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CLOSED GEODESICS AND NON-DIFFERENTIABILITY OF THE METRIC IN INFINITE-DIMENSIONAL TEICHMÜLLER SPACES

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Abstract. In this paper we construct a closed geodesic in any infinite-dimensional Teichmüller space. The construction also leads to a proof of non-differentiability of the metric in infinite-dimensional Teichmüller spaces, which provides a negative answer to a problem of Goldberg.

§1. Introduction

We begin with some basic definitions and notations.

Let S be a Riemann surface which has a universal covering $\mathbb{H}$, where $\mathbb{H}$ denotes the upper half plane. Then the Riemann surface S can be expressed as $\mathbb{H}/\Gamma$, where Γ is a torsion free Fuchsian group acting on $\mathbb{H}$. We denote by $M(\Gamma)$ the set of the Beltrami differentials of Γ with L^∞-norms less than one, that is,

$$M(\Gamma) = \{\mu(z) : \mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z), \forall\gamma \in \Gamma,\ a.e.\ z \in \mathbb{H}; \|\mu\|_\infty < 1\}.$$

Denote by $f_\mu : \mathbb{H} \to \mathbb{H}$ the quasiconformal mapping with the complex dilatation μ keeping 0, 1 and ∞ fixed. We say that μ_1 is equivalent to μ_2 iff

$$f_{\mu_1}|\mathbb{R} = f_{\mu_2}|\mathbb{R}.$$

Then the Teichmüller space of S (or Γ), denoted by $T(S)$ (or $T(\Gamma)$), is defined as the set of the equivalence classes of the elements of $M(\Gamma)$. When Γ is finitely generated and of the first kind, the Riemann surface S is conformally finite and the Teichmüller space $T(\Gamma)$ is finite-dimensional. When Γ is of the second kind or infinitely generated, $T(\Gamma)$ is infinite-dimensional.

A Beltrami differential $\mu \in M(\Gamma)$ is said to be extremal if

$$\|\mu\|_\infty \le \|\mu'\|_\infty, \quad \forall\mu' \in [\mu],$$

where $[\mu]$ is the equivalence class of μ. The Teichmüller metric for $T(\Gamma)$ is defined in terms of the extremal Beltrami differentials. For two given points $[\mu_1]$ and $[\mu_2]$ in $T(\Gamma)$, the Teichmüller distance between them is

$$d([\mu_1],\ [\mu_2]) = \frac{1}{2}\log\frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty},$$

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where μ is an extremal Beltrami differential in the equivalence class of the complex coefficient of $f_{\mu_1} \circ {f_{\mu_2}}^{-1}$. It is well known that the Teichmüller metric is complete and coincides with the Kobayashi metric (see [R] or [Ga]).

Throughout this paper, by "geodesic" we mean a curve which is locally shortest with respect to the Teichmüller metric. In the other words, we say an arc α in $T(\Gamma)$ is a geodesic if for any point $p \in \alpha$ there is a neighborhood U of p such that any subarc τ of α contained in U is the shortest among the curves in U which join its endpoints.

In a finite-dimensional Teichmüller space, a geodesic is always shortest in the large. However, we will see in this paper that this is not true any more for the infinite-dimensional case.

There are some essential differences in the geometry of the Teichmüller metric between the finite-dimensional case and the infinite-dimensional case. In the finite-dimensional case, the Teichmüller space is a straight geodesic space in the sense of Buseman, namely, for any given two points there is a unique geodesic joining them and this geodesic can be extended to a straight line (an isometric embedding from $\mathbb{R}$ into the Teichmüller space with respect to the Euclidean metric and the Teichmüller metric respectively)(see [K]). However, in the infinite-dimensional case, a geodesic joining two given points may not be unique and there may exist infinitely many geodesics joining the two points. The first example for the universal Teichmüller space was given by the author (see [L1]). For the general infinite-dimensional case, it was shown by H. Tanigawa ([T]) and Li Zhong ([L2]). This problem was further investigated in a recent paper by C. J. Earle, I. Kra and S.L.Krushkal' ([EKK]).

The purpose of this paper is to show further differences in the geometry between the infinite-dimensional Teichmüller space and the finite-dimensional Teichmüller space. We will prove the following theorem:

Theorem 1. *In any infinite-dimensional Teichmüller space, there are closed geodesics with respect to the Teichmüller metric.*

This theorem implies the non-uniqueness of geodesics and the non-convexity of spheres in an infinite-dimensional Teichmüller space (see [L3]).

