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SINGULAR SOLUTIONS FOR A CLASS OF GRUSIN TYPE OPERATORS

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Abstract. We construct singular solutions for a one-parameter family of partial differential equations with double characteristics and with complex lower order terms. The parameter belongs to a discrete set which is described in terms of the spectrum of a related differential operator.

1. Introduction

Let k be an odd positive integer, and for each $\lambda \in \mathbb{C}$ let P_λ be the operator in $\mathbb{R}^2$ defined by

$$P_\lambda = \partial_x^2 + x^{2k}\partial_y^2 + i(\lambda + k)x^{k-1}\partial_y. \tag{1.1}$$

Also let Λ_k be the set

$$\Lambda_k = \{\lambda : \lambda = 2j(k+1) \text{ or } \lambda = 2j(k+1) + 2,\ j \in \mathbb{Z}\}. \tag{1.2}$$

The local solvability of the operator P_λ was studied by Gilioli and Treves [3] by using the method of concatenations. They proved that P_λ is solvable if and only if λ does not belong to the set Λ_k. The local hypoellipticity of a similar class of operators in an abstract setting was studied by Gilioli [2] by using concatenations different from the ones used in [3]. Earlier, and in the case $k = 1$, Grusin [4] studied the hypoellipticity of P_λ by using the spectral theory of the harmonic oscillator. Using its eigenvalues he described the discrete set Λ_1. The purpose of this work is to give an explicit computation of the set Λ_k in terms of the eigenvalues of a generalized harmonic oscillator, and to present two methods of construction of singular solutions. One is by certain Fourier integrals with symbols expressed in terms of the eigenfunctions (see Theorem 1.1 and formula (2.1)). The other is by the method of concatenations of Gilioli and Treves [3] (see Theorem 3.1 and formula (3.14)). In particular we construct solutions with prescribed C^∞ and analytic wave front sets. Next we state our first result:

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Theorem 1.1. *Let k be an odd positive integer. If $\lambda \in \Lambda_k$, then the following hold*:

1. *There exists a C^∞ solution to the equation $P_\lambda u = 0$ which is not analytic.*
2. *For each $m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ there exists a C^m solution to the equation $P_\lambda u = 0$ which is not C^{m+1}.*

Remark. If $\lambda \in \Lambda_k$, then by using the same methods we can construct solutions u to $P_\lambda u = 0$ which are distributions of any finite order.

2. Proof of Theorem 1.1

Let $u(x, y)$ be defined by

$$u(x,y) = \int_0^\infty e^{i\rho^{k+1}y} A(\rho x) w(\rho) d\rho, \tag{2.1}$$

where the functions $A(x)$ and $w(\rho)$ are to be determined. We have

$$\bar{P}_\lambda u(x,y) = \int_0^\infty e^{i\rho^{k+1}y}\rho^2 \left[A''(\rho x) - x^{2k}\rho^{2k}A(\rho x) + (\lambda + k)x^{k-1}\rho^{k-1}A(\rho x)\right] w(\rho) d\rho.$$

For u to be a solution to $\bar{P}_\lambda u = 0$, it suffices that A satisfies the equation

$$\left(-\frac{d^2}{dx^2} + x^{2k} - (\lambda + k)x^{k-1}\right) A(x) = 0. \tag{2.2}$$

In the next proposition we will state the spectral theory of the last equation on $\mathbb{R}$ with boundary conditions

$$\lim_{|x|\to\infty} A(x) = 0. \tag{2.3}$$

Let $L^2_{w_k}(\mathbb{R})$ be the Hilbert space of all functions $f : \mathbb{R} \to \mathbb{C}$ with the following inner product

$$(f, g)_{w_k} = \int_{\mathbb{R}} f(x)\overline{g}(x) e^{-\frac{2}{k+1}x^{k+1}} x^{k-1} dx.$$

Proposition 2.1. *Let k be an odd positive integer. Then for the boundary value problem (2.2) - (2.3) the following hold:*

