PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 5, May 1996

SPECTRUM OF POSITIVE ENTROPY MULTIDIMENSIONAL DYNAMICAL SYSTEMS WITH A MIXED TIME

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. It is shown that if an abelian countable group $G = G_1 \oplus G_2$ is such that G_2 is a finite group and every aperiodic positive entropy action Φ of G_1 on a Lebesgue probability space $(X, \mathcal{B}, \mu)$ has a countable Haar spectrum in the subspace $L^2_0(X, \mu) \ominus L^2_0(X, \Pi(\Phi), \mu)$, where $\Pi(\Phi)$ denotes the Pinsker σ-algebra of Φ, then every aperiodic positive entropy action of G on $(X, \mathcal{B}, \mu)$ has the same property. A positive answer to the question of J.P. Thouvenot is obtained as a corollary.

1. Introduction

An important role is played in the spectral theory of dynamical systems by the theorem which says that every action Φ of the group $\mathbb{Z}^d$, $1 \le d \le \infty$, on a Lebesgue probability space $(X, \mathcal{B}, \mu)$ has a countable Lebesgue spectrum in the subspace $L^2_0(X, \mu) \ominus L^2_0(X, \Pi(\Phi), \mu)$.

This theorem has been shown by Rokhlin and Sinai ([RS]), the author ([Ka]), and the author and Liardet ([KL]) in the cases $d = 1$, $d \ge 2$, and $d = \infty$, respectively. The proofs of this theorem are based on the fact that the group $\mathbb{Z}^d, 1 \le d \le \infty$, is torsionfree.

J.P. Thouvenot asked the author whether the above theorem is true for actions of finitely generated abelian groups which contain nontrivial torsion subgroups. Corollary 1 gives the positive answer to this question.

2. Result

Let G be an abelian countable group. For a finite subset $A \subset G$ we denote by $|A|$ the cardinality of A. A sequence (A_n) of finite subsets of G is said to be a Følner sequence if for every $g \in G$

$$\lim_{n\to\infty} |A_n|^{-1} |(g + A_n) \triangle A_n| = 0.$$

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space. For a given sub-σ-algebra $\mathcal{A} \subset \mathcal{B}$ we denote by $L^2_0(X, \mathcal{A}, \mu)$ the linear subspace of $L^2(X, \mu)$ consisting of $\mathcal{A}$-measurable functions f with $\int_X f d\mu = 0$.

Received by the editors November 3, 1994.
1991 *Mathematics Subject Classification.* Primary 28D15; Secondary 60G15.
Key words and phrases. Countable Haar spectrum, entropy, Gaussian actions, spectral measure, spectrally natural.

Let Φ be an action of G on $(X,\mathcal{B},\mu)$, i.e. Φ is a homomorphism of G into the group of all measure-preserving automorphisms of $(X,\mathcal{B},\mu)$.We denote by Φ^g the automorphism of $(X,\mathcal{B},\mu)$ corresponding to $g\in G$.

Let $U=U_\Phi$ be the unitary representation of G in $L^2(X,\mu)$ induced by Φ, i.e.

$$U^g f = f\circ\Phi^g, f\in L^2(X,\mu), g\in G.$$

For a function $f\in L^2(X,\mu)$ we denote by σ_f the spectral measure of f with respect to U, i.e. a Borel measure on the dual group $\hat{G}$ such that

$$(U^g f,f)=\int_{\hat{G}}\chi(g)\sigma_f(d\chi), g\in G.$$

Similarly as in the case when $G=\mathbb{Z}$ one can prove that there exists a sequence $(f_n)\subset L^2(X,\mu)$ such that the cyclic subspaces

$$Z(f_i)=\overline{S}p\{U^g f_i, g\in G\}, i\geq 1,$$

are pairwise orthogonal,

$$L^2(X,\mu)=\bigoplus_{i=1}^\infty Z(f_i)$$

and the types $\overline{\rho}_i=\overline{\rho}_{f_i}$ of the measures $\rho_{f_i}, i\geq 1$, satisfy the condition

$$\overline{\rho}_1\gg\overline{\rho}_2\gg\dots.$$

We say that Φ has a Haar spectrum if the measure ρ_1 is equivalent to the Haar measure of $\hat{G}$. Φ is said to have a countable Haar spectrum (CHS) if all types $\overline{\rho}_i$ are non-zero, $i\geq 1$. Now we recall the concept of a mean entropy of Φ given a finite measurable partition of X (cf. [Kif1]). Let P be a finite measurable partition of X. It is shown in [Kif1] that there exists the limit

$$\lim_{n\to\infty}|A_n|^{-1}H(\bigvee_{g\in A_n}\Phi^g P)$$

where (A_n) is an arbitrary Følner set in G. The limit does not depend on the choice of (A_n) and is called the mean entropy of Φ given P.

We define the entropy $h(\Phi)$ of Φ by

$$h(\Phi)=\sup h(P,\Phi)$$

where P runs over the set of all finite measurable partitions of X.

By the Pinsker σ-algebra $\Pi(\Phi)$ of Φ we mean the smallest σ-algebra containing all finite partitions P with $h(P,\Phi)=0$.

We say that an action Φ is a Kolmogorov one (K-action) if $\Pi(\Phi)$ is a trivial σ-algebra.

Similarly (cf. [KL]) one defines the relative entropies $h(P,\Phi|\mathcal{A})$, $h(\Phi|\mathcal{A})$ and the relative Pinsker σ-algebra $\Pi(\Phi|\mathcal{A})$ of Φ with respect to a given factor σ-algebra $\mathcal{A}$.

