A CHARACTERIZATION OF INTEGRAL CURVES
WITH GORENSTEIN HYPERPLANE SECTIONS

KOHJI YANAGAWA

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Abstract. We classify a reduced, irreducible and non-degenerate curve $C \subset \mathbb{P}^r$ such that its general hyperplane section $C \cap H$ is arithmetically Gorenstein, but $C$ itself is not. These curves are contained in surface scrolls and are closely related to Castelnuovo theory on curves in projective space.

Introduction

Let $k$ be an algebraically closed field of characteristic $0$. Recently, Huneke and Ulrich [6] proved the following theorem which generalizes Strano’s result ([8]) on curves in $\mathbb{P}_k^3$.

Theorem A ([6, Theorem 3.20]). Let $C \subset \mathbb{P}_k^r$, $r \geq 3$, be a reduced, connected and non-degenerate curve which is not contained in a quadric hypersurface. If a general hyperplane section of $C$ is arithmetically Gorenstein, then $C$ itself is arithmetically Gorenstein.

In this paper, we will refine the result above under the additional assumption that $C$ is irreducible. The main results of this paper are the following.

Theorem 0.1. Let $C \subset \mathbb{P}^r$ be a reduced, irreducible, non-degenerate curve. Suppose that a general hyperplane section $\Gamma := C \cap H$ is arithmetically Gorenstein, but $C$ itself is not. Then $\Gamma \subset H \simeq \mathbb{P}^r - 1$ is contained in a rational normal curve, and $\deg C \equiv 2 \pmod{r - 1}$.

Theorem 0.2. For a given integer $d \geq r + 1$ such that $d \equiv 2 \pmod{r - 1}$, there is a smooth, irreducible curve $C \subset \mathbb{P}^r$ with $\deg C = d$ which is not arithmetically Gorenstein, but its general hyperplane section is arithmetically Gorenstein.

To prove Theorem 0.1, we use M. Green’s “Strong Castelnuovo Lemma” (cf. [3]). The curves constructed in Theorem 0.2 are closely related to Castelnuovo theory on curves in projective space. For example, let $C \subset \mathbb{P}^r$ be a nearly Castelnuovo curve (see [1] or [4] for the definition) with $\deg C \equiv 2 \pmod{r - 1}$. Then $C$ is smooth, non-degenerate and not arithmetically Cohen–Macaulay, but its general hyperplane section is arithmetically Gorenstein.

After proving these, we prove the hypersurface version of the results above.
Theorem 0.3. Let $C \subset \mathbb{P}^r$, $r \geq 3$, be a reduced, irreducible and non-degenerate curve. If its general degree $d \geq 2$ hypersurface section $Z$ is arithmetically Gorenstein, then $C$ itself is arithmetically Gorenstein.

If $C$ is not integral, there is a counterexample of Theorem 0.3. See Migliore [7].

1. Main results

Throughout this paper, $k$ is an algebraically closed field of characteristic 0.

Let $C \subset \mathbb{P}^r$ be an integral and non-degenerate curve, and $\Gamma := C \cap H$ a general hyperplane section. Denote the homogeneous coordinate ring of $\mathbb{P}^r$ (resp. $C$) by $S := k[x_0, x_1, \ldots, x_r]$ (resp. $A := S/I_C$). Set $m := (x_0, x_1, \ldots, x_r)$. We let $x$ be the linear form which defines $H$. Set $S' := S/xS$, $A' := A/xA$ and $R := A'/H^0_m(A')$. $S'$ (resp. $R$) represents the homogeneous coordinate ring of $H \simeq \mathbb{P}^{r-1}$ (resp. $\Gamma \subset H$).

We denote the Hilbert function of $C$ (resp. $\Gamma$) by $H_C$ (resp. $H_\Gamma$). In other words, $H_C(n) := \dim_k A_n$ and $H_\Gamma(n) := \dim_k R_n$ for all $n \in \mathbb{Z}$.

