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A-DIFFERENTIABILITY AND A-ANALYTICITY

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ABSTRACT. Let A be a finite-dimensional commutative algebra over $\mathbb{R}$ and let $C_A^r(U)$, $C^\omega(U, A)$ and $\mathcal{O}_A(U)$ be the ring of A-differentiable functions of class C^r, $0 \le r \le \infty$, the ring of real analytic mappings with values in A and the ring of A-analytic functions, respectively, defined on an open subset U of A^n. We prove two basic results concerning A-differentiability and A-analyticity: 1^{st}) $\mathcal{O}_A(U) = C_A^\infty(U) \bigcap C^\omega(U, A)$, 2^{nd}) $\mathcal{O}_A(U) = C_A^\infty(U)$ if and only if A is defined over $\mathbb{C}$.

1. Preliminaries and statement of the main results

Let A be a finite-dimensional commutative $\mathbb{R}$-algebra. Let us denote by $G(A)$ the group of units of A. Clearly, $G(A)$ is an open dense subset of A and hence it is endowed with a canonical structure of Lie group. Let U be an open subset of A. A function $f \colon U \longrightarrow A$ is said to be A-differentiable if for every $a \in U$ there exists the limit $f'(a) = \lim_{x \to a}(f(x) - f(a))/(x - a)$, $x - a \in G(A)$. We say that f is A-differentiable of class C^r and set $f \in C_A^r(U)$ if for every $a \in U$, $f'(a), f''(a), ..., f^{(r)}(a)$ exist and are continuous. For further properties on A-differentiability we refer to [2, 3, 4].

As is well known, if V is a finite-dimensional $\mathbb{R}$-vector space, for each open subset $U \subset V$ we have a canonical isomorphism $\zeta_x \colon V \longrightarrow T_x(U)$ given by $\zeta_x(v)(f) = \lim_{t \to 0} \frac{1}{t}(f(x + tv) - f(x))$ (directional derivative). In the particular case $V = A$ we can define a family of endomorphisms on each tangent space $h_a \colon T_x(U) \longrightarrow T_x(U)$, $a \in A$, $h_a(X) = \zeta_x(a \cdot \zeta_x^{-1}(X))$. Then, it is proved that the definition of the analyst coincides with that of the differential geometer (this is not the case for non-commutative algebras, see [2]); *i.e.*, *a function $f \colon U \longrightarrow A$ of class C^∞ is A-differentiable if and only if for every $a \in A$, $x \in U$, the mapping $f_* \colon T_x(U) \longrightarrow T_x(U)$ commutes with h_a.*

The product in A induces an $\mathbb{R}$-bilinear mapping $\mu \colon A \times A \longrightarrow A$. Let $e_0 = 1, e_1, ..., e_{m-1}$ be a basis of A as an $\mathbb{R}$-vector space. Then, $\mu(e_i, e_j) = \sum_{k=0}^{m-1} \mu_{ij}^k e_k$, $0 \le i, j \le m-1$. Each function $f \colon U \longrightarrow A$ can be written as $f(x) = \sum_{i=0}^{m-1} f_i(x) e_i$ and f is A-differentiable if and only if $\partial f/\partial x_j = (\partial f/\partial x_0)\, e_j$ for every $j = 1, ..., m-1$; or more explicitly, $\partial f_i/\partial x_j = \sum_{h=0}^{m-1} \mu_{hj}^i (\partial f_h/\partial x_0)$, $i = 0, ..., m-1$; $j = 1, ..., m-1$ (*Cauchy-Riemann equations*). It follows that $C_A^r(U)$ is a closed A-subalgebra of $C^r(U, A)$ with respect to its natural topology of Fréchet algebra.

