ON HANKEL OPERATORS NOT IN THE TOEPLITZ ALGEBRA

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Abstract. In this paper we exhibit a class of Hankel operators, which is contained in the essential commutant of the unilateral shift, but disjoint from the Toeplitz algebra.

In [2] it is proved that if \( E \) is the essential commutant of the unilateral shift and \( T \) is the Toeplitz algebra, then the set \( E \setminus T \) contains Hankel operators (the definitions of \( E \) and \( T \) are given below). The proof given in [2] makes use of the theory of maximal ideals of function algebras; since this method is not constructive, it does not yield concrete operators. The purpose of this paper is to exhibit a concrete class of Hankel operators which is contained in \( E \setminus T \). The result in [2], and ours, answers one of the questions raised in [1] concerning membership in the Toeplitz algebra.

The underlying Hilbert space is \( H^2 \) of the unit circle. Let \( \phi \) be in \( L^\infty \) of the unit circle. The Toeplitz operator \( T_\phi \) on \( H^2 \) is defined by \( T_\phi f = P(\phi f) \), where \( P \) is the orthogonal projection from \( L^2 \) to \( H^2 \). The Hankel operator \( H_\phi \) on \( H^2 \) is defined by the Hankel matrix \((c_{-i-j-1})_{i,j=0}^\infty \), where \( \{c_n\}_{n=-\infty}^{\infty} \) is the sequence of Fourier coefficients of \( \phi \). The unilateral shift \( S \) is the Toeplitz operator \( T_\phi \) with \( \phi(z) = z \). The essential commutant of \( S \) is the set \( E \) of all operators \( T \) on \( H^2 \) for which \( TS - ST \in K \), where \( K \) is the ideal of all compact operators on \( H^2 \). The Toeplitz algebra \( T \) is the \( C^* \)-algebra generated by \( \{T_\phi : \phi \in L^\infty\} \). The following result describes a class of Hankel operators in \( E \setminus T \).

Theorem. Let \( b \) be an infinite Blaschke product whose zero set \( Z \) has its cluster points in \( \{-1, 1\} \). We further assume that there exists a sequence \( \{a_n\}_{n=1}^{\infty} \) in \( Z \) such that

(i) \( \lim_{n \to \infty} a_n = \lambda \);

(ii) \( \lim_{n \to \infty} \frac{\lambda - a_n}{1 - |a_n|} = +\infty \).

Then \( H_b \) is in the essential commutant of \( S \), but \( H_b \) does not belong to the Toeplitz algebra.

The first assertion of the theorem follows from the result [2, Proposition 3.5], which we state below as a proposition. This proposition is proved in [2] using the theory of maximal ideals; the proof given below is based on function theory. First
we introduce notation and facts which are needed. For \( \phi \in L^\infty \), we write \( \tilde{\phi} \) for the function defined by \( \phi(z) = \phi(\bar{z}) \). For \( \phi \) and \( \psi \) in \( L^\infty \) the following identities hold [1]:

\[
(T_{\phi \psi} = T_\phi T_\psi + H_{\phi \psi};
\]

(2)

\[
H_{\phi \psi} = H_\phi T_\psi + T_\phi H_\psi.
\]

For \( f \in L^1 \) and \( z \) in the open unit disk \( D \), we write \( P_z(f) \) for the Poisson integral of \( f \):

\[
P_z(f) = \frac{1}{2\pi} \int_{\partial D} f \, P_z \, d\theta,
\]

where \( P_z \) denotes the Poisson kernel for the point \( z \). We end this introduction with the algebra \( H^\infty + C \), whose properties are important in our proof. Here \( H^\infty \) is the algebra of boundary functions for bounded analytic functions in \( D \), and \( C \) is the algebra of continuous complex valued functions on \( \partial D \).

**Proposition.** Let \( b \) be a Blaschke product with zero set \( Z \). Then \( H_b \) is in the essential commutant of \( S \) if and only if \( Z \) is finite or \( Z \) has its cluster points in \( \{-1, 1\} \).