Note that $R$ is a 1-dimensional Cohen–Macaulay ring, and the Hilbert series of $R$ is given by

$$F(R, \lambda) = \sum_{n \geq 0} H_\Gamma(n) \lambda^n = (h_0 + h_1 \lambda + \cdots + h_s \lambda^s)/(1 - \lambda),$$

where $h_0, h_1, \cdots, h_s$ are certain positive integers. We call the vector $(h_0, h_1, \cdots, h_s)$ the $h$-vector of $R$ (or $\Gamma$). It is well known that $h_0 = 1$ and $\deg C = \deg \Gamma = \sum_{i=0}^s h_i$.

Lemma 1.1. Let the notation be as above. For a general hyperplane $H$, we have that

(a) $h_i \geq h_1 = r - 1$ for all $2 \leq i \leq s - 1$.

(b) If $\Gamma \subset H = \mathbb{P}^{r-1}$ is contained in a rational normal curve, then $h_1 = h_2 = \cdots = h_{s-1}$ and $h_s \leq h_1$.

(c) $\Gamma$ is arithmetically Gorenstein if and only if $h_i = h_{s-i}$ for each $i$.

Proof. The assertion follows from “Uniform Position Lemma” (cf. [1]) and the Cayley-Bacharach characterization of arithmetically Gorenstein zero-dimensional schemes (cf. [2]).

Suppose that $R$ is Gorenstein. In this case, $A$ is Gorenstein if and only if it is Cohen–Macaulay. Consider the minimal free resolution of $R$ over $S'$:

$$0 \to S'(-n_{r-1}) \oplus \bigoplus_{i=1}^{b_{r-2}} S'(-n_{r-2,i}) \to \cdots \oplus \bigoplus_{i=1}^{b_1} S'(-n_{1,i}) \to S' \to R \to 0.$$

We need the following result due to Huneke and Ulrich.

Lemma 1.2 ([6, Corollary 3.24]). If $C$ is not arithmetically Cohen–Macaulay, we have

$$n_{r-1} = \min\{i | H^0_m(A')_i \neq 0\} + r - 1 = \max\{n_{1,i}\} + r - 1.$$

Now we can prove the following theorem.
Theorem 1.3. Let \( C \subset \mathbb{P}^r, r \geq 3 \), be a reduced, irreducible and non-degenerate curve. Suppose that \( \Gamma = C \cap H \) is arithmetically Gorenstein for a generic hyperplane \( H \), but \( C \) itself is not arithmetically Gorenstein (equivalently, Cohen–Macaulay). Then,

(a) \( \Gamma \subset H \cong \mathbb{P}^{r-1} \) is contained in a rational normal curve.
(b) \( \deg C \equiv 2 \pmod{r-1} \).
(c) If \( \deg C \geq 2r \), the intersection of the quadrics containing \( C \) is a surface scroll.

Proof. (a) From Lemma 1.2 and the “duality” of the free resolution of \( R \), it is easy to see that \( \min\{n_{r-3, i}\} = r - 1 \). Since we may assume that \( \Gamma \subset H \) is in linearity general position, \( \Gamma \) is contained in a rational normal curve by M. Green’s “Strong Castelnuovo’s Lemma” (Corollary 3.c.6. of [3]).

(b) Since \( \Gamma \subset \mathbb{P}^{r-1} \) is arithmetically Gorenstein and contained in a rational normal curve, the \( h \)-vector of \( \Gamma \) is given by \( (1, r - 1, r - 1, \ldots, r - 1, 1) \). Hence, we have \( \deg C = \deg \Gamma = \sum_{i=0}^{r-1} h_i \equiv 2 \pmod{r-1} \).