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The above definitions and results can also be extended to several variables. Let U be an open subset of A^n. A function $F\colon U \longrightarrow A$ is said to be A-differentiable if for every $x = (x_1, ..., x_n) \in U$ and every $i = 1, ..., n$ the function

$$y \mapsto F(x_1, ..., x_{i-1}, y, x_{i+1}, ..., x_n),$$

defined on a neighbourhood of x_i, is A-differentiable. Set $x_i = \sum_{j=0}^{m-1} x_{ij} e_j$, $i = 1, ..., n$. We obtain: *F is A-differentiable if and only if*

$$\partial F / \partial x_{ij} = (\partial F / \partial x_{i0})\, e_j, \qquad i = 1, ..., n\,;\; j = 1, ..., m-1.$$

Finally, F is said to be A-analytic and then we set $F \in \mathcal{O}_A(U)$ if for every $x^0 = (x_1^0, ..., x_n^0) \in U$ there exists a series $\sum_{\alpha \in \mathbb{N}^n} a_\alpha\, (x_1 - x_1^0)^{\alpha_1} \cdots (x_n - x_n^0)^{\alpha_n}$ absolutely convergent on $|x_i| \leq r_i$, $i = 1, ..., n$, such that for every $x = (x_1, ..., x_n) \in U$, $|x_i - x_i^0| \leq r_i$, $i = 1, ..., n$, we have

$$F(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha\, (x_1 - x_1^0)^{\alpha_1} \cdots (x_n - x_n^0)^{\alpha_n}.$$

Our basic results are the following two theorems:

Theorem I. *The ring of A-analytic functions on an open set $U \subset A^n$ is the intersection of the ring of A-differentiable functions of class C^∞ on U and the ring of real analytic A-valued mappings defined on U; i.e., $\mathcal{O}_A(U) = C_A^\infty(U) \cap C^\omega(U, A)$.*

Theorem II. *$\mathcal{O}_A(U) = C_A^\infty(U)$ if and only if A is defined over $\mathbb{C}$.*

The latter theorem can also be restated as follows:

Corollary. *$\mathcal{O}_A(U) = C_A^\infty(U)$ if and only if $G(A)$ is connected.*

2. Proof of Theorem I

Lemma 1. *Let $F\colon U \longrightarrow A$, $U \subset A^n$, be an A-differentiable function of class C^r. For every system of indices $i_1, ..., i_k, j_1, ..., j_k$, such that $1 \leq i_h \leq n$, $0 \leq j_h \leq m-1$, $1 \leq h \leq k \leq r$, we have*

$$\frac{\partial^k F}{\partial x_{i_1 j_1} ... \partial x_{i_k j_k}} = \frac{\partial^k F}{\partial x_{i_1 0} ... \partial x_{i_k 0}}\, e_{j_1} \cdots e_{j_k}.$$

Proof. We proceed by recurrence on k. For $k = 1$, the above formula is nothing but the Cauchy-Riemann equations for F (cf. §1), and by virtue of the recurrence hypothesis for every $k > 1$, we obtain

$$\begin{aligned} \partial^k F / \partial x_{i_1 j_1} \ldots \partial x_{i_k j_k} &= \partial / \partial x_{i_1 j_1} \left(\partial^{k-1} F / \partial x_{i_2 j_2} \ldots \partial x_{i_k j_k} \right) \\ &= \partial / \partial x_{i_1 j_1} \left((\partial^{k-1} F / \partial x_{i_2 0} \ldots \partial x_{i_k 0}) e_{j_2} \cdots e_{j_k} \right) \\ &= (\partial^k F / \partial x_{i_1 0} \ldots \partial x_{i_k 0}) e_{j_1} e_{j_2} \cdots e_{j_k}. \end{aligned}$$

