SPECTRUM OF POSITIVE ENTROPY MULTIDIMENSIONAL DYNAMICAL SYSTEMS WITH A MIXED TIME

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Abstract. It is shown that if an abelian countable group $G = G_1 \oplus G_2$ is such that $G_2$ is a finite group and every aperiodic positive entropy action $\Phi$ of $G_1$ on a Lebesgue probability space $(X, \mathcal{B}, \mu)$ has a countable Haar spectrum in the subspace $L^2_0(X, \mu) \ominus L^2_0(X, \Pi(\Phi), \mu)$, where $\Pi(\Phi)$ denotes the Pinsker $\sigma$-algebra of $\Phi$, then every aperiodic positive entropy action of $G$ on $(X, \mathcal{B}, \mu)$ has the same property. A positive answer to the question of J.P. Thouvenot is obtained as a corollary.

1. Introduction

An important role is played in the spectral theory of dynamical systems by the theorem which says that every action $\Phi$ of the group $\mathbb{Z}^d$, $1 \leq d \leq \infty$, on a Lebesgue probability space $(X, \mathcal{B}, \mu)$ has a countable Lebesgue spectrum in the subspace $L^2_0(X, \mu) \ominus L^2_0(X, \Pi(\Phi), \mu)$.

This theorem has been shown by Rokhlin and Sinai ([RS]), the author ([Ka]), and the author and Liardet ([KL]) in the cases $d = 1$, $d \geq 2$, and $d = \infty$, respectively. The proofs of this theorem are based on the fact that the group $\mathbb{Z}^d$, $1 \leq d \leq \infty$, is torsionfree.

J.P. Thouvenot asked the author whether the above theorem is true for actions of finitely generated abelian groups which contain nontrivial torsion subgroups. Corollary 1 gives the positive answer to this question.

2. Result

Let $G$ be an abelian countable group. For a finite subset $A \subset G$ we denote by $|A|$ the cardinality of $A$. A sequence $(A_n)$ of finite subsets of $G$ is said to be a Følner sequence if for every $g \in G$

$$\lim_{n \to \infty} |A_n|^{-1}|(g + A_n) \triangle A_n| = 0.$$  

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space. For a given sub-$\sigma$-algebra $\mathcal{A} \subset \mathcal{B}$ we denote by $L^2_0(X, \mathcal{A}, \mu)$ the linear subspace of $L^2(X, \mu)$ consisting of $\mathcal{A}$-measurable functions $f$ with $\int_X f d\mu = 0$. 

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Let $\Phi$ be an action of $G$ on $(X, \mathcal{B}, \mu)$, i.e. $\Phi$ is a homomorphism of $G$ into the group of all measure-preserving automorphisms of $(X, \mathcal{B}, \mu)$. We denote by $\Phi^g$ the automorphism of $(X, \mathcal{B}, \mu)$ corresponding to $g \in G$.

Let $U = U_\Phi$ be the unitary representation of $G$ in $L^2(X, \mu)$ induced by $\Phi$, i.e.

$$U^g f = f \circ \Phi^g, f \in L^2(X, \mu), g \in G.$$ 

For a function $f \in L^2(X, \mu)$ we denote by $\sigma_f$ the spectral measure of $f$ with respect to $U$, i.e. a Borel measure on the dual group $\hat{G}$ such that

$$(U^g f, f) = \int_{\hat{G}} \chi(g) \sigma_f(d\chi), g \in G.$$ 

Similarly as in the case when $G = \mathbb{Z}$ one can prove that there exists a sequence $(f_n) \subset L^2(X, \mu)$ such that the cyclic subspaces

$$Z(f_i) = \overline{\rho(U^g f_i, g \in G), i \geq 1},$$

are pairwise orthogonal,

$$L^2(X, \mu) = \bigoplus_{i=1}^{\infty} Z(f_i)$$

and the types $\rho_i = \rho_{f_i}$ of the measures $\rho_{f_i}, i \geq 1$, satisfy the condition