(c) By (a), \( \Gamma \) is contained in a rational normal curve \( X_\Gamma \). Since \( n_{r-3, i} \geq 2r + 2 \) (note that \( \deg \Gamma \geq 2r \) and \( r \geq 3 \)), we have \( \min\{i|H^0_m(A')_i \neq 0\} \geq 3 \) and \( A'_2 = R_2 \) by Lemma 1.2. Hence the intersection of the quadrics containing \( C \) meets \( H \) exactly in \( X_\Gamma \) (note that the defining ideal of a rational normal curve is generated by quadrics). Thus the intersection of the quadrics containing \( C \) is a surface \( X \) whose general hyperplane section is a rational normal curve, in particular \( \deg X = r - 1 \). So \( X \) is a Veronese surface of \( \mathbb{P}^5 \) or a surface scroll (cf. [1]). But easy calculation shows that a non-degenerate curve contained in a Veronese surface of \( \mathbb{P}^5 \) is always projectively normal. So \( X \) is a surface scroll. \( \square \)

Corollary 1.4. Let \( C \subset \mathbb{P}^r \) be a reduced, irreducible and non-degenerate curve with \( \dim_k(I_C)_2 < \binom{r-1}{2} \). If a general hyperplane section of \( C \) is arithmetically Gorenstein, then \( C \) itself is arithmetically Gorenstein.

Proof. If \( \deg C \geq 2r \) (equivalently \( \deg C \neq r + 1 \)), the assertion follows from Theorem 1.3.(c), since a surface scroll in \( \mathbb{P}^r \) is defined by \( \binom{r-1}{2} \) quadrics. So we may suppose that \( \deg C = r + 1 \). Then we have \( \dim_k(I_\Gamma)_2 = \binom{r+1}{2} - H_\Gamma(2) = \binom{r+1}{2} - (r + 1) \geq \binom{r-1}{2} + 1 \). By the same argument as in the proof of [6, Theorem 3.20], we see that \( \dim_k H^0_m(A')_2 \leq 1 \) in this case. Hence we have \( \dim_k(I_C)_2 \geq \dim_k(I_\Gamma)_2 - 1 \). It contradicts the assumption \( \dim_k(I_C)_2 < \binom{r-1}{2} \).

Next, we describe a curve with Gorenstein hyperplane sections as a divisor of a smooth scroll. For each integer \( e \geq 0 \), we denote the rational ruled surface \( \mathbb{P}(O_{\mathbb{P}^3} \oplus O_{\mathbb{P}^1}(-e)) \) by \( X_e \). There is an embedding of \( X_e \subset \mathbb{P}^r \) as a rational normal scroll if and only if there exists an integer \( n > e \) such that \( r = 2n - e + 1 \).

Let \( X_e \subset \mathbb{P}^r \) be a smooth surface scroll. It is well known that the divisor class group of \( X_e \) is free of rank 2, having as generators the classes \( H \) and \( L \) of a hyperplane section and a line of the ruling, respectively. The intersection pairing is given by

\[ H \cdot H = r - 1, \quad H \cdot L = 1, \quad L \cdot L = 0, \]

and the canonical class is

\[ K_{X_e} \sim -2H + (r - 3)L. \]
Let $\alpha, \beta$ be two integers. The existence of a reduced and irreducible curve $C \sim \alpha H + \beta L$ on a scroll $X_e \subset P^r$ depends on $e$. But, when $e$ is smallest possible (i.e., $e = 0$ if $r$ is odd, and $e = 1$ if $r$ is even), the condition for these curves to exist is mildest. With this hypothesis, we have the following fact.

**Lemma 1.5.** Let $\alpha, \beta$ be two integers with $\alpha > 0$. Then the following are equivalent.

(a) There exists a smooth, irreducible curve $C \sim \alpha H + \beta L$ on $X_e$.

(b) There exists a reduced, irreducible curve $C \sim \alpha H + \beta L$ on $X_e$.

(c) $\alpha \left[ \frac{r-1}{2} \right] + \beta \geq 0$.

**Proof.** Follows from [5, V, Corollary 2.18 and 19]. See also [4].

Let $C \subset P^r$ be a reduced and irreducible curve contained in a smooth scroll $X$ such that $C \sim \alpha H + \beta L$. Then we have that

$$\deg C = \alpha (r-1) + \beta$$

and

$$g = \frac{\alpha(\alpha-1)}{2} (r-1) + (\alpha-1)(\beta-1),$$

where $g$ is the (arithmetic) genus of $C$. It is easy to see that $C$ is non-degenerate if and only if $\deg C = \alpha (r-1) + \beta \geq r$.