Proof of Theorem I. First we prove $\mathcal{O}_A(U) \subset C_A^\infty(U) \bigcap C^\omega(U, A)$. If $F\colon U \longrightarrow A$ is A-analytic, given $x^0 = (x_1^0, ..., x_n^0) \in U$, there exists a series $\sum_\alpha a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ converging absolutely on $|x_i| \leq r_i$, $1 \leq i \leq n$, such that for every $x = (x_1, \ldots, x_n) \in U$, $|x_i - x_i^0| \leq r_i$, $1 \leq i \leq n$, we have $F(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha\, (x_1 - x_1^0)^{\alpha_1} \cdots (x_n - x_n^0)^{\alpha_n}$. With the above notations, *i.e.*, $x_i = \sum_{j=0}^{m-1} x_{ij} e_j$, $1 \leq i \leq n$, it is not difficult to see that $F(x)$ can be expressed as an absolutely convergent series of the real variables $x_{ij} - x_{ij}(x^0)$ taking values in A. Therefore, $F \in C^\omega(U, A)$. Moreover, from the standard properties of power series it follows that we can calculate $\partial F / \partial x_{ij}$ simply taking derivatives term by term. As $(\partial/\partial x_{ij})(x_h - x_h^0)^k = k(x_h - x_h^0)^{k-1}\,\delta_{hi}\,e_j$, we have:

$$\frac{\partial F}{\partial x_{ij}} = \sum_{\alpha\in\mathbb{N}^n} a_\alpha \alpha_i (x_1 - x_1^0)^{\alpha_1}\cdots(x_i - x_i^0)^{\alpha_i - 1}\cdots(x_n - x_n^0)^{\alpha_n}\cdot e_j = \frac{\partial F}{\partial x_{i0}}\,e_j\,.$$

Thus, F is A-differentiable.

Next we shall give two proofs of the second part of Theorem I.

1^{st}) Assume $F\colon U \longrightarrow A$ is a function of class $\mathcal{C}^\omega$. Then, its components $F = \sum_{k=0}^{m-1} F_k\,e_k$ can be expanded in power series on a neighbourhood of $x^0 \in U$:

$$F_k(x) = \sum_{|\beta_1|,\dots,|\beta_n|=0}^{\infty} \frac{1}{\beta_1!\cdots\beta_n!}\frac{\partial^{|\beta_1|+\dots+|\beta_n|}}{\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}}(x^0)(x_1 - x_1(x^0))^{\beta_1}\cdots(x_n - x_n(x^0))^{\beta_n},$$

where we have set:

$$\beta_i = (\beta_{i0},\dots,\beta_{i,m-1})\,,\ i = 1,\dots,n\,,\quad \partial x_i^{\beta_i} = \partial x_{i0}^{\beta_{i0}}\cdots\partial x_{i,m-1}^{\beta_{i,m-1}},$$
$$(x_i - x_i(x^0))^{\beta_i} = (x_{i0} - x_{i0}(x^0))^{\beta_{i0}}\cdots(x_{i,m-1} - x_{i,m-1}(x^0))^{\beta_{i,m-1}},$$
$$|\beta_i| = \beta_{i0} + \dots + \beta_{i,m-1}\,,\quad \beta_i! = \beta_{i0}!\cdots\beta_{i,m-1}!.$$

Furthermore, we set:

$$\beta = (\beta_1,\dots,\beta_n),\quad \beta! = \beta_1!\cdots\beta_n!,\quad (x - x(x^0))^\beta = \prod_{i=1}^n (x_i - x_i(x^0))^{\beta_i},$$
$$\partial^{|\beta|}F_k/\partial x^\beta = \partial^{|\beta_1|+\dots+|\beta_n|}F_k/\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}.$$

Hence,

$$F(x) = \sum_{k=0}^{m-1}\sum_{|\beta|=0}^{\infty}\frac{1}{\beta!}(\partial^{|\beta|}F_k/\partial x^\beta)(x^0)\left(x - x(x^0)\right)^\beta\ e_k,$$

and by virtue of Lemma 1,

$$(\partial^{|\beta|}F/\partial x^\beta)(x^0) = \sum_{k=0}^{m-1}(\partial^{|\beta|}F_k/\partial x^\beta)(x^0)\,e_k$$
$$= (\partial^{|\beta|}F/\partial x_{10}^{|\beta_1|}\cdots\partial x_{n0}^{|\beta_n|})(x^0)\,e_0^{\beta_{10}}\cdots e_{m-1}^{\beta_{1,m-1}}\cdots e_0^{\beta_{n0}}\cdots e_{m-1}^{\beta_{n,m-1}}.$$