$$\rho_1 \gg \rho_2 \gg \ldots.$$ 

We say that $\Phi$ has a Haar spectrum if the measure $\rho_1$ is equivalent to the Haar measure of $\hat{G}$. $\Phi$ is said to have a countable Haar spectrum (CHS) if all types $\rho_i$ are non-zero, $i \geq 1$. Now we recall the concept of a mean entropy of $\Phi$ given a finite measurable partition of $X$ (cf. [Kif1]). Let $P$ be a finite measurable partition of $X$. It is shown in [Kif1] that there exists the limit

$$\lim_{n \to \infty} |A_n|^{-1} H(\bigvee_{g \in A_n} \Phi^g P)$$

where $(A_n)$ is an arbitrary Følner set in $G$. The limit does not depend on the choice of $(A_n)$ and is called the mean entropy of $\Phi$ given $P$.

We define the entropy $h(\Phi)$ of $\Phi$ by

$$h(\Phi) = \sup h(P, \Phi)$$

where $P$ runs over the set of all finite measurable partitions of $X$.

By the Pinsker $\sigma$-algebra $\Pi(\Phi)$ of $\Phi$ we mean the smallest $\sigma$-algebra containing all finite partitions $P$ with $h(P, \Phi) = 0$.

We say that an action $\Phi$ is a Kolmogorov one (K-action) if $\Pi(\Phi)$ is a trivial $\sigma$-algebra.

Similarly (cf. [KL]) one defines the relative entropies $h(P, \Phi|A)$, $h(\Phi|A)$ and the relative Pinsker $\sigma$-algebra $\Pi(\Phi|A)$ of $\Phi$ with respect to a given factor $\sigma$-algebra $A$.

In the sequel we need the following

**Relative Sinai Theorem.** If $\Phi$ is aperiodic with $h(\Phi|A) > 0$ and $I$ is a probability vector such that $H(I) \leq h(\Phi|A)$, then there exists a finite partition $P$ of $X$ such that the distribution of $P$ is equal to $I$, the partitions $\Phi^g P, g \in G$, are independent, and the factor $\sigma$-algebras $\bigvee_{g \in G} \Phi^g P$ and $A$ are independent.

This theorem has been shown in [T] for $G = \mathbb{Z}$. The proof for arbitrary countable abelian groups $G$ is similar by the use of methods of [Kif2].

We say that $G$ is spectrally natural if every action $\Phi$ of $G$ on $(X, \mathcal{B}, \mu)$ with $h(\Phi) > 0$ has a countable Haar spectrum in the subspace $L^2_0(X, \mu) \otimes L^2_0(X, \Pi(\Phi)), \mu)$.

It follows from [Ka] and [KL] that $G = \mathbb{Z}^d$, $1 \leq d \leq \infty$, is spectrally natural.

**Theorem.** If a countable abelian group $G$ is a direct sum of two subgroups the one of which is spectrally natural and the second is finite, then $G$ is spectrally natural.
Proof. Let \( G = G_1 \oplus G_2 \) where \( G_1 \) is spectrally natural and \( G_2 \) is finite. Let \( \Phi \) be an action of \( G \) on \( (X, \mathcal{B}, \mu) \) with \( h(\Phi) > 0 \) and let \( T \) and \( S \) denote the restrictions of \( \Phi \) to \( G_1 \) and \( G_2 \), respectively.

First, let us observe that

\[
\Pi(\Phi) = \Pi(T).
\]

Indeed, let \( P \) be a finite measurable partition of \( X \) and let \((A_n)\) be a Følner sequence in \( G_1 \). It is clear that the sequence \((B_n)\) of subsets of \( G \) defined by \( B_n = A_n + G_2, n \geq 1 \), is a Følner sequence. We have

\[
\begin{align*}
\lim_{n \to \infty} |B_n|^{-1} \cdot H(\bigvee_{g \in B_n} \Phi^g) &= |G_2|^{-1} \cdot \lim_{n \to \infty} |A_n|^{-1} \cdot H(\bigvee_{g_1 \in A_n} T^{g_1}(\bigvee_{g_2 \in G_2} S^{g_2}P)) \\
&= |G_2|^{-1} \cdot h(\bigvee_{g_2 \in G_2} S^{g_2}P, T).
\end{align*}
\]

Since \( h(S^{g_2}P, T) = h(P, T), g_2 \in G_2 \), the last equality gives

\[
(1)
\]

i.e., (1) is satisfied.

