

ON HANKEL OPERATORS NOT IN THE TOEPLITZ ALGEBRA

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ABSTRACT. In this paper we exhibit a class of Hankel operators, which is contained in the essential commutant of the unilateral shift, but disjoint from the Toeplitz algebra.

In [2] it is proved that if \mathbf{E} is the essential commutant of the unilateral shift and \mathbf{T} is the Toeplitz algebra, then the set $\mathbf{E} \setminus \mathbf{T}$ contains Hankel operators (the definitions of \mathbf{E} and \mathbf{T} are given below). The proof given in [2] makes use of the theory of maximal ideals of function algebras; since this method is not constructive, it does not yield concrete operators. The purpose of this paper is to exhibit a concrete class of Hankel operators which is contained in $\mathbf{E} \setminus \mathbf{T}$. The result in [2], and ours, answers one of the questions raised in [1] concerning membership in the Toeplitz algebra.

The underlying Hilbert space is \mathbf{H}^2 of the unit circle. Let ϕ be in \mathbf{L}^∞ of the unit circle. The Toeplitz operator T_ϕ on \mathbf{H}^2 is defined by $T_\phi f = P(\phi f)$, where P is the orthogonal projection from \mathbf{L}^2 to \mathbf{H}^2 . The Hankel operator H_ϕ on \mathbf{H}^2 is defined by the Hankel matrix $(c_{-i-j-1})_{i,j=0}^\infty$, where $\{c_n\}_{n=-\infty}^\infty$ is the sequence of Fourier coefficients of ϕ . The unilateral shift S is the Toeplitz operator T_ϕ with $\phi(z) = z$. The essential commutant of S is the set \mathbf{E} of all operators T on \mathbf{H}^2 for which $TS - ST \in \mathbf{K}$, where \mathbf{K} is the ideal of all compact operators on \mathbf{H}^2 . The Toeplitz algebra \mathbf{T} is the C^* -algebra generated by $\{T_\phi: \phi \in \mathbf{L}^\infty\}$. The following result describes a class of Hankel operators in $\mathbf{E} \setminus \mathbf{T}$.

Theorem. *Let b be an infinite Blaschke product whose zero set Z has its cluster points in $\{-1, 1\}$. We further assume that there exists a sequence $\{a_n\}_{n=1}^\infty$ in Z such that*

- (i) $\lim_{n \rightarrow \infty} a_n = \lambda;$
(ii) $\lim_{n \rightarrow \infty} \frac{|\lambda - a_n|}{1 - |a_n|} = +\infty.$

Then $H_{\bar{b}}$ is in the essential commutant of S , but $H_{\bar{b}}$ does not belong to the Toeplitz algebra.

The first assertion of the theorem follows from the result [2, Proposition 3.5], which we state below as a proposition. This proposition is proved in [2] using the theory of maximal ideals; the proof given below is based on function theory. First

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we introduce notation and facts which are needed. For $\phi \in \mathbf{L}^\infty$, we write $\tilde{\phi}$ for the function defined by $\tilde{\phi}(z) = \phi(\bar{z})$. For ϕ and ψ in \mathbf{L}^∞ the following identities hold [1]:

$$\begin{aligned} (1) \quad & T_{\phi\psi} = T_\phi T_\psi + H_{\tilde{\phi}} H_\psi; \\ (2) \quad & H_{\phi\psi} = H_\phi T_\psi + T_{\tilde{\phi}} H_\psi. \end{aligned}$$

For f in \mathbf{L}^1 and z in the open unit disk D , we write $P_z(f)$ for the Poisson integral of f :

$$P_z(f) = \frac{1}{2\pi} \int_{\partial D} f P_z d\theta,$$

where P_z denotes the Poisson kernel for the point z . We end this introduction with the algebra $\mathbf{H}^\infty + \mathbf{C}$, whose properties are important in our proof. Here \mathbf{H}^∞ is the algebra of boundary functions for bounded analytic functions in D , and \mathbf{C} is the algebra of continuous complex valued functions on ∂D .