Hence,

$$\begin{aligned}F(x) &= \sum_{|\beta|=0}^{\infty}\frac{1}{\beta!}\frac{\partial^{|\beta|}F}{\partial x_{10}^{|\beta_1|}\cdots\partial x_{n0}^{|\beta_n|}}(x^0)(x_{10} - x_{10}(x^0))^{\beta_{10}}\cdots\\ &\quad\cdot(x_{1,m-1} - x_{1,m-1}(x^0))^{\beta_{1,m-1}}e_0^{\beta_{10}}\cdots e_{m-1}^{\beta_{1,m-1}}\cdots\\ &\quad\cdot(x_{n0} - x_{n0}(x^0))^{\beta_{n0}}\cdots(x_{n,m-1} - x_{n,m-1}(x^0))^{\beta_{n,m-1}}\,e_0^{\beta_{n0}}\cdots e_{m-1}^{\beta_{n,m-1}}\\ &= \frac{1}{m!^n}\sum_{|\beta|=0}^{\infty}\frac{1}{\beta!}\frac{\partial^{|\beta|}F}{\partial x_{10}^{|\beta_1|}\cdots\partial x_{n0}^{|\beta_n|}}(x^0)\prod_{i=1}^n(x_i - x_i^0)^{|\beta_i|},\end{aligned}$$

as follows by applying to $(x_i - x_i^0)^{|\beta_i|} = (\sum_{k=0}^{m-1}(x_{ik} - x_{ik}(x^0))e_k)^{|\beta_i|}$ the Leibniz's formula.

By setting

$$\alpha_i = |\beta_i|\,,\ i = 1,...,n\,,\ \alpha = (\alpha_1,\dots,\alpha_n),$$
$$a_\alpha = 1/m!^n\beta!(\partial^{|\beta|}F/\partial x_{10}^{\alpha_1}\cdots\partial x_{n0}^{\alpha_n})(x^0),$$

we have $F(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha (x_1 - x_1^0)^{\alpha_1}\cdots(x_n - x_n^0)^{\alpha_n}$, thus proving that F is A-analytic.

2^{nd}) The second proof relies on the Taylor formula below which is of interest by itself. Essentially this formula tells us that an A-differentiable function of class C^{r+1} can be approximated around a point by a suitable A-polynomial of degree r plus a remainder.

Proposition 1. *Let F be an A-differentiable function of class $\mathcal{C}^{r+1}$ defined on a neighbourhood $|x - x^0| < \epsilon$ of a point $x^0 \in A^n$. Then we have:*

$$F(x) = \sum_{|\alpha|=0}^{r} \frac{1}{\alpha!}\frac{\partial^{|\alpha|}F}{\partial x^\alpha}(x^0)(x-x^0)^\alpha + \sum_{|\alpha|=r+1}\int_0^1 \frac{(1-t)^r}{\alpha!}\frac{\partial^{r+1}F}{\partial x^\alpha}\left(x^0 + t(x-x^0)\right)(x-x^0)^\alpha dt\,.$$

