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A CHARACTERIZATION OF INTEGRAL CURVES WITH GORENSTEIN HYPERPLANE SECTIONS

KOHJI YANAGAWA

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ABSTRACT. We classify a reduced, irreducible and non-degenerate curve $C \subset \mathbb{P}^r$ such that its general hyperplane section $C \cap H$ is arithmetically Gorenstein, but C itself is not. These curves are contained in surface scrolls and are closely related to Castelnuovo theory on curves in projective space.

INTRODUCTION

Let k be an algebraically closed field of characteristic 0. Recently, Huneke and Ulrich [6] proved the following theorem which generalizes Strano's result ([8]) on curves in $\mathbb{P}^3_k$.

Theorem A ([6, Theorem 3.20]). *Let $C \subset \mathbb{P}^r_k$, $r \geq 3$, be a reduced, connected and non-degenerate curve which is not contained in a quadric hypersurface. If a general hyperplane section of C is arithmetically Gorenstein, then C itself is arithmetically Gorenstein.*

In this paper, we will refine the result above under the additional assumption that C is irreducible. The main results of this paper are the following.

Theorem 0.1. *Let $C \subset \mathbb{P}^r$ be a reduced, irreducible, non-degenerate curve. Suppose that a general hyperplane section $\Gamma := C \cap H$ is arithmetically Gorenstein, but C itself is not. Then $\Gamma \subset H \simeq \mathbb{P}^{r-1}$ is contained in a rational normal curve, and $\deg C \equiv 2 \pmod{r-1}$.*

Theorem 0.2. *For a given integer $d \geq r+1$ such that $d \equiv 2 \pmod{r-1}$, there is a smooth, irreducible curve $C \subset \mathbb{P}^r$ with $\deg C = d$ which is not arithmetically Gorenstein, but its general hyperplane section is arithmetically Gorenstein.*

To prove Theorem 0.1, we use M. Green's "Strong Castelnuovo Lemma"(cf. [3]).

The curves constructed in Theorem 0.2 are closely related to Castelnuovo theory on curves in projective space. For example, let $C \subset \mathbb{P}^r$ be a *nearly Castelnuovo curve* (see [1] or [4] for the definition) with $\deg C \equiv 2 \pmod{r-1}$. Then C is smooth, non-degenerate and *not* arithmetically Cohen–Macaulay, but its general hyperplane section is arithmetically Gorenstein.

After proving these, we prove the hypersurface version of the results above.

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Theorem 0.3. *Let $C \subset \mathbb{P}^r$, $r \geq 3$, be a reduced, irreducible and non-degenerate curve. If its general degree d (≥ 2) hypersurface section Z is arithmetically Gorenstein, then C itself is arithmetically Gorenstein.*

If C is not integral, there is a counterexample of Theorem 0.3. See Migliore [7].

1. Main results

Throughout this paper, k is an algebraically closed field of characteristic 0.

Let $C \subset \mathbb{P}^r_k$ be an integral and non-degenerate curve, and $\Gamma := C \cap H$ a general hyperplane section. Denote the homogeneous coordinate ring of $\mathbb{P}^r$ (resp. C) by $S := k[x_0, x_1, \dots, x_r]$ (resp. $A := S/I_C$). Set $m := (x_0, x_1, \dots, x_r)$. We let x be the linear form which defines H. Set $S' := S/xS$, $A' := A/xA$ and $R := A'/H^0_m(A')$. S' (resp. R) represents the homogeneous coordinate ring of $H \simeq \mathbb{P}^{r-1}$ (resp. $\Gamma \subset H$).

We denote the Hilbert function of C (resp. Γ) by H_C (resp. H_Γ). In other words, $H_C(n) := \dim_k A_n$ and $H_\Gamma(n) := \dim_k R_n$ for all $n \in \mathbb{Z}$.

Note that R is a 1-dimensional Cohen–Macaulay ring, and the Hilbert series of R is given by

$$F(R, \lambda) = \sum_{n \geq 0} H_\Gamma(n)\lambda^n = (h_0 + h_1\lambda + \cdots + h_s\lambda^s)/(1 - \lambda),$$

where $h_0, h_1, \cdots, h_s$ are certain *positive* integers. We call the vector $(h_0, h_1, \cdots, h_s)$ the h-vector of R (or Γ). It is well known that $h_0 = 1$ and $\deg C = \deg \Gamma = \sum_{i=0}^{s} h_i$.

Lemma 1.1. *Let the notation be as above. For a general hyperplane H, we have that*

(a) $h_i \geq h_1 = r - 1$ *for all* $2 \leq i \leq s - 1$.