1. *The set of eigenvalues is equal to*

$$k + \Lambda_k^+, \text{ where } \Lambda_k^+ = \{\lambda : \lambda = 2J(k+1), \text{ or } \lambda = 2[J(k+1)+1], \ J \in \mathbb{N}_0\}. \tag{2.4}$$

2. *If an eigenvalue is of the form $2J(k+1) + k$ for some $J \in \mathbb{N}_0$, then the corresponding eigenspace is generated by the even function*

$$B^0_{k,J}(x) e^{-\frac{1}{k+1}x^{k+1}}, \text{ with } B^0_{k,J}(x) = \sum_{j=0}^{J} b_{j(k+1)} x^{j(k+1)}, \tag{2.5}$$

where for $0 \le j \le J$

$$b_{j(k+1)} = (-1)^{J+j}(k+1)^J 2^j \binom{J}{j} [J(k+1)-1][(J-1)(k+1)-1] \dots [(j+1)(k+1)-1].$$

If an eigenvalue is of the form $2[J(k+1)+1]+k$ *for some* $J \in \mathbb{N}_0$*, then the corresponding eigenspace is generated by the odd function*

$$B^1_{k,J}(x)e^{-\frac{1}{k+1}x^{k+1}}, \text{ with } B^1_{k,J}(x) = \sum_{j=0}^{J} b_{j(k+1)+1}x^{j(k+1)+1}, \tag{2.6}$$

where for $0 \le j \le J$

$$b_{j(k+1)+1} = (-1)^{J+j}(k+1)^J 2^j \binom{J}{j} [J(k+1)+1][(J-1)(k+1)+1] \dots [(j+1)(k+1)+1].$$

3. *The set of generalized Hermite polynomials* $\{B^\ell_{k,J}\}^{\ell=0,1}_{J\in\mathbb{N}_0}$ *is an orthogonal basis for the Hilbert space* $L^2_{w_k}(\mathbb{R})$.

Remark. If $k=1$, then $1+\Lambda_1 = \{2J+1, \ J \in \mathbb{N}_0\}$ is the set of eigenvalues for the classical harmonic oscillator. Also, $B^0_{1,J}$ are the even Hermite polynomials, and $B^1_{1,J}$ are the odd Hermite polynomials.

The set Λ_k appears naturally as an expression of the eigenvalues of this problem. The singular solutions are integral expressions of the corresponding eigenfunctions.

The proof of Proposition 2.1 can be obtained by transforming equation (2.2) to an equation satisfied by the orthogonal polynomials corresponding to the weight function $e^{-x^2}|x|^{2\kappa}$, for $\kappa = \frac{1}{2}\frac{k-1}{k+1}$. Then one is reduced to Problem 25 in Szegö's book [11]. Here we shall give a direct and self-contained proof of Proposition 2.1. The eigenfunctions are computed explicitly and are shown to be an orthogonal basis. These facts, although not needed here, are of independent interest.

Proof of 1. In this case we choose $w(\rho) = e^{-\rho}$. Then for $\lambda \in \Lambda^+_k$ by Proposition 2.1 there exists a function $A \in \mathcal{S}(\mathbb{R})$, with $A(0) \neq 0$ or $A'(0) \neq 0$, satisfying equation (2.2). Since $A \in \mathcal{S}(\mathbb{R}^n)$, (2.1) defines a C^∞ solution to the equation $P_\lambda \bar{u} = 0$. Without loss of generality we assume that $A(0) \neq 0$. Since

$$\begin{aligned} \partial^j_y u(0,0) &= i^j A(0) \int_0^\infty \rho^{j(k+1)} e^{-\rho} d\rho \\ &= i^j A(0)[j(k+1)]!, \end{aligned}$$

we conclude that $\bar{u}$ is not real analytic in $\mathbb{R}$.

