

## SIMPLE PROOF OF CALABI-BERNSTEIN'S THEOREM ON MAXIMAL SURFACES

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*Dedicated to the memory of Professor P. Bobillo*

The following uniqueness result, called Calabi-Bernstein's Theorem on maximal surfaces, is well known ([Ca], [C-Y], [K], [E-R]).

**Theorem.** *The only entire solutions to the maximal surface equation*

$$\operatorname{div} \left( \frac{Du}{\sqrt{1 - |Du|^2}} \right) = 0, \quad |Du| < 1,$$

*are affine functions.*

In the references cited above, this result appears as either a particular case of some much more general theorems or stated in terms of local complex representation of the surface. However, a direct simple proof would be desirable to be easily understood for beginning researchers. The proof we present here uses only Liouville's Theorem on harmonic functions on  $\mathbb{R}^2$ . Thus, it is simple and complex function theory is not needed. This proof is inspired by [Ch]. Roughly, the key steps of our proof are: (1) On any maximal surface there exists a positive harmonic function, which is constant if and only if the surface is totally geodesic. (2) The metric of any spacelike graph is globally conformally related to a metric  $g^*$ , which is complete when the graph is entire. (3) On any maximal graph the metric  $g^*$  is flat.

### 1. PRELIMINARIES

Consider the Lorentz-Minkowski space  $\mathbb{L}^3$  with its Lorentzian metric  $\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 - dx_3^2$ , given in the usual coordinate system. Let  $x : M \rightarrow \mathbb{L}^3$  be a spacelike immersion of a two-dimensional manifold  $M$  in  $\mathbb{L}^3$ . Note that  $M$  must be orientable. Let  $N$  be a globally defined unit timelike normal vector field on  $M$ . Suppose that the mean curvature of  $x$  vanishes; then  $M$  is called a maximal surface in  $\mathbb{L}^3$ .

For each vector  $a \in \mathbb{L}^3$  we can consider on  $M$  the smooth function  $\langle N, a \rangle$ . Let  $a^T$  be the vector field on  $M$  induced from the tangential component of  $a$  at any

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point of  $M$ , that is,  $a^T = a + \langle N, a \rangle N$ . Standard computations give the identities

$$(1) \quad \Delta \langle N, a \rangle = \operatorname{tr}(A^2) \langle N, a \rangle,$$

$$(2) \quad \nabla \langle N, a \rangle = -A(a^T),$$

where  $A$  is the Weingarten endomorphism associated to  $N$ , and  $\Delta$  and  $\nabla$  are respectively the Laplacian and the gradient relative to the induced Riemannian metric  $g$  on  $M$ .

From (2) we obtain

$$(3) \quad |\nabla \langle N, a \rangle|^2 = (1/2) \operatorname{tr}(A^2) \{ \langle N, a \rangle^2 + \langle a, a \rangle \}.$$

Now, choose  $a \neq 0$  satisfying  $\langle a, a \rangle = 0$ . Note that in this case  $\langle N, a \rangle$  never vanishes and thus we may assume  $\langle N, a \rangle > 0$ . From (1) and (3) we get

$$(4) \quad \Delta(1/\langle N, a \rangle) = 0.$$

On the other hand, if we choose now  $b \in \mathbb{L}^3$  such that  $\langle b, b \rangle = -1$  and  $\langle N, b \rangle > 0$ , then, using again (1) and (3) we obtain

$$(5) \quad \Delta \log(\langle N, b \rangle + 1) = (1/2) \operatorname{tr}(A^2)$$

which means, taking into account the Gauss equation  $2K = \operatorname{tr}(A^2)$  for  $x$ , that the metric  $g^* = (\langle N, b \rangle + 1)^2 g$  on  $M$  is flat.

## 2. PROOF OF THE THEOREM

We begin by observing that on a spacelike graph in  $\mathbb{L}^3$  of any entire function  $u = u(x, y)$ ,  $|Du| < 1$ , the metric  $g' = (1/(1 - |Du|^2))g$  is complete. This easily follows from the inequality  $L' \geq (1/\sqrt{2})L_0$ , where  $L'$  (resp.  $L_0$ ) denotes the length of a curve on the graph with respect to  $g'$  (resp. the metric induced from the usual Riemannian one of  $\mathbb{R}^3$ ). Now take  $b = (0, 0, -1)$  so that  $\langle N, b \rangle = 1/\sqrt{1 - |Du|^2}$ . Consider now the Riemannian metric  $g^*$  introduced above. This metric is complete on a spacelike graph of any entire function. Assume the graph is maximal. On the other hand, from (5) we see that  $g^*$  is flat. Using Cartan's Theorem we have a global isometry from the Euclidean plane  $\mathbb{R}^2$  onto the maximal entire graph endowed with the metric  $g^*$ . The invariance of harmonic functions under conformal changes of metric and this isometry permit us to induce  $1/\langle N, a \rangle$  on a positive harmonic function on  $\mathbb{R}^2$  which must be constant by Liouville's Theorem. Hence  $\langle N, a \rangle$  is constant on the graph, which implies, using (1), that the maximal graph is totally geodesic.

## 3. REMARK

Consider as in §1, a maximal surface  $M$  in  $\mathbb{L}^3$ . The metric  $g^*$  on  $M$  is also complete if the induced metric  $g$  is complete. Thus, the same reasoning as in the previous section gives a proof of the well-known parametric version of this Theorem: "The only complete maximal surfaces in  $\mathbb{L}^3$  are the spacelike planes."

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