(b) *If* $\Gamma \subset H = \mathbb{P}^{r-1}$ *is contained in a rational normal curve, then* $h_1 = h_2 = \cdots = h_{s-1}$ *and* $h_s \leq h_1$.

(c) Γ *is arithmetically Gorenstein if and only if* $h_i = h_{s-i}$ *for each* i.

Proof. The assertion follows from "Uniform Position Lemma" (cf. [1]) and the Caylay-Bacharach characterization of arithmetically Gorenstein zero-dimensional schemes (cf. [2]). □

Suppose that R is Gorenstein. In this case, A is Gorenstein if and only if it is Cohen–Macaulay. Consider the minimal free resolution of R over S' :

$$0 \to S'(-n_{r-1}) \to \bigoplus_{i=1}^{b_{r-2}} S'(-n_{r-2,i}) \to \cdots \to \bigoplus_{i=1}^{b_1} S'(-n_{1,i}) \to S' \to R \to 0.$$

We need the following result due to Huneke and Ulrich.

Lemma 1.2 ([6, Corollary 3.24.]). *If C is not arithmetically Cohen–Macaulay, we have*

$$n_{r-1} = \min\{i | [H^0_m(A')]_i \neq 0\} + r - 1 = \max\{n_{1,i}\} + r - 1.$$

Now we can prove the following theorem.

Theorem 1.3. *Let $C \subset \mathbb{P}^r$, $r \geq 3$, be a reduced, irreducible and non-degenerate curve. Suppose that $\Gamma = C \cap H$ is arithmetically Gorenstein for a generic hyperplane H, but C itself is not arithmetically Gorenstein (equivalently, Cohen–Macaulay). Then,*

(a) $\Gamma \subset H \simeq \mathbb{P}^{r-1}$ *is contained in a rational normal curve.*

(b) $\deg C \equiv 2 \pmod{r-1}$.

(c) *If* $\deg C \geq 2r$, *the intersection of the quadrics containing C is a surface scroll.*

Proof. (a) From Lemma 1.2 and the "duality" of the free resolution of R, it is easy to see that $\min\{n_{r-2,i}\} = r-1$. Since we may assume that $\Gamma \subset H$ is in linearly general position, Γ is contained in a rational normal curve by M. Green's "Strong Castelnuovo's Lemma" (Corollary 3.c.6. of [3]).

(b) Since $\Gamma \subset \mathbb{P}^{r-1}$ is arithmetically Gorenstein and contained in a rational normal curve, the h-vector of Γ is given by $(1, r-1, r-1, \ldots, r-1, 1)$. Hence, we have $\deg C = \deg \Gamma = \sum_{i=0}^{s} h_i \equiv 2 \pmod{r-1}$.

(c) By (a), Γ is contained in a rational normal curve X_Γ. Since $n_{r-1} \geq r+2$ (note that $\deg \Gamma \geq 2r$ and $r \geq 3$), we have $\min\{i | [H^0_m(A')]_i \neq 0\} \geq 3$ and $A'_2 = R_2$ by Lemma 1.2. Hence the intersection of the quadrics containing C meets H exactly in X_Γ (note that the defining ideal of a rational normal curve is generated by quadrics). Thus the intersection of the quadrics containing C is a surface X whose general hyperplane section is a rational normal curve, in particular $\deg X = r-1$. So X is a Veronese surface of $\mathbb{P}^5$ or a surface scroll (cf. [1]). But easy calculation shows that a non-degenerate curve contained in a Veronese surface of $\mathbb{P}^5$ is always projectively normal. So X is a surface scroll. □

Corollary 1.4. *Let $C \subset \mathbb{P}^r$ be a reduced, irreducible and non-degenerate curve with $\dim_k(I_C)_2 < \binom{r-1}{2}$. If a general hyperplane section of C is arithmetically Gorenstein, then C itself is arithmetically Gorenstein.*

Proof. If $\deg C \geq 2r$ (equivalently $\deg C \neq r+1$), the assertion follows from Theorem 1.3.(c), since a surface scroll in $\mathbb{P}^r$ is defined by $\binom{r-1}{2}$ quadrics. So we may suppose that $\deg C = r+1$. Then we have $\dim_k(I_\Gamma)_2 = \binom{r+1}{2} - H_\Gamma(2) = \binom{r+1}{2} - (r+1) \geq \binom{r-1}{2} + 1$. By the same argument as in the proof of [6, Theorem 3.20], we see that $\dim_k H^0_m(A')_2 \leq 1$ in this case. Hence we have $\dim_k(I_C)_2 \geq \dim_k(I_\Gamma)_2 - 1$. It contradicts the assumption $\dim_k(I_C)_2 < \binom{r-1}{2}$. □

Next, we describe a curve with Gorenstein hyperplane sections as a divisor of a smooth scroll. For each integer $e \geq 0$, we denote the rational ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \bigoplus \mathcal{O}_{\mathbb{P}^1}(-e))$ by X_e. There is an embedding of $X_e \hookrightarrow \mathbb{P}^r$ as a rational normal scroll if and only if there exists an integer $n > e$ such that $r = 2n - e + 1$.