Proof of Proposition 1. From Taylor's formula with integral remainder we obtain

$$F(x) = \sum_{k=0}^{r}\frac{1}{k!}D^kF(x^0)(x-x^0,\overset{(k}{\dots},x-x^0) + \int_0^1 \frac{(1-t)^r}{r!}D^{r+1}F\left(x^0+t(x-x^0)\right)(x-x^0,\overset{(r+1}{\dots},x-x^0)dt\,,$$

and we can conclude as in the proof of Theorem 2.3 of [4]; *i.e.*, since F is A-differentiable, we have

$$DF(x^0)(x-x^0) = \sum_i (\partial F/\partial x_i)(x^0)(x_i - x_i^0),$$

$$\begin{aligned} D^2F(x^0)(x-x^0,x-x^0) &= D(DF)(x^0)(x-x^0)(x-x^0) \\ &= \sum_{i,j}(\partial^2F/\partial x_i\partial x_j)(x^0)(x_i-x_i^0)(x_j-x_j^0), \end{aligned}$$

and so on.

The proof of the second part of Theorem I is then a simple consequence of the above formula. In fact, we know that if F is real analytic, then there exist a neighbourhood N of x and two positive constants M,ρ, such that for every $x\in N$ and every $\alpha\in\mathbb{N}^n$, we have $\left|(\alpha!)^{-1}\left(\partial^{|\alpha|}F/\partial x^\alpha\right)(x)\right| \le M\,\rho^{|\alpha|}$, and the result follows by simply bounding the remainder term in Taylor's formula.

3. Proof of Theorem II and of the Corollary

Lemma 2. *Let A be a finite-dimensional commutative local $\mathbb{R}$-algebra and let $\mathfrak{m}$ be its maximal ideal. If $A/\mathfrak{m}\cong\mathbb{C}$, then A admits a structure of $\mathbb{C}$–algebra compatible with its structure of $\mathbb{R}$-algebra.*

Proof. As $\mathfrak{m}$ is nilpotent, it is clear that A is $\mathfrak{m}$-adic complete. From a classical result by I. S. Cohen (see e.g., [1, II.8.25A]) there is a subfield $K \subset A$, containing $\mathbb{R}$, such that $K \to A/\mathfrak{m}$ is an isomorphism.

Lemma 3. *Let A be a finite-dimensional commutative local $\mathbb{R}$-algebra such that $A/\mathfrak{m} \cong \mathbb{R}$. There exist a system of elements $e_1, \dots, e_r \in \mathfrak{m}$ and a set of multi-indices $M \subset \mathbb{N}^r$ such that:*

(i) *The cosets of $e^\alpha = e_1^{\alpha_1} \cdots e_r^{\alpha_r}$, $|\alpha| = l$, $\alpha \in M$, modulo $\mathfrak{m}^{l+1}$ are a basis of the $\mathbb{R}$–vector space $\mathfrak{m}^l/\mathfrak{m}^{l+1}$ for every $l \leq n-1$, n being the least integer such that $\mathfrak{m}^n = 0$.*

(ii) *The elements e^α, $\alpha \in M$, with $e^0 = 1$, are a basis of A as an $\mathbb{R}$-vector space.*

Proof. Let $e_1, \dots, e_r$ be elements of $\mathfrak{m}$ whose cosets are a basis of the $\mathbb{R}$-vector space $\mathfrak{m}/\mathfrak{m}^2$. From Nakayama's lemma, $e_1, \dots, e_r$ also generate $\mathfrak{m}$ as an A-module; that is, every $a \in A$ can be written as $a = a_1 e_1 + \dots + a_r e_r$ for some $a_i \in A$. Assume we have constructed a set of multi-indices $M_l \subset \mathbb{N}^r$ of order $|\alpha| = l$ so that the cosets of e^α, $\alpha \in M_l$, are a basis of $\mathfrak{m}^l/\mathfrak{m}^{l+1}$. As $e_1, \dots, e_r$ generate the A-module $\mathfrak{m}$, it is clear that all the elements e^α, $|\alpha| = l+1$, generate $\mathfrak{m}^{l+1}$. Hence each $a \in \mathfrak{m}^{l+1}$ can be written as $a = \sum_{|\alpha|=l+1} a_\alpha e^\alpha$ and since $A/\mathfrak{m} \cong \mathbb{R}$, there exist scalars $\lambda_\alpha \in \mathbb{R}$ such that $a_\alpha - \lambda_\alpha \in \mathfrak{m}$. Hence, $a \equiv \sum_{|\alpha|=l+1} \lambda_\alpha e^\alpha \pmod{\mathfrak{m}^{l+2}}$. Accordingly, the cosets of e^α, $|\alpha| = l+1$, generate $\mathfrak{m}^{l+1}/\mathfrak{m}^{l+2}$ as a vector space and from them we can select a basis. In other words, there exist a set of multi-indices M_{l+1} of order $|\alpha| = l+1$ such that the cosets of the elements e^α, $\alpha \in M_{l+1}$, are a basis for $\mathfrak{m}^{l+1}/\mathfrak{m}^{l+2}$. Proceeding by recurrence, it will suffice to put $M = \bigcup_{l=0}^{n-1} M_l$.