Let \( f \in L^2_0(X, \mu) \otimes L^2_0(X, \Pi(\Phi), \mu) \). By the Pontryagin duality theorem we may write

\[
(2)
\]

\[
(U^g f, f) = \int_{\hat{G}} g(\chi) \rho_f(d\chi), g \in G.
\]

Let \( \varphi : \hat{G}_1 \times \hat{G}_2 \to \hat{G} \) be defined as

\[
[\varphi(\chi_1, \chi_2)](g) = \chi_1(g_1) \cdot \chi_2(g_2),
\]

\( g = g_1 + g_2, g_i \in G_i, i = 1, 2 \). It is well known that \( \varphi \) is an isomorphism.

It follows from (3) that

\[
(U^g f, f) = \int_{\hat{G}_1 \times \hat{G}_2} g(\varphi(\chi_1, \chi_2))(\rho_f \circ \varphi)(d\chi_1 d\chi_2)
\]

\[
= \int_{\hat{G}_1 \times \hat{G}_2} [\varphi(\chi_1, \chi_2)](g)(\rho_f \circ \varphi)(d\chi_1 d\chi_2)
\]

\[
= \int_{\hat{G}_1 \times \hat{G}_2} \chi_1(g_1) \cdot \chi_2(g_2)(\rho_f \circ \varphi)(d\chi_1 d\chi_2)
\]

for every \( g = g_1 + g_2 \in G, g_i \in G_i, i = 1, 2 \).

In particular we have

\[
(3)
\]

\[
(U^{g_1} f, f) = \int_{\hat{G}_1 \times \hat{G}_2} \chi_1(g_1)(\rho_f \circ \varphi)(d\chi_1 d\chi_2), g_1 \in G_1.
\]

Hence

\[
(4)
\]

\[
\int_{\hat{G}_1} \chi_1(g_1)\sigma_f(d\chi_1) = \int_{\hat{G}_1 \times \hat{G}_2} \chi_1(g_1)(\rho_f \circ \varphi)(d\chi_1 d\chi_2)
\]

for every \( g_1 \in G_1 \).

Let \( \hat{\rho}_f \) be the Borel measure on \( \hat{G}_1 \) defined as

\[
\hat{\rho}_f(A) = (\rho_f \circ \varphi)(A \times \hat{G}_2), A \subset \hat{G}_1.
\]
Since
\[ \int_{\hat{G}_1 \times \hat{G}_2} \chi_1(g_1)(\rho_f \circ \varphi)(d\chi_1 d\chi_2) = \int_{\hat{G}_1} \chi_1(g_1)\hat{\rho}_f(d\chi_1), g_1 \in G, \]
the equality (5) implies
\[ \sigma_f(A) = \hat{\rho}_f(A) = (\rho_f \circ \varphi)(A \times \hat{G}_2) \]
for every Borel set \( A \subset \hat{G}_1 \).

Let \( \lambda, m, \) and \( \delta \) denote the normalized Haar measures on \( \hat{G}, \hat{G}_1, \) and \( \hat{G}_2 \), respectively, and let \( f \) be a function of the maximal spectral type in \( L^2_0(X, \mu) \oplus L^2_0(X, \Pi(\Phi), \mu) \) for \( U \).

From (2) we have \( h(T) \geq h(\Phi) > 0 \). Hence and from the assumption, the action \( T \) has a CHS in \( L^2_0(X, \mu) \oplus L^2_0(X, \Pi(\Phi), \mu) \), i.e. \( \sigma_f \ll m \).