Let $X_e \subset \mathbb{P}^r$ be a smooth surface scroll. It is well known that the divisor class group of X_e is free of rank 2, having as generators the classes H and L of a hyperplane section and a line of the ruling, respectively. The intersection pairing is given by

$$H \cdot H = r-1, \qquad H \cdot L = 1, \qquad L \cdot L = 0,$$

and the canonical class is

$$K_{X_e} \sim -2H + (r-3)L.$$

Let α, β be two integers. The existence of a reduced and irreducible curve $C \sim \alpha H + \beta L$ on a scroll $X_e \subset \mathbb{P}^r$ depends on e. But, when e is smallest possible (i.e., $e = 0$ if r is odd, and $e = 1$ if r is even), the condition for these curves to exist is mildest. With this hypothesis, we have the following fact.

Lemma 1.5. *Let α, β be two integers with $\alpha > 0$. Then the following are equivalent.*

(a) *There exists a smooth, irreducible curve $C \sim \alpha H + \beta L$ on X_e.*
(b) *There exists a reduced, irreducible curve $C \sim \alpha H + \beta L$ on X_e.*
(c) $\alpha[\frac{r-1}{2}] + \beta \geq 0$.

Proof. Follows from [5, V, Corollary 2.18 and 19]. See also [4]. □

Let $C \subset \mathbb{P}^r$ be a reduced and irreducible curve contained in a smooth scroll X such that $C \sim \alpha H + \beta L$. Then we have that

$$\deg C = \alpha(r-1) + \beta$$

and

$$g = \frac{\alpha(\alpha-1)}{2}(r-1) + (\alpha-1)(\beta-1),$$

where g is the (arithmetic) genus of C. It is easy to see that C is non-degenerate if and only if $\deg C = \alpha(r-1) + \beta \geq r$.

Lemma 1.6. *Let $C \subset \mathbb{P}^r$ be as above. Then C is arithmetically Cohen–Macaulay if and only if $-(r-2) \leq \beta \leq 1$.*

Proof. C is arithmetically Cohen–Macaulay if and only if the genus of C is equal to $\pi(d,r)$ (see [1] or [4] for the definition). Hence the assertion follows from easy calculation. See also [1, III Exercise I -4]. □

If C satisfies the conditions of the previous lemma, C is called a *Castelnuovo curve*. A Castelnuovo curve $C \subset \mathbb{P}^r$ with $\deg C = d$ has the largest genus among integral and non-degenerate curves in $\mathbb{P}^r$ with degree d.

Let $d \geq r+1$ be an integer such that $d \equiv 2 \pmod{r-1}$. From Lemma 1.5, we can take a smooth, irreducible and non-degenerate curve

$$C \sim nH + 2L, \qquad \text{where} \quad n = \frac{d-2}{r-1}$$

on some surface scroll. Easy calculation shows that the arithmetic genus of C is equal to $\pi(d,r) - 1$, and hence C is a typical example of a "nearly Castelnuovo curve" (see [1] or [4] for detail). From Lemma 1.6, C is not arithmetically Cohen–Macaulay. On the other hand, since C is contained in a surface scroll, a general hyperplane section $\Gamma = C \cap H$ is contained in a rational normal curve. The h-vector of Γ is given by $(1, r-1, r-1, \cdots, r-1, 1)$ by Lemma 1.1.(b). So Γ is arithmetically Gorenstein by Lemma 1.1.(c).

Now, we obtain the following.

Theorem 1.7. *For a given integer $d \geq r+1$ such that $d \equiv 2 \pmod{r-1}$, there is a smooth, irreducible curve $C \subset \mathbb{P}^r$ with $\deg C = d$ which is not arithmetically Gorenstein, but its general hyperplane section is arithmetically Gorenstein.*

By Lemmas 1.5 and 1.6, we can determine the set of pairs (α, β) such that the curve $C \sim \alpha H + \beta L$ satisfies the assumption of Theorem 1.7.

Example 1.8 ([6, Example 3.26]). Let $B = k[x^8, x^7y, x^4y^4, x^3y^5, y^8]$ and $C := \operatorname{Proj} B \subset \mathbb{P}^4$. The generic hyperplane section $C \cap H$ is arithmetically Gorenstein, but C is not arithmetically Cohen–Macaulay. Note that $\deg C = 8 \equiv 2 \pmod 3$ and $g = 3$. There exists a smooth surface scroll $X \subset \mathbb{P}^4$ which contains C as $C \sim 4H - 4L$.