As for (ii), first let us assume we have a relation $\sum_{\alpha \in M} \lambda_\alpha e^\alpha = 0$. Reducing it modulo $\mathfrak{m}$ we obtain $\lambda_0 = 0$. Reducing it modulo $\mathfrak{m}^2$ we obtain $\lambda_1 = \dots = \lambda_r = 0$, and so on. Moreover such elements generate A as a vector space, because given $a \in A$ there is a scalar λ_0 such that $a' = a - \lambda_0 \in \mathfrak{m}$. Again $a' = a_1 e_1 + \dots + a_r e_r$ and there are scalars $\lambda_1, \dots, \lambda_r$ so that $a_i = \lambda_i + a'_i$, $a'_i \in \mathfrak{m}$; hence $a = \lambda_0 + \lambda_1 e_1 + \dots + \lambda_r e_r + a''$, $a'' \in \mathfrak{m}^2$, and this process finishes after a finite number of steps because $\mathfrak{m}$ is nilpotent.

Proof of Theorem II. Let $A = A_1 \times \dots \times A_l$ be the decomposition of A in local algebras. If $A_i/\mathfrak{m}_i \cong \mathbb{C}$ for every i, then, by virtue of Lemma 2, A is a $\mathbb{C}$-algebra and Cauchy's integral formula holds for $\mathbb{C}$-algebras (see [2]); hence the statement is true in that case. In fact, in [2] the author works with algebras of the form $\mathbb{C} \otimes_{\mathbb{R}} B$, where B is a $\mathbb{R}$-algebra, because he assumes the μ's are real, but his theory also applies to arbitrary $\mathbb{C}$-algebras. Therefore, it will suffice to prove that our statement does not hold if an index i exists at least such that $A_i/\mathfrak{m}_i \cong \mathbb{R}$. We can assume $i = 1$.

It will suffice to construct a function $f\colon A_1 \longrightarrow A_1$ which is A_1-differentiable but not C^ω-analytic, because prolongating f by zero on the rest of components we will obtain a function $\bar{f}\colon A \longrightarrow A$ which is A-differentiable but not $\mathcal{C}^\omega$-analytic and we could conclude by virtue of Theorem I. Consequently, we can assume that A is local and $A/\mathfrak{m} \cong \mathbb{R}$.

Let $f_0\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a C^∞ but not C^ω function. Identifying $\mathbb{R}$ with $A/\mathfrak{m}$ and taking into account that $\mathfrak{m}^n = 0$, it will suffice to construct a $(A/\mathfrak{m}^l)$-differentiable function $f_{l-1} : A/\mathfrak{m}^l \longrightarrow A/\mathfrak{m}^l$, for every $l = 1, \dots, n-1$, making commutative the diagram below for $l \geq 2$:

$$\begin{array}{ccc} A/\mathfrak{m}^l & \longrightarrow & A/\mathfrak{m}^{l-1} \\ {\scriptstyle f_{l-1}}\downarrow & & \downarrow{\scriptstyle f_{l-2}} \\ A/\mathfrak{m}^l & \longrightarrow & A/\mathfrak{m}^{l-1} \end{array}$$

