e., if $\xi \neq \eta$, then $a_\xi \cap a_\eta$ is finite). Now let $\mathcal{A} = \{a_\xi : \xi < \omega_1\}$ and construct $X = \Psi(\mathcal{A})$, where $\Psi(\mathcal{A})$ is the Mrówka-Isbell space induced by $\mathcal{A}$. To be specific, let

$$X = \Psi(\mathcal{A}) = \omega \cup \{\hat{a}_\xi : \xi < \omega_1\}$$

Each $n \in \omega$ is isolated in X, and the basic neighborhoods of $\hat{a}_\xi$ are the sets

$$V_{\xi,k} = \{\hat{a}_\xi\} \cup a_\xi \setminus k.$$

We will call any Mrówka-Isbell space built along a ω_1-p-ultrafilter as above a *p-Ψ-space.*

The exposition of Example 3.1 now splits into two parts. We first show that if X is any p-Ψ-space, then $C_p(X)$ is not normal. We then employ $\diamondsuit$ to build a particular p-Ψ-space X such that $C_p(X)$ also has countable extent.

Let us now describe $C_p(X)$ for X as above. Any basic open set U in $C_p(X)$ has a finite support $\mathit{supp}(U) \subseteq X$. For every $x \in \mathit{supp}(U)$ there is an open interval with rational endpoints I_x such that

$$U = \{f : \forall x \in \mathit{supp}(U), \; f(x) \in I_x\}.$$

If $\mathit{supp}(U) = \{x_0, ..., x_k\}$, then we write

$$U = U(x_0, I_{x_0}, ..., x_k, I_{x_k}) = U(x_i, I_{x_i} : i \leq k).$$

Claim 3.4. Let $r \in \mathbb{R}$, C be a countable subset of ω_1, F be a finite subset of ω, and let $f : C \cup F \to \mathbb{R}$ be any function into the reals. Then there exists $g \in C_p(X)$ such that $g(\hat{a}_\xi) = f(\xi)$ for each $\xi \in C$, $g(n) = f(n)$ for each $n \in F$, and $g(\hat{a}_\xi) = r$ for each $\xi \notin C$.

Proof. Without loss of generality, $C = \alpha$ for some ordinal α. For every $\xi \in C$, choose n_ξ such that if $a'_\xi = (a_\xi \setminus n_\xi)$, then $a'_\xi \cap p_\alpha = \emptyset$ and $a'_\xi \cap a'_\eta = \emptyset$ for each $\xi < \eta < \alpha$.

Define $g : \Psi(\mathcal{A}) \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(\xi), & \text{if } x = \hat{a}_\xi \text{ or } x \in a'_\xi \setminus F \text{ and } \xi < \alpha; \\ f(x), & \text{if } x \in F; \\ r, & \text{otherwise.} \end{cases}$$

It is easy to check that g is continuous and is as required.

Claim 3.5. Let $f \in C_p(X)$. Then there exist an $\alpha < \omega_1$ and an $r \in \mathbb{R}$ such that $f(\hat{a}_\xi) = r$ for each $\xi \geq \alpha$.

Proof. Let $f \in C_p(X)$ and consider the set $S = \{f(n) : n \in \omega\}$. For every basic open interval $(a, b) \subset \mathbb{R}$, let $D(a, b) = \{n : f(n) \in (a, b)\}$ and let

$$E(a, b) = \{\xi \in \omega_1 : \hat{a}_\xi \in cl(D(a, b))\}.$$

We leave it to the reader to check that for reals $a < b < c < d$, at most one of the sets $E(a, b)$ and $E(c, d)$ is uncountable. This completes the proof of 3.5.

Lemma 3.6. *If X is any p-Ψ-space as above, then $C_p(X)$ is not normal.*

Proof. By claims 3.4 and 3.5, the proof is identical to the proof of 2.3.