**Lemma 1.6.** Let $C \subset P^r$ be as above. Then $C$ is arithmetically Cohen–Macaulay if and only if $-(r-2) \leq \beta \leq 1$.

**Proof.** $C$ is arithmetically Cohen–Macaulay if and only if the genus of $C$ is equal to $\pi(d,r)$ (see [1] or [4] for the definition). Hence the assertion follows from easy calculation. See also [1, III Exercise I -4].

If $C$ satisfies the conditions of the previous lemma, $C$ is called a Castelnuovo curve. A Castelnuovo curve $C \subset P^r$ with $\deg C = d$ has the largest genus among integral and non-degenerate curves in $P^r$ with degree $d$.

Let $d \geq r+1$ be an integer such that $d \equiv 2 \pmod{r-1}$. From Lemma 1.5, we can take a smooth, irreducible and non-degenerate curve

$$C \sim nH + 2L, \quad \text{where} \quad n = \frac{d-2}{r-1}$$

on some surface scroll. Easy calculation shows that the arithmetic genus of $C$ is equal to $\pi(d,r) - 1$, and hence $C$ is a typical example of a “nearly Castelnuovo curve” (see [1] or [4] for detail). From Lemma 1.6, $C$ is not arithmetically Cohen–Macaulay. On the other hand, since $C$ is contained in a surface scroll, a general hyperplane section $\Gamma = C \cap H$ is contained in a rational normal curve. The $h$-vector of $\Gamma$ is given by $(1, r-1, r-1, \cdots, r-1, 1)$ by Lemma 1.1.(b). So $\Gamma$ is arithmetically Gorenstein by Lemma 1.1.(c).

Now, we obtain the following.

**Theorem 1.7.** For a given integer $d \geq r+1$ such that $d \equiv 2 \pmod{r-1}$, there is a smooth, irreducible curve $C \subset P^r$ with $\deg C = d$ which is not arithmetically Gorenstein, but its general hyperplane section is arithmetically Gorenstein.

By Lemmas 1.5 and 1.6, we can determine the set of pairs $(\alpha, \beta)$ such that the curve $C \sim \alpha H + \beta L$ satisfies the assumption of Theorem 1.7.
Example 1.8 ([6, Example 3.26]). Let \( B = k[x^6, x^7y, x^4y^4, x^3y^5, y^8] \) and \( C := \text{Proj} B \subset \mathbb{P}^4 \). The generic hyperplane section \( C \cap H \) is arithmetically Gorenstein, but \( C \) is not arithmetically Cohen–Macaulay. Note that \( \deg C = 8 \equiv 2 \pmod{3} \) and \( g = 3 \). There exists a smooth surface scroll \( X \subset \mathbb{P}^4 \) which contains \( C \) as \( C \sim 4H - 4L \).

2. Hypersurface cases

In this section, we study a hypersurface version of the results in the previous section. As in Section 1, let \( C \subset \mathbb{P}^r \) be a reduced, irreducible, non-degenerate curve, and \( A \) the projective coordinate ring of \( C \).

Let \( f \in S_d, d \geq 2, \) be a general degree \( d \) element which defines a hypersurface \( F \). Set \( A' := A/fA \) and \( B := A'/H_m^0(A') \). Then \( B \) is the projective coordinate ring of hypersurface section \( Z := C \cap F \) in \( \mathbb{P}^r \).

Remark 2.1. The general degree \( d \) \( (\geq 2) \) hypersurface section of \( C \) is in linearly general position in \( \mathbb{P}^r \). The proof of the “Uniform Position Lemma” given in [1] is also applicable in the hypersurface case.