Next we apply P_λ to $u(x,y)$ in (2.1) and we see that $P_\lambda u = 0$ if A satisfies the equation

$$\left(-\frac{d^2}{dx^2} + x^{2k} + (\lambda+k)x^{k-1}\right)A(x) = 0.$$

Since $\lambda + k = -[(-\lambda - 2k) + k]$, by Proposition 2.1 the last equation has a solution $A \in \mathcal{S}(\mathbb{R}) - 0$ if $-\lambda - 2k \in \Lambda^+_k$, which is equivalent to

$$\lambda \in \{\lambda : \lambda = 2j(k+1) \text{ or } \lambda = 2j(k+1)+2, \quad j = -1, -2, \dots\}.$$

Then, as before, u is a C^∞ but not an analytic solution to equation $P_\lambda u = 0$.

Proof of 2. It follows by choosing $w(\rho) = (1+\rho^{k+1})^N$, for an appropriate $N = N(m) < 0$. Also for any $m \in \mathbb{N}_0$, by choosing an appropriate $N = N(m) > 0$, we can construct a solution to $P_\lambda u = 0$ which is a distribution of order m.

The proof of Theorem 1.1 will be complete if we prove Proposition 2.1.

Proof of Proposition 2.1. First we make the following substitution:

$$A(x) = e^{-\frac{1}{k+1}x^{k+1}} B(x). \tag{2.7}$$

Then equation (2.2) takes the form

$$-B'' + 2x^k B' - \lambda x^{k-1} B = 0. \tag{2.8}$$

If $k = 1$, then (2.8) becomes the Hermite equation.

If in equation (2.8) we let

$$B(x) = \sum_{j=0}^{\infty} b_j x^j, \tag{2.9}$$

then the coefficients b_j satisfy the relations

(2.10)

$$b_{j+2} = 0, \text{ for } 0 \le j \le k-2, \text{ and } b_{j+k+1} = \frac{2j - \lambda}{(j+k)(j+k+1)} b_j, \ j = 0, 1, 2, \ldots.$$

We shall need the following lemma.

Lemma 2.1. *For a fixed $\lambda \in \mathbb{R}$ let $B^0_{k,\lambda}$ be the solution to equation (2.8) with initial conditions*

$$B^0_{k,\lambda}(0) = 1 \text{ and } \frac{dB^0_{k,\lambda}}{dx}(0) = 0, \tag{2.11}$$

and let $B^1_{k,\lambda}$ be the solution of equation (2.8) with initial conditions

$$B^1_{k,\lambda}(0) = 0 \text{ and } \frac{dB^1_{k,\lambda}}{dx}(0) = 1. \tag{2.12}$$

Then $B^0_{k,\lambda}$ is a polynomial if and only if

$$\lambda = 2J(k+1), \text{ for some } J \in \mathbb{N}_0, \tag{2.13}$$

and $B^1_{k,\lambda}$ is a polynomial if and only if

$$\lambda = 2J(k+1) + 2, \ J \in \mathbb{N}_0. \tag{2.14}$$

Proof. By (2.10) and (2.11) we obtain $b_j = 0$, if $j \ne J(k+1)$ for some $J \in \mathbb{N}_0$, and

$$b_0 = 1 \text{ and } b_{(j+1)(k+1)} = \frac{2j(k+1) - \lambda}{[j(k+1)+k](j+1)(k+1)} b_{j(k+1)}, \ j \in \mathbb{N}_0. \tag{2.15}$$

Also by (2.10) and (2.12) we obtain $b_j = 0$, if $j \ne J(k+1) + 1$ for some $J \in \mathbb{N}_0$, and

(2.16)

$$b_1 = 1, \ b_{(j+1)(k+1)+1} = \frac{2[j(k+1)+1] - \lambda}{(j+1)(k+1)[(j+1)(k+1)+1]} b_{j(k+1)+1}, \ j \in \mathbb{N}_0.$$

Then Lemma 2.1 follows from (2.15) and (2.16).

For λ as in equalities (2.13) or (2.14) the corresponding $B^0_{k,\lambda} e^{-x^{k+1}/(k+1)}$ and $B^1_{k,\lambda} e^{-x^{k+1}/(k+1)}$ decay exponentially fast near $\pm\infty$. In the next lemma we will show that this is not the case for all the other λ.