2. Hypersurface cases

In this section, we study a hypersurface version of the results in the previous section. As in Section 1, let $C \subset \mathbb{P}^r$ be a reduced, irreducible, non-degenerate curve, and A the projective coordinate ring of C.

Let $f \in S_d$, $d \geq 2$, be a general degree d element which defines a hypersurface F. Set $A' := A/fA$ and $B := A'/H^0_m(A')$. Then B is the projective coordinate ring of hypersurface section $Z := C \cap F$ in $\mathbb{P}^r$.

Remark 2.1. The general degree d (≥ 2) hypersurface section of C is in linearly general position in $\mathbb{P}^r$. The proof of the "Uniform Position Lemma" given in [1] is also applicable in the hypersurface case.

We need a result of J. C. Migliore which is a hypersurface version of Lemma 1.2.

Suppose that Z is arithmetically Gorenstein, and consider the minimal free resolution of B over S :

$$0 \to S(-n_r) \to \bigoplus_{i=1}^{b_{r-1}} S(-n_{r-1,i}) \to \cdots \to \bigoplus_{i=1}^{b_1} S(-n_{1,i}) \to S \to B \to 0.$$

Lemma 2.2. *Let the notation be as above. Suppose that C is not arithmetically Cohen–Macaulay and Z is arithmetically Gorenstein. Set*

$$b := \min\{n | H^0_m(A')_n \neq 0\} = \min\{n | A'_n \neq B_n\}.$$

Then we have

$$n_r = b + r = \max\{n_{1,i}\} + r.$$

Proof. The assertion follows from [7, Proposition 2.2]. See also the argument following (3.2) of [7].

Theorem 2.3. *Let $C \subset \mathbb{P}^r$, $r \geq 3$, be a reduced, irreducible and non-degenerate curve. If its general degree d (≥ 2) hypersurface section Z is arithmetically Gorenstein, then C itself is arithmetically Gorenstein.*

Proof. Assume the contrary (i.e., C is not arithmetically Cohen–Macaulay). Note that $\deg C \geq r+1$ (if $\deg C \leq r$, then C is a rational normal curve and projectively normal).

From Lemma 2.2 and an argument similar to our proof of Theorem 1.3 (a), we can see that Z is contained in a rational normal curve of $\mathbb{P}^r$. Hence we have that $H_Z(n) = \min\{\deg Z, nr + 1\}$ for all $n \in \mathbb{Z}$.

Case 1. $b = \min\{n | A'_n \neq B_n\} > 2$: Since the defining ideal of a rational normal curve is generated by quadrics, we have $(I_Z)_2 \neq (I_C)_2$ (i.e., $A_2 \neq B_2$). By the assumption that $b \geq 3$, we have $d = 2$. Easy calculation shows that $H_C(2) = H_Z(2) + 1 = 2r + 2$.

Let $x \in S_1$ be a general linear form which defines a hyperplane H, and set $\Gamma := C \cap H$. Then,

$$\begin{aligned} H_\Gamma(2) &= H_C(2) - H_C(1) - \dim_k[H^0_m(A/xA)]_2 \\ &= 2r + 2 - (r+1) - \dim_k[H^0_m(A/xA)]_2 \\ &= r + 1 - \dim_k[H^0_m(A/xA)]_2. \end{aligned}$$

On the other hand, $\Gamma \subset H$ is in linearly general position, and hence $H_\Gamma(2) \geq \min\{\deg\Gamma, 2r-1\}$. So we have $\deg C = \deg\Gamma = H_\Gamma(2) = r+1$ and $[H^0_m(A/xA)]_2 = 0$. Since Γ is non-degenerate in H, we have $[H^0_m(A/xA)]_i = 0$ for all $i \leq 1$. Furthermore, since the defining ideal of $\Gamma \subset H$ is generated by quadrics (note that Γ is arithmetically Gorenstein with h-vector $(1, r-1, 1)$), we have $[H^0_m(A/xA)]_i = 0$ for all $i \geq 3$. So we have $H^0_m(A/xA) = 0$, and A is Cohen–Macaulay. This is a contradiction.

Case 2. $b = 2$: Since $\max\{n_{1,i}\} = b = 2$ and Z is contained in a rational normal curve, the h-vector of Z is (1, r, 1) and $\deg Z = d \deg C = r + 2$. It contradicts the facts that $d \geq 2$ and $\deg C \geq r + 1$. □

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Department of Mathematics, School of Science, Nagoya University Chikusa-ku, Nagoya 464 Japan

E-mail address: `yanagawa@math.nagoya-u.ac.jp`