We proceed by recurrence assuming $f_0, \dots, f_{l-1}$ have been constructed and we shall then construct f_l. To do this we shall use the basis $\{e^\alpha, \alpha \in M\}$ of Lemma 3 and we shall denote by x_α the coordinates in this basis; *i.e.*, $a = \sum_{\alpha \in M} x_\alpha(a) e^\alpha$, so that A-differentiability is expressed as $\partial f/\partial x_\alpha = (\partial f/\partial x_0)\, e^\alpha$, $\alpha \in M$, $|\alpha| > 0$. Let us denote by $[a]_{l-1}$ the coset of an element $a \in A$ in $A/\mathfrak{m}^l$. We note that $\{[e^\alpha]_{l-1}\,;\, \alpha \in M, |\alpha| \leq l-1\}$ is a basis for $A/\mathfrak{m}^l$. We set $M_l = \{\alpha \in M\,;\, |\alpha| \leq l\}$. Thus, $f_{l-1} = \sum_{\alpha \in M_{l-1}} f_{l-1,\alpha}[e^\alpha]_{l-1}$. The commutativity of the above diagram means $f_{l-1,\alpha} = f_{l-2,\alpha}$ for every $\alpha \in M_{l-2}$; hence $f_{l-1,\alpha}$ only depends on the variables x_β, $\beta \in M_{l-1}$, $|\beta| \leq |\alpha|$. As f_{l-1} is $(A/\mathfrak{m}^l)$-differentiable, for every $|\beta| > 0$, $\beta \in M_{l-1}$, we have

$$\frac{\partial f_{l-1}}{\partial x_\beta} = \sum_{\alpha \in M_{l-1}} \frac{\partial f_{l-1,\alpha}}{\partial x_\beta}[e^\alpha]_{l-1} = \sum_{\alpha \in M_{l-1}} \frac{\partial f_{l-1,\alpha}}{\partial x_0}[e^{\alpha+\beta}]_{l-1}.$$

Hence,

$$\frac{\partial f_{l-1,\alpha}}{\partial x_\beta} = \frac{\partial f_{l-1,\alpha-\beta}}{\partial x_0}, \quad \alpha - \beta \in M_{l-1}, |\beta| > 0, \alpha, \beta \in M_{l-1}, \tag{3.1}$$

$$\frac{\partial f_{l-1,\alpha}}{\partial x_\beta} = 0, \quad \alpha - \beta \notin M_{l-1}, |\beta| > 0, \alpha, \beta \in M_{l-1}. \tag{3.2}$$

Let us define f_l by giving its components:

$$\begin{cases} f_{l,\alpha} = f_{l-1,\alpha} & \text{if } \alpha \in M_{l-1}, \\ f_{l,\sigma} = \frac{\partial f_{l-1,0}}{\partial x_0}\, x_\sigma & \text{if } \sigma \in M_l - M_{l-1}, \end{cases} \tag{3.3}$$

thus also proving the commutativity of the above diagram for the order $l+1$. Let us check that f_l is $(A/\mathfrak{m}^{l+1})$-differentiable as well. For every $\gamma \in \mathbb{N}^r$ we have:

$$\begin{aligned} \frac{\partial f_l}{\partial x_\gamma} = \sum_{\alpha \in M_{l-1}} \frac{\partial f_{l-1,\alpha}}{\partial x_\gamma}\,[e^\alpha]_l \\ + \sum_{\sigma \in M_l - M_{l-1}} \left(\frac{\partial^2 f_{l-1,0}}{\partial x_\gamma \partial x_0}\, x_\sigma + \frac{\partial f_{l-1,0}}{\partial x_0}\, \delta_{\gamma,\sigma} \right) [e^\sigma]_l. \end{aligned} \tag{3.4}$$