It is well known that the Teichmüller metric in a finite-dimensional Teichmüller space is differentiable. This was proved by C. J. Earle ([E]). It is of interest to know whether or not the metric in an infinite-dimensional Teichmüller space is differentiable. This is an open problem posed by L. R. Goldberg (Problem 5, p.1184, [Go]). The following theorem will provide a negative answer to this problem:

Theorem 2. *In any infinite-dimensional Teichmüller space there are points* $[\mu_1] \neq [\mu_2]$ *and tangent vectors* ϕ *at* $[\mu_1]$ *such that the directional derivative of* $\rho([\mu]) = d([\mu], [\mu_2])$ *at* $[\mu_1]$ *in the direction* ϕ *does not exist.*

Theorem 2 and its proof were suggested by C. J. Earle, after he read the manuscript of this paper.

We will prove Theorem 1 by constructing a closed geodesic in given Teichmüller space (§3). Theorem 2 is an application of such a construction (§4). A special holomorphic quadratic differential constructed in [L3] will play an important role. For completeness we repeat the construction of the holomorphic quadratic differential here (§2).

§2. Construction of a special quatratic differential

We denote by $Q(\Gamma)$ the set of holomorphic quadratic differentials of Γ with finite L^1-norm, that is,

$$Q(\Gamma) = \{\phi : \phi(z) = \phi(\gamma(z))\gamma'(z)^2, \forall \gamma \in \Gamma \text{ and } z \in \mathbb{H}; \\ \phi \text{ is holomorphic in } \mathbb{H}; \text{ and } \|\phi\| < \infty\},$$

where

$$\|\phi\| = \int_{\mathbb{H}/\Gamma} |\phi|.$$

Now we suppose that the surface $S = \mathbb{H}/\Gamma$ is not of conformally finite type, namely the Fuchsian group Γ is of the second kind or infinitely generated. In this case, the Banach space $Q(\Gamma)$, as well as the Teichmüller space $T(\Gamma)$, are infinite-dimensional.

Since the unit sphere of an infinite-dimensional Banach space is non-compact, there is a sequence $\{\phi_n\}$ in the unit sphere of $Q(\Gamma)$ that has no norm-convergent subsequence. We may assume (by passing to a subsequence) that $\phi_n \to \phi \in Q(\Gamma)$ on compact subsets of $\mathbb{H}$. Replacing ϕ_n by $\psi_n = (\phi_n - \phi)/\|\phi_n - \phi\|$, we obtain a sequence $\{\psi_n\}$ of unit vectors in $Q(\Gamma)$ that converges to zero uniformly on compact sets in $\mathbb{H}$.

Set $D_k = \{z \in \mathbb{H} : \rho(z, i) \leq k\}$ for each $k \geq 1$, and set $E_k = D_k \cap \omega$, where ω is a fundamental polygon and ρ is the Poincaré distance in $\mathbb{H}$. Now put $D_{k_0} = \emptyset$ and choose strictly increasing sequences of positive integers n_l and k_l, $l \geq 1$, so that $n_1 = 1$ and for all $l \geq 1$ we have

$$E_{k_{l-1}} \subset E_{k_l},$$

$$|\psi_{n_l}(z)| < 2^{-l}, \text{for all } z \in D_{k_{l-1}}, \tag{1}$$

$$\int_{E_{k_l} \setminus E_{k_{l-1}}} |\psi_{n_l}| > 1 - 2^{-l}, \tag{2}$$

and

$$\int_{\omega \setminus E_{k_l}} |\psi_{n_j}| < 2^{-l}, \quad \text{if } 1 \leq j \leq l. \tag{3}$$

For simplicity, from now on, we write ψ_l, D_l, and E_l instead of ψ_{n_l}, D_{k_l}, and E_{k_l}.

Let

$$\psi = \sum_{l=1}^{\infty} \psi_l.$$

It is easy to see from (1) that the series is uniformly convergent in any compact subset of $\mathbb{H}$ and hence defines a holomorphic quadratic differential of Γ.

Lemma 1. *The holomorphic quadratic differentials ψ_l and $\psi = \sum_{l=1}^{\infty} \psi_l$ have the following properties:*

$$\int_{E_{l-1}} |\psi_l| = O(2^{-l}), \quad as \ l \to \infty, \tag{4}$$

$$\int_{E_l \setminus E_{l-1}} |\psi - \psi_l| = O(l2^{-l}), \quad as \ l \to \infty, \tag{5}$$

$$\int_{E_l \setminus E_{l-1}} |\psi| = 1 + O(l2^{-l}), \quad as \ l \to \infty. \tag{6}$$