Lemma 2.2. *Let α be a real number such that $\frac{1}{k+1} < \alpha < \frac{2}{k+1}$. We have the following:*

If $\lambda \neq 2J(k+1),\ J \in \mathbb{N}_0$, then there exist $C_\lambda > 0$ and $R_\lambda > 0$ such that

$$|B^0_{k,\lambda}(x)e^{-\frac{1}{k+1}x^{k+1}}| \geq C_\lambda e^{(\alpha-\frac{1}{k+1})x^{k+1}},\ |x| \geq R_\lambda.$$

If $\lambda \neq 2J(k+1)+2,\ J \in \mathbb{N}_0$, then there exist $C_\lambda > 0$ and $R_\lambda > 0$ such that

$$|B^1_{k,\lambda}(x)e^{-\frac{1}{k+1}x^{k+1}}| \geq C_\lambda e^{(\alpha-\frac{1}{k+1})x^{k+1}}, |x| \geq R_\lambda.$$

If $\lambda \notin \Lambda_k^+$, then every non-zero solution A to equation (1.1) satisfies the estimate

$$|A(x)| \geq Ce^{(\alpha-\frac{1}{k+1})x^{k+1}}$$

near at least one of $+\infty$ or $-\infty$, where C depends on A.

Proof. It follows by comparing the coefficients of $B^l_{k,\lambda}$ given by (2.15) and (2.16) with the coefficients of the exponential $e^{\alpha x^{k+1}}$.

Lemmas 2.1 and 2.2 give the first two parts of Proposition 2.1. The even polynomials are computed by letting in (2.15) $\lambda = 2J(k+1)$ for some $J \in \mathbb{N}_0$. The odd polynomials are computed by letting in (2.16) $\lambda = 2J(k+1)+2$ for $J \in \mathbb{N}_0$.

Now we prove the last part of Proposition 2.1.

Proof of part 3 in Proposition 2.1. A short computation yields the orthogonality. Next we shall show the completeness. Let f be a function in $L^2_{w_k}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} f(x)B^\ell_{k,J}(x)e^{-\frac{2}{k+1}x^{k+1}}x^{k-1}dx = 0, \tag{2.17}$$

for all $\ell \in \{0,1\}$ and $J \in \mathbb{N}_0$. We will show that $f = 0$ a.e. in $\mathbb{R}$. By (2.17) and the form of the $B^\ell_{k,J}$ we obtain the relations:

$$\int_{\mathbb{R}} f(x)p\left(x^{k+1}\right)e^{-\frac{2}{k+1}x^{k+1}}x^{k-1}dx = 0,\ p \in \mathcal{P}, \tag{2.18}$$

$$\int_{\mathbb{R}} f(x)xp\left(x^{k+1}\right)e^{-\frac{2}{k+1}x^{k+1}}x^{k-1}dx = 0,\ p \in \mathcal{P}, \tag{2.19}$$

where $\mathcal{P}$ is the set of all polynomials in one variable.

By the Cauchy-Schwarz inequality we obtain that if $f \in L^2_{w_k}(\mathbb{R})$, then

$$f(x)e^{|\xi||x|^{(k+1)/2}}e^{-\frac{2}{k+1}x^{k+1}}x^{k-1} \in L^1(\mathbb{R}) \tag{2.20}$$

for every $\xi \in \mathbb{R}$. If we write $\cos(\xi x^{(k+1)/2})$ as a series and use (2.20), then by the dominated convergence theorem we can pass the integral inside the sum and by also using (2.18) we obtain

$$\int_{\mathbb{R}} \cos\left(\xi x^{(k+1)/2}\right)f(x)e^{-\frac{2}{k+1}x^{k+1}}x^{k-1}dx = 0,\ \xi \in \mathbb{R}.$$