Now we use $\diamondsuit$ to construct an almost disjoint family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ is a p-Ψ-space and so that $C_p(\Psi(\mathcal{A}))$ will have countable extent. We fix a $\diamondsuit$-sequence designed to capture potential uncountable discrete subsets of our space. Fix $\{D_\alpha : \alpha < \omega_1\}$ such that for each $\alpha < \omega_1$, D_α is a countable collection of real-valued functions and $dom(f) = \omega \times \{0\} \cup \alpha \times \{1\}$ for each $f \in D_\alpha$. For every collection $D = \{d_\alpha : \alpha < \omega_1\}$ of real-valued functions on $\omega \times \{0\} \cup \omega_1 \times \{1\}$,

$$\{\alpha \in \omega_1 : D_\alpha = \{d_\beta \restriction (\omega \times \{0\} \cup \alpha \times \{1\}) : \beta < \alpha\}\}$$

is stationary.

We construct $\{p_\alpha : \alpha < \omega_1\}$ and $\{a_\alpha : \alpha < \omega_1\}$ by recursion on $\alpha < \omega_1$. We will let $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ and we will require that $\{p_\alpha : \alpha < \omega_1\}$ be a base for an ω_1-p-ultrafilter as in (b) of Definition 3.3. Also, we will require that $a_\alpha \subseteq p_\alpha \setminus p_{\alpha+1}$ for each $\alpha < \omega_1$.

Fix an enumeration $\{x_\alpha : \alpha < \omega_1\}$ of $[\omega]^\omega$. Suppose that α is a limit ordinal and that $\{p_\xi : \xi < \alpha\}$ and $\{a_\xi : \xi < \alpha\}$ have been defined. Let $\mathcal{A}_\alpha = \{a_\xi : \xi < \alpha\}$. Each $f \in D_\alpha$ induces a $g_f : \Psi(\mathcal{A}_\alpha) \to \mathbb{R}$ defined by

(a) $g_f(n) = f((n, 0))$, and

(b) $g_f(\hat{a}_\xi) = f((\xi, 1))$.

Let $E_\alpha = \{g_f : f \in D_\alpha\}$.

Case 1. If g_f is not continuous for some $f \in D_\alpha$, then choose p_α arbitrary so that $p_\alpha \subseteq^* p_\xi$ for each $\xi < \alpha$.

Case 2. If g_f is continuous for each $f \in D_\alpha$, then we ask whether the following statement holds:

($*$) *For each $g \in E_\alpha$ there is an $\eta < \alpha$ and a real number r_g such that $g \restriction \{\hat{a}_\xi : \eta \leq \xi < \alpha\}$ is constant with value r_g. Furthermore, there exist $h \in C_p(\Psi(\mathcal{A}_\alpha))$, $S \subset \omega$, and a real number r such that*

(c) $\forall \xi < \alpha \;\; S \subseteq^* p_\xi$,
(d) $h(S)$ *is a sequence convergent to* r, *and*
(e) *for every neighborhood* U *of* h *and each interval* I *containing* r, *the set* $\{g \in E_\alpha : g \in U \text{ and } r_g \in I\}$ *is infinite.*

If $(*)$ holds, fix h_α, r_α and S_α which witness it, and let $p_\alpha \subseteq S_\alpha$.

If $(*)$ doesn't hold, as in Case 1 choose p_α an arbitrary pseudo-intersection of the p_ξ's. In all cases we make sure that either $p_\alpha \subseteq x_\alpha$ or $p_\alpha \cap x_\alpha = \emptyset$. This guarantees that $\{p_\alpha : \alpha < \omega_1\}$ will be a base for an ultrafilter. Also, we let a_α be any infinite co-infinite subset of p_α.

For successor ordinals $\alpha = \beta + 1$, as long as α_β was chosen to be a co-infinite subset of p_β we can let $p_{\beta+1} = p_\beta \setminus a_\beta$.

This completes the recursive construction of $\mathcal{A}$. We let $X = \Psi(\mathcal{A})$.

Lemma 3.7. $e(C_p(X)) = \aleph_0$.