We claim that \( \rho_f \ll \lambda \). Indeed, let \( \lambda(E) = 0 \) for a certain Borel set \( E \subset \hat{G} \). The uniqueness of the Haar measure implies \( \lambda \circ \varphi = m \times \delta \). Therefore we have
\[ 0 = (m \times \delta)(\varphi^{-1}E) = \int_{\hat{G}_2} m((\varphi^{-1}E)_{\chi_2}) \delta(d\chi_2) \]
where \( (\varphi^{-1}E)_{\chi_2} \) denotes the section of \( \varphi^{-1}E \) determined by \( \chi_2 \in \hat{G}_2 \). Since \( \delta \) is a discrete Haar measure, we get \( m((\varphi^{-1}E)_{\chi_2}) = 0 \) and so
\[ 0 = \sigma_f((\varphi^{-1}E)_{\chi_2}) = (\rho_f \circ \varphi)((\varphi^{-1}E)_{\chi_2} \times \hat{G}_2) \geq (\rho_f \circ \varphi)((\varphi^{-1}E)_{\chi_2}) \]
for every \( \chi_2 \in \hat{G}_2 \). Thus \( \rho_f(E) = 0 \), i.e. \( \rho_f \ll \lambda \).

Now we shall show that \( \rho_f \gg \lambda \), i.e. \( \Phi \) has a Haar spectrum in \( L^2_0(X, \mu) \oplus L^2_0(X, \Pi(\Phi), \mu) \), and that the multiplicity of the spectrum is infinite.

Since \( h(\Phi) > 0 \), we also have \( h(\Phi(\Pi(\Phi))) > 0 \). In view of the Relative Sinai Theorem there exists a non-trivial finite measurable partition \( P \) of \( X \) such that \( \mathcal{A} = \bigvee_{g \in G} \Phi^g P \) is a Bernoulli factor \( \sigma \)-algebra of \( \Phi \) independent of \( \Pi(\Phi) \). From [Kir] there exists an orthonormal basis \( \{f_{g,i}, g \in G, i \geq 1\} \) in \( L^2_0(X, \mathcal{A}, \mu) \) such that
\[ U^h f_{g,i} = f_{h+g,i}, g, h \in G, i \geq 1. \]

It is clear that \( f_{g,i} \) are orthogonal to \( L^2_0(X, \Pi(\Phi), \mu), g \in G, i \geq 1 \). Since \( f \) is a function of the maximal spectral type in \( L^2_0(X, \mu) \oplus L^2_0(X, \Pi(\Phi), \mu) \), we have \( \rho_f \gg \lambda \), i.e. the maximal spectral type is equal to the type of the Haar measure \( \lambda \). Moreover, the multiplicity of the spectrum of \( \Phi \) is infinite. \( \square \)

**Corollary 1.** Every finitely generated abelian (FGA) group is spectrally natural.

**Proof.** Let \( G \) be an FGA group. It is well known that \( G \) is a direct sum of a finite number of cyclic groups of infinite and prime power order. Let \( G_1, G_2 \) denote the subgroup which is the direct sum of cyclic groups of infinite (prime power) order. From [Ka] \( G_1 \) is spectrally natural and so, in view of the Theorem, \( G \) is also spectrally natural. \( \square \)

The following corollaries are easy consequences of Corollary 1.

**Corollary 2.** For actions of an FGA group the following properties are satisfied:

(a) every \( K \)-action has CHS,

(b) every action with a singular spectrum or a spectrum with finite multiplicity has zero entropy.
Using arguments similar to those used by Parry in [Pa] one obtains from Corollary 1 the following

**Corollary 3.** Every Gaussian action of a FGA group with a singular spectral measure has zero entropy.

We refer the reader to [Kir] for the definition of general Gaussian actions.

The result given in Corollary 3 for $G = \mathbb{Z}^d$ has been shown in [FK] and [Ru]. By [KL] it can be extended to $G = \mathbb{Z}^\infty$ and then, in view of the Theorem, to every group which is a direct sum of $\mathbb{Z}^\infty$ and an arbitrary finite group.

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