**Proof.** Let \( A = H_b S - SH_b \). Applying (2) twice we obtain

\[
H_{az} = H_b T_z + T_b H_z = H_b T_z = H_b S \quad \text{and} \quad H_{zb} = H_z T_b + T_z H_b = H_z T_b + SH_b,
\]

therefore \( A = H_{(z-\bar{z})b} + H_z T_b \). Since \( H_z \) is of finite rank, \( A \) is compact if and only if \( (z - \bar{z})b \in H^\infty + C \) [7, p. 101].

If \( Z \) is finite, then \( b \in C \) and therefore \( (\bar{z} - z)b \in C \). If \( Z \) has its cluster points in \( \{-1, 1\} \), then \( b \) is continuous in \( \partial D \setminus \{-1, 1\} \) [5, p. 68], and because \( b \) is bounded it follows that \( (\bar{z} - z)b \in C \). In either case we have \( (z - \bar{z})b \in H^\infty + C \).

For the converse we assume that \( (z - \bar{z})b - f \) for some \( f \in H^\infty + C \). We further assume that \( Z \) is infinite. Let \( \lambda \) be a cluster point of \( Z \). Then \( |\lambda| = 1 \). Let \( a_n \in Z \) such that \( a_n \to \lambda \). Since the Poisson integral is asymptotically multiplicative on \( H^\infty + C \) [3, p. 169], given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |P_z(bf) - P_z(b)P_z(f)| < \varepsilon \) for \( 1 - |z| < \delta \). But \( P_z(bf) = b(z) \) for \( z \in D \), and from \( bf = z - \bar{z} \) on \( \partial D \) we have \( P_z(bf) = 2\mathcal{J}(z) \) for \( z \) in \( D \). Then \( 2|\mathcal{J}(a_n)| < \varepsilon \) for \( 1 - |a_n| < \delta \). This shows that \( \mathcal{J}(a_n) \to 0 \) and therefore \( \Re(a_n) \to \lambda \). Hence \( \lambda \) is real and \( \lambda \in \{-1, 1\} \). \( \square \)

**Lemma.** Let \( g \in H^\infty + C \) such that its conjugate \( \bar{g} \) is also in \( H^\infty + C \). Then \( TT_g - T_{\bar{g}}T \in K \) for all \( T \) in the Toeplitz algebra.

**Proof.** For \( f \in L^\infty \), let \( A_f \) be the set of all operators \( T \) on \( H^2 \) for which \( TT_f - T_f T \in K \). Clearly \( A_f \) is a Banach space, and from \( T_1 T_2 T_f - T_f T_1 T_2 = T_1 (T_2 T_f - T_f T_2) + (T_1 T_f - T_f T_1) T_2 \) it follows that \( A_f \) is a Banach algebra. Now we consider a function \( g \) satisfying the hypothesis of the lemma. Since \( H_g \) is compact [7, p. 101], from (1) we conclude that \( T_{\phi \bar{g}} - T_{\phi \bar{g}} T \in K \) for all \( \phi \in L^\infty \). The same argument applied to \( \phi \) and \( \bar{g} \) gives that \( T_{\bar{g} \phi} - T_{\bar{g} \phi} T \in K \). Therefore \( T_{\phi \bar{g}} T_{\bar{g} \phi} = T_{\phi \bar{g} \phi} T_{\bar{g} \phi} \in K \) for all \( \phi \) in \( L^\infty \). Since we can interchange the roles of \( g \) and \( \bar{g} \), the set \( \{ T_\phi : \phi \in L^\infty \} \) is contained in \( A_g \cap A_{\bar{g}} \). Because \( A_g \cap A_{\bar{g}} \) is closed under the adjoint operation, \( A_g \cap A_{\bar{g}} \) is a C*-algebra, and therefore \( T \in A_g \cap A_{\bar{g}} \). \( \square \)
Now to prove the second part of the theorem it is enough to exhibit a function \( g \) satisfying the hypothesis of the lemma, for which the operator \( H_k T_g - T_g H_k \) is not compact. The construction of \( g \) will be carried out in the Banach space BMO of functions of bounded mean oscillation. For the definition and properties of this space we refer the reader to [7]. A key part in the construction of \( g \) is played by the following result [4, Lemma 1].