Proof. Fix an uncountable $D \subseteq C_p(X)$. We will show that D cannot be closed and discrete by constructing a function $h \in C_p(X)$ such that every neighborhood of h intersects D in an infinite set.

Enumerate D as $\{f_\alpha : \alpha < \omega_1\}$. For each $f \in D$ there is a function $f' : \omega \times \{0\} \cup \omega_1 \times \{1\} \to \mathbb{R}$ coding f as in (a) and (b) above. Let $D' = \{f'_\alpha : \alpha < \omega_1\}$ be the set of codes for the elements in D. It is sets of the form D' that our $\diamondsuit$-sequence was designed to capture. Fix an elementary submodel M of some $H(\lambda)$ for λ large enough so that M contains everything in sight (e.g., D, D', $\mathcal{A}, \ldots$). By $\diamondsuit$ we can also require that if $\alpha = M \cap \omega$; then $D_\alpha = \{f'_\xi \mid (\omega \times \{0\} \cup \alpha \times \{1\}) : \xi < \alpha\}$. Note that, following our previous notation, $E_\alpha = \{f_\xi \mid (\omega \cup \mathcal{A}_\alpha) : \xi < \alpha\}$.

Claim 3.8. For M, D, and α as above, $(*)$ held at stage α of the recursive construction.

Proof. Let M, D, and α be as in the assumptions. Fix a sequence of positive reals $(\epsilon_n) \to 0$. We construct $S = \{m_n : n < \omega\} \subset \omega$ by induction on n. Enumerate $\{a_\xi : \xi < \alpha\}$ as $\{a_{\xi_n} : n < \omega\}$. By adding or removing a finite set from each a_ξ, let $\{a'_n : n < \omega\}$ be pairwise disjoint such that $a_{\xi_n} =^* a'_n$ for each $n < \omega$, and such that $\omega = \bigcup_{n<\omega} a'_n$.

Note that for each n, both a'_n and $\{a'_k : k < n\}$ are elements of M. Also enumerate $\{p_\xi : \xi < \alpha\}$ as $\{b_n : n < \omega\}$. Each $d \in D$ is eventually constant on $\mathcal{A}$, so fix $\eta(d) \in \omega_1$ and $r_d \in \mathbb{R}$ such that $d(\hat{a}_\xi) = r_d$ for each $\xi \geq \eta(d)$. For each open interval I containing r_d, let $S_d(I) = \{n \in p_{\eta(d)} : d(n) \in I\}$ and note that $S_d(I) \supset^* a_\xi$ for each $\xi \geq \eta(d)$. Therefore $S_d(I)$ is in the ultrafilter (otherwise there would be a $\xi \geq \eta(d)$ for which $p_\xi \cap S_d(I)$ is finite, contradicting $a_\xi \subseteq p_\xi$).

By induction on $n < \omega$ we construct the following.

(1) Integers $k_0 < k_1 < \ldots < k_n$;
(2) Integers $m_0 < m_1 < \ldots < m_n$;
(3) Basic open intervals $P_n, \{P_n^i : i \leq n\}$, $\{I_n^i : i \leq n\}$ and $\{J_{x,n}^i : i \leq n,\ x \in (a'_i \cap k_n) \setminus \{m_i : i \leq n\}\}$;
(4) Uncountable subsets D_n of D such that $D_n \in M$.

The objects constructed will have the following properties:

(h) If $U_n = U(\hat{a}_{\xi_0}, I_n^0, \ldots \hat{a}_{\xi_n}, I_n^n) \cap U(x, J_{x,n}^i : i \leq n, x \in (a'_i \cap k_n) \setminus \{m_i : i \leq n\}) \cap U(m_0, P_n^0, \ldots, m_n, P_n^n)$, then $D_n \subseteq U_n$, and moreover $r_d \in P_n$ for each $d \in D_n$;