Proposition. *Let b be a Blaschke product with zero set Z . Then $H_{\bar{b}}$ is in the essential commutant of S if and only if Z is finite or Z has its cluster points in $\{-1, 1\}$.*

Proof. Let $A = H_{\bar{b}}S - SH_{\bar{b}}$. Applying (2) twice we obtain

$$H_{\bar{b}z} = H_{\bar{b}}T_z + T_{\tilde{b}}H_z = H_{\bar{b}}T_z = H_{\bar{b}}S \quad \text{and} \quad H_{z\bar{b}} = H_zT_{\bar{b}} + T_zH_{\bar{b}} = H_zT_{\bar{b}} + SH_{\bar{b}},$$

therefore $A = H_{(z-\bar{z})\bar{b}} + H_zT_{\bar{b}}$. Since H_z is of finite rank, A is compact if and only if $(z-\bar{z})\bar{b} \in \mathbf{H}^\infty + \mathbf{C}$ [7, p. 101].

If Z is finite, then $b \in \mathbf{C}$ and therefore $(\bar{z}-z)b \in \mathbf{C}$. If Z has its cluster points in $\{-1, 1\}$, then b is continuous in $\partial D \setminus \{-1, 1\}$ [5, p. 68], and because b is bounded it follows that $(\bar{z}-z)b \in \mathbf{C}$. In either case we have $(z-\bar{z})\bar{b} \in \mathbf{H}^\infty + \mathbf{C}$.

For the converse we assume that $(z-\bar{z})\bar{b} = f$ for some f in $\mathbf{H}^\infty + \mathbf{C}$. We further assume that Z is infinite. Let λ be a cluster point of Z . Then $|\lambda| = 1$. Let $a_n \in Z$ such that $a_n \rightarrow \lambda$. Since the Poisson integral is asymptotically multiplicative on $\mathbf{H}^\infty + \mathbf{C}$ [3, p. 169], given $\varepsilon > 0$ there exists $\delta > 0$ such that $|P_z(bf) - P_z(b)P_z(f)| < \varepsilon$ for $1 - |z| < \delta$. But $P_z(b) = b(z)$ for z in D , and from $bf = z - \bar{z}$ on ∂D we have $P_z(bf) = 2i\mathfrak{I}(z)$ for z in D . Then $2|\mathfrak{I}(a_n)| < \varepsilon$ for $1 - |a_n| < \delta$. This shows that $\mathfrak{I}(a_n) \rightarrow 0$ and therefore $\Re(a_n) \rightarrow \lambda$. Hence λ is real and $\lambda \in \{-1, 1\}$. \square

Lemma. *Let $g \in \mathbf{H}^\infty + \mathbf{C}$ such that its conjugate \bar{g} is also in $\mathbf{H}^\infty + \mathbf{C}$. Then $TT_g - T_gT \in \mathbf{K}$ for all T in the Toeplitz algebra.*

Proof. For $f \in \mathbf{L}^\infty$, let \mathcal{A}_f be the set of all operators T on \mathbf{H}^2 for which $TT_f - T_fT \in \mathbf{K}$. Clearly \mathcal{A}_f is a Banach space, and from $T_1T_2T_f - T_fT_1T_2 = T_1(T_2T_f - T_fT_2) + (T_1T_f - T_fT_1)T_2$ it follows that \mathcal{A}_f is a Banach algebra. Now we consider a function g satisfying the hypothesis of the lemma. Since H_g is compact [7, p. 101], from (1) we conclude that $T_{\phi g} - T_\phi T_g \in \mathbf{K}$ for all ϕ in \mathbf{L}^∞ . The same argument applied to $\tilde{\phi}$ and \bar{g} gives that $T_{\tilde{\phi}\bar{g}} - T_{\tilde{\phi}}T_{\bar{g}} \in \mathbf{K}$ and by taking the adjoint we conclude that $T_{\phi g} - T_gT_\phi \in \mathbf{K}$. Therefore $T_\phi T_g - T_gT_\phi \in \mathbf{K}$ for all ϕ in \mathbf{L}^∞ . Since we can interchange the roles of g and \bar{g} , the set $\{T_\phi : \phi \in \mathbf{L}^\infty\}$ is contained in $\mathcal{A}_g \cap \mathcal{A}_{\bar{g}}$. Because $\mathcal{A}_g \cap \mathcal{A}_{\bar{g}}$ is closed under the adjoint operation, $\mathcal{A}_g \cap \mathcal{A}_{\bar{g}}$ is a C^* -algebra, and therefore $\mathbf{T} \subset \mathcal{A}_g \cap \mathcal{A}_{\bar{g}}$. \square