We need a result of J. C. Migliore which is a hypersurface version of Lemma 1.2. Suppose that \( Z \) is arithmetically Gorenstein, and consider the minimal free resolution of \( B \) over \( S \):

\[
0 \to S(-n_r) \to \bigoplus_{i=1}^{b_{r-1}} S(-n_{r-1,i}) \to \cdots \to \bigoplus_{i=1}^{b_1} S(-n_{1,i}) \to S \to B \to 0.
\]

Lemma 2.2. Let the notation be as above. Suppose that \( C \) is not arithmetically Cohen–Macaulay and \( Z \) is arithmetically Gorenstein. Set

\[
b := \min\{n|H_m^0(A')_n \neq 0\} = \min\{n|A'_n \neq B_n\}.
\]

Then we have

\[
n_r = b + r = \max\{n_{1,i}\} + r.
\]

Proof. The assertion follows from [7, Proposition 2.2]. See also the argument following (3.2) of [7].

Theorem 2.3. Let \( C \subset \mathbb{P}^r, r \geq 3, \) be a reduced, irreducible and non-degenerate curve. If its general degree \( d \) \( (\geq 2) \) hypersurface section \( Z \) is arithmetically Gorenstein, then \( C \) itself is arithmetically Gorenstein.

Proof. Assume the contrary (i.e., \( C \) is not arithmetically Cohen–Macaulay). Note that \( \deg C \geq r + 1 \) (if \( \deg C \leq r \), then \( C \) is a rational normal curve and projectively normal).

From Lemma 2.2 and an argument similar to our proof of Theorem 1.3 (a), we can see that \( Z \) is contained in a rational normal curve of \( \mathbb{P}^r \). Hence we have that \( H_Z(n) = \min\{\deg Z, nr + 1\} \) for all \( n \in \mathbb{Z} \).

Case 1. \( b = \min\{n|A'_n \neq B_n\} > 2 \): Since the defining ideal of a rational normal curve is generated by quadrics, we have \( (I_Z)_2 \neq (I_C)_2 \) (i.e., \( A_2 \neq B_2 \)). By the assumption that \( b \geq 3 \), we have \( d = 2 \). Easy calculation shows that \( H_C(2) = H_Z(2) + 1 = 2r + 2 \).
Let \( x \in S_1 \) be a general linear form which defines a hyperplane \( H \), and set \( \Gamma := C \cap H \). Then,

\[
H_\Gamma(2) = H_C(2) - H_C(1) - \dim \langle H^0_m(A/xA) \rangle_2 \\
= 2r + 2 - (r + 1) - \dim \langle H^0_m(A/xA) \rangle_2 \\
= r + 1 - \dim \langle H^0_m(A/xA) \rangle_2.
\]

On the other hand, \( \Gamma \subset H \) is in linearly general position, and hence \( H_\Gamma(2) \geq \min \{ \deg \Gamma, 2r - 1 \} \). So we have \( \deg C = \deg \Gamma = H_\Gamma(2) = r + 1 \) and \( \langle H^0_m(A/xA) \rangle_2 = 0 \). Since \( \Gamma \) is non-degenerate in \( H \), we have \( \langle H^0_m(A/xA) \rangle_i = 0 \) for all \( i \leq 1 \). Furthermore, since the defining ideal of \( \Gamma \subset H \) is generated by quadrics (note that \( \Gamma \) is arithmetically Gorenstein with \( h \)-vector \( (1, r - 1, 1) \)), we have \( \langle H^0_m(A/xA) \rangle_i = 0 \) for all \( i \geq 3 \). So we have \( H^0_m(A/xA) = 0 \), and \( A \) is Cohen–Macaulay. This is a contradiction.

**Case 2.** \( b = 2 \): Since \( \max \{ n_1, i \} = b = 2 \) and \( Z \) is contained in a rational normal curve, the \( h \)-vector of \( Z \) is \( (1, r, 1) \) and \( \deg Z = d \deg C = r + 2 \). It contradicts the facts that \( d \geq 2 \) and \( \deg C \geq r + 1 \).

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**References**

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**Department of Mathematics, School of Science, Nagoya University Chikusa-ku, Nagoya 464 Japan**

E-mail address: yanagawa@math.nagoya-u.ac.jp