(i) $i < j \leq n \Rightarrow U_j \subset U_i$ and $D_j \subset D_i$;
(j) For each $d \in D_n$ and each $i \leq n$, $m_i \in p_{\eta(d)}$, $d(m_i) \in P_n^i$ and $r_d \in P_n$;
(k) $P_n,\ P_n^i,\ I_n^i,\ J_{x,n}^i$ are all of length $\leq \epsilon_n$;
(l) For each $i \leq n' < n$ and each $x \in (a_i \cap k_n) \setminus \{m_i : i \leq n\}$, we have $cl(P_n) \subset P_{n'},\ cl(P_n^i) \subseteq P_{n'}^i,\ cl(I_n^i) \subseteq I_{n'}^i$, and $cl(J_{x,n}^i) \subseteq J_{x,n'}^i$. Also let $P_n^n = P_n$.

The Inductive Step. Assume that for each $i < j \leq n,\ k_j,\ m_j$, and $P_j,\ \{I_j^i : i \leq j\},\ \{J_{x,j}^i : i \leq j,\ x \in (a_i \cap k_j) \setminus \{m_i : i \leq j\}\},\ \{P_j^i : i \leq j\}$ and D_j have been defined. We apply the pigeonhole principle repeatedly to D_n.

Step 1. Choose a basic open interval $P_{n+1} \subset cl(P_{n+1}) \subset P_n$ of length ϵ_{n+1} such that

$$D'_{n+1} = \{d \in D_{n+1} : r_d \in P_{n+1}\}$$

is uncountable.

Step 2. Since $r_d \in P_{n+1}$ for each $d \in D'_{n+1}$,

$$S'_d = S_d(P_{n+1}) \cap \bigcap_{i \leq n} b_i \setminus \bigcup_{i \leq n} a'_i$$

is an element of our ultrafilter. Therefore, since D'_{n+1} is uncountable, there is an $m_{n+1} > m_n$ such that

$$D''_{n+1} = \{d \in D'_{n+1} : m_{n+1} \in S'_d\}$$

is uncountable. Clearly D'_{n+1} and hence D''_{n+1} are both in M. We let $P_{n+1}^{n+1} = P_{n+1}$.

Step 3. Choose a basic open interval I_{n+1}^{n+1} of length $< \epsilon_{n+1}$ such that

$$U'_{n+1} = U_n \cap U(\hat{a}_{\xi_{n+1}}, I_{n+1}^{n+1})$$

has uncountable intersection with D''_{n+1}.

Step 4. For each $i \leq n$ choose basic open intervals $I_{n+1}^i \subset cl(I_{n+1}^i) \subset I_n^i$ of length $< \epsilon_{n+1}$ such that

$$U''_{n+1} = U'_n \cap U(\hat{a}_{\xi_0}, I_{n+1}^0, ..., \hat{a}_{\xi_n}, I_{n+1}^n)$$

has uncountable intersection with D''_{n+1}.

Step 5. For each $i \leq n$ choose intervals P_{n+1}^i of length $< \epsilon_{n+1}$ so that $cl(P_{n+1}^i) \subset P_n^i$ and so that

$$U'''_{n+1} = U''_{n+1} \cap U(m_i, P_{n+1}^i : i \leq n)$$

has uncountable intersection with D''_{n+1}.

Step 6. By Step 4 we can choose a $k_{n+1} > k_n$ such that

$$D'''_{n+1} = \{d \in D''_{n+1} \cap U'''_{n+1} : \forall i \leq n+1\ d(a'_i \setminus k_{n+1}) \subset I_{n+1}^i\}$$

is uncountable.

Step 7. For each $i \leq n+1$ and each $x \in (a'_i \cap k_{n+1} \setminus \{m_i : i \leq n+1\})$ choose a basic open interval $J_{x,n+1}^i$ of length $< \epsilon_{n+1}$ so that

(7a) If $i \leq n$ and $x \in a_i \cap k_n$, then $cl(J_{x,n+1}^i) \subset J_{x,n}^i$;
(7b) If $i \leq n$ and $x \in a_i \cap [k_n, k_{n+1})$, then $cl(J_{x,n+1}^i) \subset I_n^i$;

(7c) Letting $U_{n+1} = U'''_{n+1} \cap U(x, J^i_{x,n+1} : x \in a'_i \cap k_{n+1} \setminus \{m_i : i \leq n+1\})$, then U_{n+1} has uncountable intersection with D'''_{n+1}.