By the last relation and the change of variables $t = x^{\frac{k+1}{2}}$ we obtain that

$$\int_0^\infty \cos(\xi t) f_E\left(t^{\frac{2}{k+1}}\right) e^{-\frac{2}{k+1}t^2} t^{\frac{k-1}{k+1}} dt = 0, \ \xi \in \mathbb{R}, \tag{2.21}$$

where $f_E(x) = \frac{1}{2}[f(x) + f(-x)]$. By (2.21) and the Fourier inversion theorem we obtain

$$f_E\left(t^{\frac{2}{k+1}}\right) e^{-\frac{2}{k+1}t^2} t^{\frac{k-1}{k+1}} = 0 \quad \text{a.e. } t \in (0, \infty),$$

which implies that

$$f(x) + f(-x) = 0 \text{ a.e. for } x \in \mathbb{R}. \tag{2.22}$$

Similarly by using (2.19) we show that

$$xf(x) - xf(-x) = 0 \text{ a.e. for } x \in \mathbb{R}. \tag{2.23}$$

By using (2.22) and (2.23) we obtain that $f = 0$ a.e. in $\mathbb{R}$. This completes the proof of Proposition 2.1.

3. The method of concatenations

Let the vector field L in $\mathbb{R}^2$ be defined by

$$L = \partial_x + ix^k \partial_y. \tag{3.1}$$

Then the operator P_λ in (1.1) can be written as

$$P_\lambda = \bar{L}L + i\lambda x^{k-1}\partial_y. \tag{3.2}$$

Also for $\mu \in \mathbb{R}$ let

$$V_\mu = xL + \mu. \tag{3.3.}$$

Following Gilioli and Treves [3] we have the following concatenation formulas.

Lemma 3.1. *For any λ in $\mathbb{R}$ we have:*

$$\bar{V}_{\frac{\lambda}{2}+k+2}\ P_\lambda = P_{\lambda+2(k+1)}\ \bar{V}_{\frac{\lambda}{2}+k} \tag{3.4}$$

and

$$V_{-\frac{\lambda}{2}+2}\ P_\lambda = P_{\lambda-2(k+1)}\ V_{-\frac{\lambda}{2}}. \tag{3.5}$$

Also simple computations yield the additional formulas

$$xP_2 = \bar{L}(xL - 1) = \bar{L}V_{-1} \tag{3.6}$$

and

$$xP_{-2k-2} = L(x\bar{L} - 1) = L\bar{V}_{-1}. \tag{3.7}$$

Next we shall use the method of concatenations to construct singular solutions to the operator P_λ with prescribed microlocal singularities.

Theorem 3.1. *Let P_λ be the operator (1.1) and $\gamma_0 = (0, y_0; 0, \eta_0)$, $\eta_0 \neq 0$, a characteristic point of P_λ. Then the following hold:*

(1) *P_λ is not hypoelliptic, or analytic hypoelliptic, at γ_0, with $\eta_0 < 0$, if $\lambda \in \Lambda_k^+$.*

(2) *P_λ is not hypoelliptic, or analytic hypoelliptic, at γ_0, with $\eta_0 > 0$, if $\lambda \in \Lambda_k^-$.*

Here Λ_k^+ is as in (2.4) and $\Lambda_k^- = \Lambda_k - \Lambda_k^+$, where Λ_k is defined in (1.2).

Proof of (1). Let γ_0 be a characteristic point with $\eta_0 < 0$. First we observe that the function

$$Z(x, y) = y - \frac{i}{k+1}\, x^{k+1} \tag{3.8}$$

is a solution to $LZ = 0$. In addition if we define

$$u_0(x, y) = (Z(x, y))^{m+\frac{1}{2}},\ m \in \mathbb{N}, \tag{3.9}$$

then u is a C^m solution to equation $Lu = 0$ which does not belong to C^{m+1}. In particular u is singular at γ_0. Since

$$P_0 u_0 = \bar{L} L u_0 = 0, \tag{3.10}$$

u is a solution to $P_0 u = 0$ which is singular at γ_0. Next we will show how starting from u_0 we can construct a singular solution at γ_0 for P_λ, $\lambda = 2j(k + 1)$, $j \in \mathbb{N}_0$. Suppose that a distribution u, which is singular at γ_0, is a solution to the equation

$$P_{2j(k+1)}\ u = 0. \tag{3.11}$$

Then by (3.11) and concatenation formula (3.4) applied for $\lambda = 2j(k+1)$ we obtain

$$P_{2(j+1)(k+1)}\ w = 0,\ \text{where } w = \bar{V}_{j(k+1)+k}\ u. \tag{3.12}$$

Then by the following lemma we obtain that w is singular at γ_0.