Now to prove the second part of the theorem it is enough to exhibit a function g satisfying the hypothesis of the lemma, for which the operator $H_{\bar{b}}T_g - T_gH_{\bar{b}}$ is not compact. The construction of g will be carried out in the Banach space BMO of functions of bounded mean oscillation. For the definition and properties of this space we refer the reader to [7]. A key part in the construction of g is played by the following result [4, Lemma 1].

Proposition. *Let I be a subarc of ∂D and J a subarc of I with the same center. There is a continuous function u with values in $[0, 1]$ such that $u = 1$ on J , $u = 0$ off I , and $\|u\|_{\text{BMO}} \leq \text{const}/\log(|I|/|J|)$.*

Here, $|I|$ denotes the length of I , and $\|\cdot\|_{\text{BMO}}$ is the norm in the space BMO.

Construction of a function. Now we are ready to construct a particular real-valued function g in $\mathbf{H}^\infty + \mathbf{C}$. By hypothesis we have the sequence $\{a_n\}_{n=1}^\infty$ in \mathbf{Z} satisfying conditions (i) and (ii), and $\lambda \in \{-1, 1\}$. We write $a_n = |a_n|e^{i\theta_n}$, with $-\pi < \theta_n \leq \pi$. By (ii) we may assume that $\theta_n \neq 0$ and $\theta_n \neq \pi$ for all n . For $a = |a|e^{i\theta}$ in D we have the identity

$$\frac{|\lambda - a|^2}{(1 - |a|)^2} = 1 + 2|a| \cdot \frac{1 - \lambda \cos \theta}{(1 - |a|)^2}.$$

Then from (ii) it follows that

$$(iii) \quad \lim_{n \rightarrow \infty} \frac{1 - \lambda \cos \theta_n}{(1 - |a_n|)^2} = +\infty.$$

For the rest of the construction we assume that the set $\{n: \theta_n > 0\}$ is infinite. (If $\theta_n < 0$ for all n , then the modifications that are necessary are obvious.) Then from (i) it follows that $\theta_n \rightarrow 0$ (if $\lambda = 1$) or $\theta_n \rightarrow \pi$ (if $\lambda = -1$). First we consider the case $\lambda = 1$. Since $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2} = \frac{1}{2}$, from (iii) it follows that

$$\lim_{n \rightarrow \infty} \frac{\theta_n}{1 - |a_n|} = +\infty.$$

Then we can choose a subsequence $\{a_{n_k}\}_{k=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ such that

$$(iv) \quad 2\pi \exp(2^k) < \frac{\theta_{n_k}}{1 - |a_{n_k}|} \quad \text{and} \quad 3\theta_{n_{k+1}} < \theta_{n_k} \quad \text{for } k = 1, \dots, \infty.$$

Let I_k be the subarc of ∂D with center $e^{i\theta_{n_k}}$ and length θ_{n_k} . From the second inequality it follows that $I_k \cap I_l = \emptyset$ for $k \neq l$. Let J_k be the subarc of I_k with the same center as I_k and whose length is $2\pi(1 - |a_{n_k}|)$. Then $|J_k| \exp(2^k) < |I_k|$, and therefore $1/\log(|I_k|/|J_k|) < 2^{-k}$ for $k \geq 1$. Now we apply the proposition above to I_k and J_k to obtain a continuous function u_k with values in $[0, 1]$ such that $u_k = 1$ on J_k , $u_k = 0$ off I_k , and $\|u_k\|_{\text{BMO}} \leq (\text{const})2^{-k}$. Let $u = \sum_{k=1}^\infty u_k$. The series converges in BMO and the terms are continuous.