Letting $D_{n+1} = U_{n+1} \cap D'''_{n+1}$ completes the inductive construction of $S = \{m_n : n < \omega\}$. By Step 1 of the construction, $S \subseteq^* b_n$ for each $n < \omega$. Therefore S satisfies clause (c) of $(*)$.

Now we define $h \in C_p(\Psi(\mathcal{A}_\alpha))$ and $r \in \mathbb{R}$. For each $m < \omega$, let $h(\hat{a}_{\xi_m})$ be the unique real number in $\bigcap_{m<n<\omega} I^m_n$. The stipulation that $(\epsilon_n) \to 0$ and the inductive hypothesis (l) guarantee that $h \restriction \mathcal{A}_\alpha$ is well defined. For each $m_i \in S$ let $h(m_i)$ be the unique element of $\bigcap_{n<\omega} P^i_n$, and for each $x \in a'_i \setminus S$ let $h(x)$ be the unique element of $\bigcap_{n<\omega} J^i_{x,n}$. Again, inductive hypothesis (l) assures that h is well defined.

From the construction, namely from inductive hypothesis (l) and clauses (7a) and (7b), it is easy to verify that for each n, $(h(k))_{k \in a_{\xi_n}} \to h(\hat{a}_{\xi_n})$ and that $(h(m_i)) \to r$. This implies that h is a continuous function on $\Psi(\mathcal{A}_\alpha)$. The fact that $(h(m_i)) \to r$ implies that clause (d) of $(*)$ holds. By construction, $\{U_n : n < \omega\}$ forms a local neighborhood base at h in $C_p(\Psi(\mathcal{A}_\alpha))$. Therefore by (h), for each neighborhood U of h and each interval I containing r, the set $\{g \in E_\alpha : g \in U$ and $r_g \in I\}$ is infinite (in fact, if U is basic open and I has rational endpoints, then $U, I \in M$ and M models that this set is uncountable). Therefore $(*)$ holds, as required.

Claim 3.9. For M and α as above, there is an $h \in C_p(X)$ such that every neighborhood of h intersects D in an infinite set.

Proof. If $(*)$ held, then we fix a function h_α and a real r_α. Define $h \in C_p(X)$ as follows:

$h \restriction (\{\hat{a}_\xi : \xi < \alpha\} \cup \omega) = h_\alpha$, and
$h \restriction \{\hat{a}_\xi : \xi \geq \alpha\}$ is constant with value r_α.

We claim that every neighborhood of h intersects D in an infinite set. Fix a basic open set U containing h. Then $U = U_0 \cap U_1$, where

(f) *supp*$(U_1) \subseteq \{\hat{a}_\xi : \xi \geq \alpha\}$ and
(g) $U_0 \in M$, hence *supp*$(U_0) \subseteq \{\hat{a}_\xi : \xi < \alpha\} \cup \omega$.

By shrinking U if necessary we can assume that *supp*(U_1) is nonempty. Recall our notation: for each $x \in$ *supp*(U) there is an interval I_x with rational endpoints such that