Lemma 3.2. *Let $\mu \in \mathbb{N}$ and $u \in \mathcal{D}'(\mathbb{R}^2)$. If $P_\lambda u$ is regular at γ_0, with $\eta_0 < 0$, for some $\lambda \in \mathbb{R}$, and $\bar{V}_\mu u$ is regular at γ_0, then u is regular at γ_0.*

Proof of Lemma 3.2. If $\bar{V}_\mu u$ is regular at γ_0, then $x^{\mu-1}\bar{V}_\mu u$ is also regular at γ_0. By applying the formula

$$\bar{L}(x^\mu u) = x^{\mu-1}\bar{V}_\mu u, \tag{3.13}$$

we obtain that $\bar{L}(x^\mu u)$ is regular at γ_0. Since $\bar{L}$ is hypoelliptic for $\eta_0 < 0$ (see Hörmander [7], vol. IV, p. 84), we obtain that $x^\mu u$ is regular at γ_0. Now by using the assumption that $P_\lambda u$ is regular at γ_0, which implies additional regularity on u, we obtain that u is regular at γ_0 (see Hörmander [7], vol. I, p. 267). This completes the proof of Lemma 3.2.

To summarize, so far we have shown that if u_0 is a singular solution of P_0 at γ_0, say u_0 is given by (3.9), then

$$u_{2(j+1)(k+1)} = \bar{V}_{j(k+1)+k}\ \bar{V}_{(j-1)(k+1)+k} \cdots \bar{V}_k\ u_0 \tag{3.14}$$

is a singular solution to $P_{2(j+1)(k+1)}u = 0$. What remains now is the construction of singular solutions at γ_0 for $\lambda = 2j(k+1)+2$. For this we first show that P_2 is not hypoelliptic at γ_0. Let

$$w_0 = xu_0. \tag{3.15}$$

Then we obtain that

$$P_2 w_0 = 0 \text{ and } w_0 \text{ is singular at } \gamma_0. \tag{3.16}$$

Now by starting with (3.16) and by applying concatenation formula (3.4) we construct singular solutions at γ_0 to $P_\lambda u = 0$ for all $\lambda = 2j(k+1)+2,\ j \in \mathbb{N}_0$. The formula of the singular solutions is very similar to (3.14) with u_0 replaced by w_0 given by (3.15).

Proof of (2). It is analogous to the proof of (1). We start with a singular solution to $\bar{L}u = 0$ and then by using concatenation formula (3.5) we construct singular solutions at γ_0 with $\eta_0 > 0$, to $P_\lambda u = 0,\ \lambda \in \Lambda_k^-$. This completes the proof of Theorem 3.1.

Remark. By using the method of concatenations we can also construct a C^∞ solution to $P_\lambda u = 0$ with a prescribed analytic wave front set. All we need to do is to start with a function h which is holomorphic for $\mathrm{Im}z < 0$ or $\mathrm{Im}z > 0$, and C^∞ up to $\mathrm{Im}z = 0$, and then replace u_0 in (3.9) by $u_0 = h(Z)$ or $u_0 = h(\bar{Z})$.

In a forthcoming article we will show that the necessary conditions of Theorem 3.1 for hypoellipticity and analytic hypoellipticity of P_λ are also sufficient. The P_λ will be models for a wide class of non-transversally elliptic operators with double characteristics. Such operators exhibit discrete phenomena in analogy with classic work on transversally elliptic operators of Grusin [4], Boutet de Monvel - Treves [1] and Hörmander [5]. The work of these authors stands in contrast to the work on operators with real first order term which do not exhibit discrete phenomena (e.g. Hörmander [6], Kohn [8], Oleinik - Radkevic [9] and Rothschild - Stein [10]).

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