Then from [7, p. 49], u can be written as $u = f + v$ where f is in \mathbf{C} and v is the harmonic conjugate of a function h in \mathbf{C} . Then $h + iv$ is analytic in D , and since u is bounded ($\|u\|_\infty = 1$), it follows that $h + iv \in \mathbf{H}^\infty$. Since $f + ih \in \mathbf{C}$, the equality $u = -i(h + iv) + (f + ih)$ shows that $u \in \mathbf{H}^\infty + \mathbf{C}$. Now we define $g = u - \bar{u}$. Then g is a real-valued function and $g \in \mathbf{H}^\infty + \mathbf{C}$. So g satisfies the hypothesis of the lemma.

Finally, we consider the case $\lambda = -1$. Then (iii) becomes

$$\lim_{n \rightarrow \infty} \frac{1 - \cos(\pi - \theta_n)}{(1 - |a_n|)^2} = +\infty,$$

and from this it follows that

$$\lim_{n \rightarrow \infty} \frac{\pi - \theta_n}{1 - |a_n|} = +\infty.$$

Now one gets inequalities similar to (iv), where θ_{n_k} is replaced by $\pi - \theta_{n_k}$. Also, in the definition of the subarc I_k we need to change the length (but not the center) to $\pi - \theta_{n_k}$. Then the construction of g proceeds as above.

Last part of the theorem. For the function g constructed above we show that $H_{\bar{b}}T_g - T_gH_{\bar{b}}$ is not compact. From this and the lemma it will follow that $H_{\bar{b}}$ is not in the Toeplitz algebra. Let $A = H_{\bar{b}}T_g - T_gH_{\bar{b}}$. Applying (2) to $H_{\bar{b}g}$ and $H_{\bar{g}\bar{b}}$ (as we did in the proof of the first proposition) we obtain

$$A = H_{(g-\tilde{g})\bar{b}} + H_{\tilde{g}}T_{\bar{b}} - T_{\bar{b}}H_g.$$

Since g and \tilde{g} are in $\mathbf{H}^\infty + \mathbf{C}$, H_g and $H_{\tilde{g}}$ are compact, and therefore A is not compact if and only if $(g - \tilde{g})\bar{b}$ does not belong to $\mathbf{H}^\infty + \mathbf{C}$. To arrive at a contradiction, let us assume that $(g - \tilde{g})\bar{b} = f$ for some $f \in \mathbf{H}^\infty + \mathbf{C}$. Then $g - \tilde{g} = bf$. From the definition of g it follows that $\tilde{g} = -g$, so $bf = 2g$. Since the Poisson integral is asymptotically multiplicative on $\mathbf{H}^\infty + \mathbf{C}$, there exists $\delta > 0$ such that

$$|P_z(bf) - P_z(b)P_z(f)| < 1 \quad \text{for } 1 - |z| < \delta.$$

Therefore, $|P_{a_{n_k}}(g)| < \frac{1}{2}$ for all k 's in the complement of a finite set. Now a contradiction will be obtained by showing that $P_{a_{n_k}}(g) \rightarrow 1$ as $k \rightarrow \infty$. For this we use a result in [6] concerning the Poisson integral and certain averaging functionals. For $z \neq 0$ in D , we let I_z denote the closed subarc of ∂D whose center is $z/|z|$ and whose length is $2\pi(1 - |z|)$. Then from [6, Lemma 5] applied to g , we conclude that given $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$\left| P_z(g) - \frac{1}{|I_z|} \int_{I_z} g(e^{i\theta}) d\theta \right| < \varepsilon \quad \text{for } 1 - |z| < \delta'.$$

But from the definition of J_k and I_z , with $z = a_{n_k}$, we have $I_{a_{n_k}} = J_k$. Also, from the definition of g , because the functions u_k have disjoint supports, we have

$$\int_{J_k} g(e^{i\theta}) d\theta = \int_{J_k} u_k(e^{i\theta}) d\theta = \int_{J_k} 1 d\theta = |J_k|.$$

Then, from the above inequality, $|P_{a_{n_k}}(g) - 1| < \varepsilon$ for $1 - |a_{n_k}| < \delta'$. Now we have obtained the desired contradiction, and therefore A is not compact. The proof of the theorem is complete. \square

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