$$U = U(x, I_x : x \in \mathit{supp}(U)).$$

Since h has constant value r_α on *supp*(U_1), by shrinking U again we can assume without loss of generality that there is an interval I containing r_α such that $I_x = I$ for each $x \in$ *supp*(U_1). For each $\beta < \alpha$, since f_β is an element of M and $M \models$ "$f_\beta \in C_p(X)$," by Claim 5, there is a $\eta < \alpha$ and a real $r_{f_\beta} \in M$ such that $M \models \forall \xi \geq \eta \; f_\beta(\hat{a}_\xi) = r_{f_\beta}$. Therefore by elementarity this holds for all $\eta < \xi < \omega_1$. If we consider U_0 as an open neighborhood in $C_p(\Psi(\mathcal{A}_\alpha))$, then $h_\alpha \in U_0$. Therefore, $\{f \in E_\alpha : f \in U$ and $r_f \in I\}$ is infinite. Fix $\beta < \alpha$ such that both $f_\beta \restriction \Psi(\mathcal{A}_\alpha) \in E_\alpha \cap U_0$ and $r_\beta \in I$. Now this implies that $f_\beta \in U$, and thus $D \cap U$ is infinite.

4. Separable Luzin gaps

Collections A and B of subsets of ω are said to be *separated* if there exists a set $X \subseteq \omega$ such that $a \subseteq^* X$ for each $a \in A$ and $b \cap X =^* \emptyset$ for each $b \in B$. An almost disjoint family $A = (a_\alpha : \alpha < \omega_1)$ of infinite subsets of ω is said to be a *Luzin gap* if $(a_\alpha : \alpha \in B)$ and $(a_\alpha : \alpha \in \omega_1 \setminus B)$ cannot be separated for each uncountable $B \subseteq \omega_1$ such that $\omega_1 \setminus B$ is also uncountable.

Luzin constructed such a family in ZFC [L]. It is straightforward to verify that if for all $\alpha < \omega_1$ and $m < \omega$ the set $\{\beta < \alpha : a_\alpha \cap a_\beta \subseteq m\}$ is finite, then $(a_\alpha : \alpha < \omega_1)$ is a Luzin gap (see [vD]). Henceforth we call any such Luzin gap *standard*.

If A is a Luzin gap, then each continuous $f : \Psi(A) \mapsto \mathbb{R}$ is eventually constant. If A has the additional property that every countable subset of A can be separated from its complement (call such a Luzin gap *separable*), then each eventually constant function $f : A \mapsto \mathbb{R}$ can be extended to a continuous function $\hat{f} : \Psi(A) \mapsto \mathbb{R}$. This latter property implies that $C_p(\Psi(A))$ is not normal.

The proof of the following proposition can be found in [vD].

Proposition 4.1. *$\mathfrak{b} > \omega_1$ implies that every Luzin gap is separable.*

The proof of the following proposition is implicit in the proofs of Claims 3.4 and 3.5.

Proposition 4.2. *If $(p_\alpha : \alpha < \omega_1)$ is a base for an ultrafilter such that $\alpha < \beta$ implies that $p_\beta \subseteq^* p_\alpha$ and if $a_\alpha \subseteq p_\alpha \setminus p_{\alpha+1}$ is infinite for each $\alpha < \omega_1$, then $(a_\alpha : \alpha < \omega_1)$ is a separable Luzin gap.*

In the previous section we went to great lengths to construct an almost disjoint family as in 3.2 so that $e(C_p(\Psi(A))$ was countable. We don't know the answer to the following question.

Question 4.3. *Is $e(C_p(\Psi(A)) \leq \aleph_0$ whenever A is a separable Luzin gap?*

The following theorem implies that a positive answer to this question would yield a ZFC example of a separable X for which $C_p(X)$ is not normal and has countable extent. The theorem is due to J. Baumgartner and independently to P. Nyikos. However neither published the result, so we include a proof here.

Theorem 4.4 (ZFC)**.** *There exists a separable standard Luzin gap.*

Proof (Sketch). The idea is to construct inductively $(a_\alpha : \alpha < \omega_1)$, $(b_\alpha : \alpha < \omega_1)$, and $(u^\beta_\alpha : \beta < \alpha < \omega_1)$ such that for all $\beta < \beta' < \alpha < \gamma < \omega_1$:

(1) u^β_α is a finite subset of ω;
(2) $(a_\beta \setminus u^\beta_\alpha) \cap (a_{\beta'} \setminus u^{\beta'}_\alpha) = \emptyset$;
(3) $u^\beta_\alpha \cap u^{\beta'}_\alpha = \emptyset$;
(4) $|\{\beta < \alpha : u^\beta_\alpha \cap a_\beta \cap a_\alpha = \emptyset\}| < \aleph_0$;
(5) $a_\beta \subseteq^* b_\alpha$;
(6) $u^\beta_\alpha \cap b_\alpha = \emptyset$ and $|\{\delta < \gamma : u^\delta_\gamma \cap b_\alpha \neq \emptyset\}| < \aleph_0$;
(7) $a_\alpha \subset \bigcup_{\beta<\alpha} u^\beta_\alpha$.

This suffices: Consider $A = (a_\alpha : \alpha < \omega_1)$. By (1) and (2), this is a family of pairwise almost disjoint subsets of ω. By (3) and (4), A is standard. By (5), (6), and (7), b_α separates $(a_\beta : \beta < \alpha)$ from $(a_\gamma : \alpha \leq \gamma)$ for each $\alpha < \omega_1$.

We shall construct the a_α's, b_α's, and u^β_α's so that for all $\beta < \alpha < \gamma < \omega_1$:

(8) $|\{\delta < \alpha : a_\beta \cap u^\delta_\alpha \neq \emptyset\}| < \aleph_0$;
(9) $\forall n < \omega\, |\{\delta < \alpha : |a_\delta \cap u^\delta_\alpha| < n\}| < \aleph_0$;
(10) $|\{\delta < \alpha : u^\delta_\alpha \not\supseteq u^\delta_\gamma\}| < \aleph_0$.

Notice that (8) is actually a consequence of (3), (5), and (6).

Now suppose that a_β, b_β, and u^δ_β have already been constructed for $\delta < \beta < \alpha$, and conditions (1)–(10) are satisfied up to stage α. Let us assume that α is a limit ordinal (the case of a successor ordinal is easier), and let $(\beta_n)_{n\in\omega}$ be an increasing sequence of ordinals converging to α. Set $\beta_{-1} = 0$. Use diagonalization to find $(u^\beta_\alpha : \beta < \alpha)$ such that (1), (2), (3), (8) hold and such that for every $n \in \omega$:

(11) $\forall \beta_{n-1} \leq \beta < \beta_n \, |a_\beta \cap u^\beta_\alpha| > n$;
(12) $\forall m < \omega \, |\{\beta : \beta_{n-1} \leq \beta < \beta_n : |a_\beta \cap u^\beta_\alpha| < m\}| < \aleph_0$;
(13) $|\{\beta < \beta_n : u^\beta_{\beta_n} \not\supseteq u^\beta_\alpha\}| < \aleph_0$;
(14) $\forall k < n \forall \beta_{n-1} \leq \beta \, (u^\beta_\alpha \setminus u^\beta_{\beta_n} \subset a_\beta \setminus b_{\beta_k})$.

Now let $b_\alpha = \omega \setminus \bigcup_{\beta<\alpha} u^\beta_\alpha$, and choose $a_\alpha \subset \bigcup_{\beta<\alpha} u^\beta_\alpha$ in such a way that $|a_\alpha \cap u^\beta_\alpha| = 1$ for each $\beta < \alpha$.

It is obvious that the objects constructed in this way satisfy conditions (4), (7), and the first half of condition (6). Condition (5) follows from (8), condition (9) follows from (11) and (12), and (10) is a consequence of (13) and the inductive assumption (10) itself. Finally, the second part of condition (6) follows from (13) and (14).

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(W. Just and P. J. Szeptycki) Department of Mathematics, Ohio University, Athens, Ohio 45701

E-mail address: `just@ace.cs.ohiou.edu`

E-mail address:`szeptyck@ace.cs.ohiou.edu`

(O. V. Sipacheva) Chair of General Topology and Geometry, Mechanics and Mathematics Faculty, Moscow State University, 119899 Moscow, Russia

E-mail address: `sipa@glas.apc.org`