Consequently, for $|\gamma| > 0$, $\gamma \in M_{l-1}$, we have $\frac{\partial f_l}{\partial x_\gamma} = \sum_{\alpha \in M_{l-1}} \frac{\partial f_{l-1,\alpha}}{\partial x_\gamma}\,[e^\alpha]_l$. Furthermore,

$$\frac{\partial f_l}{\partial x_0} = \sum_{\alpha \in M_{l-1}} \frac{\partial f_{l-1,\alpha}}{\partial x_0}\,[e^\alpha]_l + \sum_{\sigma \in M_l - M_{l-1}} \left(\frac{\partial^2 f_{l-1,0}}{\partial x_0^2}\, x_\sigma \right) [e^\sigma]_l, \tag{3.5}$$

and since $[e^{\sigma+\gamma}]_l = 0$ because $|\sigma + \gamma| = |\sigma| + |\gamma| = l + |\gamma| \geq l+1$, we also have

$$\frac{\partial f_l}{\partial x_0}[e^\gamma]_l = \sum_{\alpha \in M_{l-1}} \frac{\partial f_{l-1,\alpha}}{\partial x_0}\,[e^{\alpha+\gamma}]_l.$$

By virtue of (3.1) and (3.2) it follows that $\partial f_l/\partial x_\gamma = (\partial f_l/\partial x_0)[e^\gamma]_l$. Assume $|\gamma| > 0$, $\gamma \in M_l - M_{l-1}$. In that case, (3.4) takes the form $\partial f_l/\partial x_\gamma = (\partial f_{l-1,0}/\partial x_0)[e^\gamma]_l$ because $f_{l-1,\alpha}$ does not depend on x_γ, and from (3.5) we obtain $(\partial f_l/\partial x_0)[e^\gamma]_l = \sum_{\alpha \in M_{l-1}} (\partial f_{l-1,\alpha}/\partial x_0)[e^{\alpha+\gamma}]_l = (\partial f_{l-1,0}/\partial x_0)[e^\gamma]_l$, since for every $|\alpha| > 0$ we have $|\alpha + \gamma| = |\alpha| + l \geq l + 1$ and $|\sigma + \gamma| = |\sigma| + |\gamma| = 2l \geq l + 1$, thus concluding that $\partial f_l/\partial x_\gamma = (\partial f_l/\partial x_0)[e^\gamma]_l$ also in this case and thus finishing the proof.

Proof of the Corollary. It follows directly from Theorem II and the following results.

Lemma 4. *Let A be a finite-dimensional commutative local $\mathbb{R}$-algebra. $G(A)$ has two connected components if $A/\mathfrak{m} \cong \mathbb{R}$ and it is connected if $A/\mathfrak{m} \cong \mathbb{C}$.*

Proof. We have a short exact sequence of Lie groups,

$$1 \longrightarrow 1 + \mathfrak{m} \longrightarrow G(A) \longrightarrow (A/\mathfrak{m})^* \longrightarrow 1 .$$

Hence $G(A)$ is a principal bundle on $(A/\mathfrak{m})^*$ with structure group $1 + \mathfrak{m}$ which is homeomorphic to the vector space $\mathfrak{m}$. Consequently, $G(A)$ and $(A/\mathfrak{m})^*$ have the same number of connected components.

Proposition 2. *Let A be a finite-dimensional commutative $\mathbb{R}$–algebra, let $A = A_1 \times \ldots \times A_l$ be its decomposition in local algebras and let $\mathfrak{m}_i$ be the maximal ideal of A_i. Then, $G(A)$ is connected if and only if $A_i/\mathfrak{m}_i \cong \mathbb{C}$ for every $i = 1, \ldots, l$.*

Proof. $G(A) \cong G(A_1) \times \ldots \times G(A_l)$. Hence $G(A)$ is connected if and only if each $G(A_i)$ is, and we can conclude by applying Lemma 4.

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