**Proposition.** Let \( I \) be a subarc of \( \partial D \) and \( J \) a subarc of \( I \) with the same center. There is a continuous function \( u \) with values in \([0,1]\) such that \( u = 1 \) on \( J \), \( u = 0 \) aff \( I \), and \( \|u\|_{BMO} \leq \text{const} \log(\|I\|/|J|) \).

Here, \( |J| \) denotes the length of \( I \), and \( \| \|_{BMO} \) is the norm in the space BMO.

**Construction of a function.** Now we are ready to construct a particular real-valued function \( g \) in \( H^\infty + C \). By hypothesis we have the sequence \( \{a_n\}_{n=1}^\infty \) in \( Z \) satisfying conditions (i) and (ii), and \( \lambda \in \{-1,1\} \). We write \( a_n = |a_n| e^{i\theta_n} \), with \(-\pi < \theta_n \leq \pi \). By (ii) we may assume that \( \theta_n \neq 0 \) and \( \theta_n \neq \pi \) for all \( n \). For \( a = |a| e^{i\theta} \) in \( D \) we have the identity

\[
\frac{|\lambda - a|^2}{(1 - |a|)^2} = 1 + 2|a| \cdot \frac{1 - \lambda \cos \theta}{(1 - |a|)^2}.
\]

Then from (ii) it follows that

\[
\lim_{n \to \infty} \frac{1 - \lambda \cos \theta_n}{(1 - |a_n|)^2} = +\infty.
\]

For the rest of the construction we assume that the set \( \{n: \theta_n > 0\} \) is infinite. (If \( \theta_n < 0 \) for all \( n \), then the modifications that are necessary are obvious.) Then from (i) it follows that \( \theta_n \to 0 \) (if \( \lambda = 1 \)) or \( \theta_n \to \pi \) (if \( \lambda = -1 \)). First we consider the case \( \lambda = 1 \). Since \( \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta^2} = \frac{1}{2} \), from (iii) it follows that

\[
\lim_{n \to \infty} \frac{\theta_n}{1 - |a_n|} = +\infty.
\]

Then we can choose a subsequence \( \{a_{n_k}\}_{k=1}^\infty \) of \( \{a_n\}_{n=1}^\infty \) such that

\[
(iv) \quad 2\pi \exp(2^k) < \frac{\theta_{n_k}}{1 - |a_{n_k}|} \quad \text{and} \quad 3\theta_{n_{k+1}} < \theta_{n_k} \quad \text{for} \quad k = 1, \ldots, \infty.
\]

Let \( I_k \) be the subarc of \( \partial D \) with center \( e^{i\theta_{n_k}} \) and length \( \theta_{n_k} \). From the second inequality it follows that \( I_k \cap I_l = \emptyset \) for \( k \neq l \). Let \( J_k \) be the subarc of \( I_k \) with the same center as \( I_k \) and whose length is \( 2\pi(1 - |a_{n_k}|) \). Then \( |J_k| \exp(2^k) < |I_k| \), and therefore \( 1/\log(\|I_k\|/|J_k|) < 2^{-k} \) for \( k \geq 1 \). Now we apply the proposition above to \( I_k \) and \( J_k \) to obtain a continuous function \( u_k \) with values in \([0,1]\) such that \( u_k = 1 \) on \( J_k \), \( u_k = 0 \) off \( I_k \), and \( \|u_k\|_{BMO} \leq (\text{const}) 2^{-k} \). Let \( u = \sum_{k=1}^\infty u_k \). The series converges in BMO and the terms are continuous.