Proof. Since $\|\psi_l\| = 1$, it is obvious that (2) implies (4) and (5) implies (6). So it is sufficient to prove (5). Noting again the fact that $\|\psi_l\| = 1$, from (2), (3), and (4) we have

$$\begin{aligned}
\int_{E_l \setminus E_{l-1}} |\psi - \psi_l| &= O\left(\sum_{j=1}^{l-1} \int_{E_l \setminus E_{l-1}} |\psi_j|\right) + O\left(\sum_{j=l+1}^{\infty} \int_{E_l \setminus E_{l-1}} |\psi_j|\right) \\
&= O\left(\sum_{j=1}^{l-1} \int_{\omega \setminus E_{l-1}} |\psi_j|\right) + O\left(\sum_{j=l+1}^{\infty} \int_{E_l} |\psi_j|\right) \\
&= O\left(\sum_{j=1}^{l-1} 2^{-l}\right) + O\left(\sum_{j=l+2}^{\infty} 2^{-j}\right) = O(l2^{-l}).
\end{aligned}$$

Then we get (5). The Lemma is proved.

§3. The proof of Theorem 1

Now we are going to construct a closed geodesic in $T(\Gamma)$ where Γ is a torsion free Fuchsian group of the second kind or infinitely generated so that $T(\Gamma)$ is infinite-dimensional. We will use the same notation as before.

First of all, we define four functions:

$$\begin{aligned}
\kappa_1(z) &= \begin{cases} k, & \text{for } z \in E_{2n+1} \setminus E_{2n}, n = 0, 1, ...; \\ 0, & \text{for } z \in E_{2n} \setminus E_{2n-1}, n = 1, 2, ..., \end{cases} \\
\kappa_2 &= -\kappa_1, \\
\kappa_3(z) &= \begin{cases} 0, & \text{for } z \in E_{2n+1} \setminus E_{2n}, n = 0, 1, ...; \\ k, & \text{for } z \in E_{2n} \setminus E_{2n-1}, n = 1, 2, ..., \end{cases} \\
\kappa_4 &= -\kappa_3,
\end{aligned}$$

where $k \in (0,\ 1)$ is a constant. They can be extended to the upper half plane $\mathbb{H}$ by defining

$$\kappa_j(\gamma(z)) = \kappa_j(z), \ \text{ for } j = 1, 2, 3, 4; \ z \in \omega.$$

Lemma 2. *Let ψ be the quadratic differential for Γ constructed in Section 2. The four Beltrami differentials $\mu_j = \kappa_j \overline{\psi}/|\psi|, j = 1, 2, 3, 4,$ are extremal.*

Proof. It is sufficient to give the proof for $j = 1$. It follows from (4) (where l is replaced by $2n + 1$) that

$$\begin{aligned}\int_\omega \mu_1 \psi_{2n+1} &= k \int_{E_{2n+1}\setminus E_{2n}} \frac{\overline{\psi}}{|\psi|}\psi_{2n+1} + O(2^{-n-1}) \\ &= k \int_{E_{2n+1}\setminus E_{2n}} |\psi| + O\left(\int_{E_{2n+1}\setminus E_{2n}} |\psi - \psi_{2n+1}|\right) + O(2^{-2n-1}).\end{aligned}$$

From (5) and (6) we get

$$\lim_{n\to\infty} \left|\int_\omega \mu_1 \psi_{2n+1}\right| = k,$$

which implies that μ_1 is extremal (see [RK], or [K]). Similarly, the other μ_j is extremal. The Lemma is proved.

By Lemma 2 the Teichmüller distance between [0] and $[\mu_j]$ is

$$d([0],\ [\mu_j]) = R, \quad \text{for } j = 1, 2, 3, 4, \tag{7}$$

where

$$R = \frac{1}{2}\log\frac{1+k}{1-k}.$$

Noting the facts that $\mu_1 = -\mu_2$ and $\mu_3 = -\mu_4$, we see

$$d([\mu_1],\ [\mu_2]) = 2R, \quad d([\mu_3],\ [\mu_4]) = 2R. \tag{8}$$

The distance formula

$$d([\mu],\, [\nu]) = \inf_{\mu'\in[\mu],\nu'\in[\nu]} \tanh^{-1}\left\|\frac{\mu' - \nu'}{1 - \overline{\nu'}\mu'}\right\|_\infty \tag{9}$$

gives immediately that

$$d([\mu_i],\, [\mu_j]) \le \tanh^{-1}\left\|\frac{\mu_i - \mu_j}{1 - \overline{\mu_i}\mu_j}\right\|_\infty = R \quad \text{if } (i,j) = (1,3), (3,2), (2,4), \text{ or } (4,1). \tag{10}$$

On the other hand, from (8) we see

$$d([\mu_1],\, [\mu_3]) + d([\mu_3],\, [\mu_2]) \ge d([\mu_1],\, [\mu_2]) = 2R.$$

It follows from (10) that

$$d([\mu_1],\, [\mu_3]) = d([\mu_3],\, [\mu_2]) = R.$$

Similarly we get

$$d([\mu_1],\ [\mu_4]) = d(\mu_4],\, [\mu_2]) = R.$$

Hence the equalities in (10) hold. By a paper ([EE]) of C.J. Earle and J. Eells, an explicit formula for the geodesic $\alpha_{i,j}$ from $[\mu_i]$ to $[\mu_j]$ (again $(i,j) = (1,3), (3,2), (2,4)$, or $(4,1)$) is

$$t \mapsto \left[\frac{t(\mu_j - \mu_i) + \mu_i(1 - \overline{\mu_i}\mu_j)}{t\overline{\mu_i}(\mu_j - \mu_i) + (1 - \overline{\mu_i}\mu_j)}\right] = \left[t\mu_j + \left(\frac{1-t}{1-k^2t}\right)\mu_i\right],\ 0 \le t \le 1.$$

Let τ_1, τ_2, τ_3 and τ_4 be the images of $\alpha_{1,3}, \alpha_{3,2}, \alpha_{2,4}$ and $\alpha_{4,1}$ respectively. It is easy to see that any pair of them can not intersect each other at an interior point (otherwise it would be in contradiction to (8)). Therefore,

$$\beta = \tau_1 \cup \tau_2 \cup \tau_2 \cup \tau_3 \cup \tau_4$$

is a closed Jordan curve.

To show that β is a geodesic, it is sufficient to show that β is locally shortest at each point μ_j $(j = 1,2,3,4)$. This is easy to see. In fact, noting (8) and the fact that the length of each τ_j is R, the arcs $\tau_1 \cup \tau_2, \tau_2 \cup \tau_3, \tau_3 \cup \tau_4$ and $\tau_4 \cup \tau_1$ are geodesics. Then we can conclude that β is a closed geodesic in $T(\Gamma)$.

The proof of Theorem 1 is completed.

§4. The proof of Theorem 2

Let μ_1, μ_2, μ_3 and μ_4 be the four Beltrami differentials of Γ constructed in Section 3. We look at the function $\rho([\mu]) = d([\mu], [\mu_3])$ on $T(\Gamma)$. (Here the definition of ρ has a formal difference with that in the statement of Theorem 2. But this is not essential.) We want to show that the function ρ does not have directional derivatives in all directions at $[\mu] = [\mu_1]$. If it did, then the function

$$f(t) = d([t\mu_1/k], [\mu_3]), \quad -1 < t < 1,$$

would be differentiable at $t = k$. But for $-k \le t \le k$ we have

$$2R = d([\mu_4], [\mu_3]) \le d([\mu_4], [t\mu_1/k]) + d([t\mu_1/k], [\mu_3]). \tag{11}$$

Using the distance formula (9) again and noting the fact that the intersection of the supports of μ_1 and μ_4 is empty, we see

$$d([\mu_4], [t\mu_1/k]) \le \tanh^{-1}\left\|\frac{\mu_4 - t\mu_1/k}{1 - (t\overline{\mu_1}/k)\mu_4}\right\|_\infty = R, \quad \text{for } -k \le t \le k. \tag{12}$$

Similarly, we have

$$f(t) = d([t\mu_1/k], [\mu_3]) \le R, \quad \text{for } -k \le t \le k. \tag{13}$$

Then it follows at once from (11-13) that

$$2R \le R + f(t) \le 2R, \quad \text{for } -k \le t \le k$$

so $f(t) = R$ for $-k \le t \le k$. For $k < t < 1$ we have

$$\begin{aligned} &\tanh^{-1} t + R = d([t\mu_1/k], [0]) + d([0], [\mu_2]) = d([t\mu_1/k], [\mu_2]) \\ &\le f(t) + d([\mu_3], [\mu_2]) = f(t) + R. \end{aligned} \tag{14}$$

Here we have used the fact that $t \mapsto [t\mu_1/k]$, $-1 < t < 1$, is a geodesic in $T(\Gamma)$. On the other hand, we have

$$\begin{aligned} f(t) &= d([t\mu_1/k], [\mu_3]) \le d([t\mu_1/k], [\mu_1]) + d([\mu_1], [\mu_3]) \\ &= d([t\mu_1/k], [\mu_1]) + R = d([t\mu_1/k], [\mu_1]) + d([\mu_1], [0]) \\ &= d([t\mu_1/k], [0]) = \tanh^{-1} t, \quad \text{for } k < t < 1. \end{aligned} \tag{15}$$

Here we have again used the fact that $t \mapsto [t\mu_1/k]$, $-1 < t < 1$, is a geodesic in $T(\Gamma)$. From (14) and (15) we see that $f(t) = \tanh^{-1} t$ for $k < t < 1$.

So the function $f(t)$ is not differentible at $t = k$. We get a contradiction which proves Theorem 2.

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