Then from [7, p. 49], \( u \) can be written as \( u = f + v \) where \( f \) is in \( C \) and \( v \) is the harmonic conjugate of a function \( h \) in \( C \). Then \( h + iv \) is analytic in \( D \), and since \( u \) is bounded (\( \|u\|_{\infty} = 1 \)), it follows that \( h + iv \in H^\infty \). Since \( f + ih \in C \), the equality \( u = -i(h + iv) + (f + ih) \) shows that \( u \in H^\infty + C \). Now we define \( g = u - \bar{u} \). Then \( g \) is a real-valued function and \( g \in H^\infty + C \). So \( g \) satisfies the hypothesis of the lemma.
Finally, we consider the case \( \lambda = -1 \). Then (iii) becomes

\[
\lim_{n \to \infty} \frac{1 - \cos(\pi - \theta_n)}{(1 - |a_n|)^2} = +\infty,
\]

and from this it follows that

\[
\lim_{n \to \infty} \frac{\pi - \theta_n}{1 - |a_n|} = +\infty.
\]

Now one gets inequalities similar to (iv), where \( \theta_{n_k} \) is replaced by \( \pi - \theta_{n_k} \). Also, in the definition of the subarc \( I_k \) we need to change the length (but not the center) to \( \pi - \theta_{n_k} \). Then the construction of \( g \) proceeds as above.

**Last part of the theorem.** For the function \( g \) constructed above we show that \( H_{\bar{g}}T_g - T_gH_{\bar{g}} \) is not compact. From this and the lemma it will follow that \( H_{\bar{g}} \) is not in the Toeplitz algebra. Let \( A = H_{\bar{g}}T_g - T_gH_{\bar{g}} \). Applying (2) to \( H_{\bar{g}}H_g \) and \( H_{\bar{g}} \) (as we did in the proof of the first proposition) we obtain

\[
A = H_{(g - \tilde{g})\bar{b}} + H_{\bar{g}}T_g - T_gH_{\bar{g}}.
\]

Since \( g \) and \( \tilde{g} \) are in \( H^\infty + C \), \( H_g \) and \( H_{\bar{g}} \) are compact, and therefore \( A \) is not compact if and only if \( (g - \tilde{g})\bar{b} \) does not belong to \( H^\infty + C \). To arrive at a contradiction, let us assume that \( (g - \tilde{g})\bar{b} = f \) for some \( f \in H^\infty + C \). Then \( g - \tilde{g} = bf \). From the definition of \( g \) it follows that \( \tilde{g} = -g \), so \( bf = 2g \). Since the Poisson integral is asymptotically multiplicative on \( H^\infty + C \), there exists \( \delta > 0 \) such that

\[
|P_z(bf) - P_z(b)P_z(f)| < 1 \quad \text{for} \quad 1 - |z| < \delta.
\]

Therefore, \( |P_{a_{n_k}}(g)| < \frac{1}{2} \) for all \( k \)'s in the complement of a finite set. Now a contradiction will be obtained by showing that \( P_{a_{n_k}}(g) \to 1 \) as \( k \to \infty \). For this we use a result in [6] concerning the Poisson integral and certain averaging functionals. For \( z \neq 0 \) in \( D \), we let \( I_z \) denote the closed subarc of \( \partial D \) whose center is \( z/|z| \) and whose length is \( 2\pi(1 - |z|) \). Then from [6, Lemma 5] applied to \( g \), we conclude that given \( \varepsilon > 0 \), there exists \( \delta' > 0 \) such that

\[
\left| P_z(g) - \frac{1}{|I_z|} \int_{I_z} g(e^{i\theta}) \, d\theta \right| < \varepsilon \quad \text{for} \quad 1 - |z| < \delta'.
\]

But from the definition of \( J_k \) and \( I_z \), with \( z = a_{n_k} \), we have \( I_{a_{n_k}} = J_k \). Also, from the definition of \( g \), because the functions \( u_k \) have disjoint supports, we have

\[
\int_{J_k} g(e^{i\theta}) \, d\theta = \int_{J_k} u_k(e^{i\theta}) \, d\theta = \int_{J_k} 1 \, d\theta = |J_k|.
\]

Then, from the above inequality, \( |P_{a_{n_k}}(g) - 1| < \varepsilon \) for \( 1 - |a_{n_k}| < \delta' \). Now we have obtained the desired contradiction, and therefore \( A \) is not compact. The proof of the theorem is complete. \( \square \)

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