In the sequel we need the following

Relative Sinai Theorem. *If Φ is aperiodic with $h(\Phi|\mathcal{A})>0$ and I is a probability vector such that $H(I)\leq h(\Phi|\mathcal{A})$, then there exists a finite partition P of X such that the distribution of P is equal to I, the partitions $\Phi^g P, g\in G$, are independent, and the factor σ-algebras $\bigvee_{g\in G}\Phi^g P$ and $\mathcal{A}$ are independent.*

This theorem has been shown in [T] for $G=\mathbb{Z}$. The proof for arbitrary countable abelian groups G is similar by the use of methods of [Kif2].

We say that G is spectrally natural if every action Φ of G on $(X,\mathcal{B},\mu)$ with $h(\Phi)>0$ has a countable Haar spectrum in the subspace $L_0^2(X,\mu)\ominus L_0^2(X,\Pi(\Phi)),\mu)$.

It follows from [Ka] and [KL] that $G=\mathbb{Z}^d$, $1\leq d\leq\infty$, is spectrally natural.

Theorem. *If a countable abelian group G is a direct sum of two subgroups the one of which is spectrally natural and the second is finite, then G is spectrally natural.*

Proof. Let $G = G_1 \oplus G_2$ where G_1 is spectrally natural and G_2 is finite. Let Φ be an action of G on $(X, \mathcal{B}, \mu)$ with $h(\Phi) > 0$ and let T and S denote the restrictions of Φ to G_1 and G_2, respectively.

First, let us observe that

$$\Pi(\Phi) = \Pi(T). \tag{1}$$

Indeed, let P be a finite measurable partition of X and let (A_n) be a Følner sequence in G_1. It is clear that the sequence (B_n) of subsets of G defined by $B_n = A_n + G_2, n \geq 1$, is a Følner sequence. We have

$$\begin{aligned} h(P, \Phi) &= \lim_{n\to\infty} |B_n|^{-1} \cdot H(\bigvee_{g \in B_n} \Phi^g P) \\ &= |G_2|^{-1} \cdot \lim_{n\to\infty} |A_n|^{-1} \cdot H(\bigvee_{g_1 \in A_n} T^{g_1}(\bigvee_{g_2 \in G_2} S^{g_2} P) \\ &= |G_2|^{-1} \cdot h(\bigvee_{g_2 \in G_2} S^{g_2} P, T). \end{aligned}$$

Since $h(S^{g_2} P, T) = h(P, T), g_2 \in G_2$, the last equality gives

$$|G_2|^{-1} \cdot h(P, T) \leq h(P, \Phi) \leq h(P, T), \tag{2}$$

i.e. (1) is satisfied.

Let $f \in L^2_0(X, \mu) \ominus L^2_0(X, \Pi(\Phi), \mu)$. By the Pontriagin duality theorem we may write

$$(U^g f, f) = \int_{\hat{G}} g(\chi) \rho_f(d\chi), g \in G. \tag{3}$$

Let $\varphi : \hat{G}_1 \times \hat{G}_2 \to \hat{G}$ be defined as

$$[\varphi(\chi_1, \chi_2)](g) = \chi_1(g_1) \cdot \chi_2(g_2),$$

$g = g_1 + g_2, g_i \in G_i, i = 1, 2$. It is well known that φ is an isomorphism.

It follows from (3) that

$$\begin{aligned} (U^g f, f) &= \int_{\hat{G}_1 \times \hat{G}_2} g(\varphi(\chi_1, \chi_2))(\rho_f \circ \varphi)(d\chi_1 d\chi_2) \\ &= \int_{\hat{G}_1 \times \hat{G}_2} [\varphi(\chi_1, \chi_2)](g)(\rho_f \circ \varphi)(d\chi_1 d\chi_2) \\ &= \int_{\hat{G}_1 \times \hat{G}_2} \chi_1(g_1) \cdot \chi_2(g_2)(\rho_f \circ \varphi)(d\chi_1 d\chi_2) \end{aligned}$$

for every $g = g_1 + g_2 \in G, g_i \in G_i, i = 1, 2$.

In particular we have

$$(U^{g_1} f, f) = \int_{\hat{G}_1 \times \hat{G}_2} \chi_1(g_1)(\rho_f \circ \varphi)(d\chi_1 d\chi_2), g_1 \in G_1. \tag{4}$$

Hence

$$\int_{\hat{G}_1} \chi_1(g_1) \sigma_f(d\chi_1) = \int_{\hat{G}_1 \times \hat{G}_2} \chi_1(g_1)(\rho_f \circ \varphi)(d\chi_1 d\chi_2) \tag{5}$$

for every $g_1 \in G_1$.

Let $\tilde{\rho}_f$ be the Borel measure on $\hat{G}_1$ defined as

$$\tilde{\rho}_f(A) = (\rho_f \circ \varphi)(A \times \hat{G}_2), A \subset \hat{G}_1.$$

Since

$$\int_{\hat{G}_1\times\hat{G}_2} \chi_1(g_1)(\rho_f\circ\varphi)(d\chi_1 d\chi_2) = \int_{\hat{G}_1} \chi_1(g_1)\tilde{\rho_f}(d\chi_1), g_1 \in G_1,$$

the equality (5) implies

$$\sigma_f(A) = \tilde{\rho_f}(A) = (\rho_f\circ\varphi)(A\times\hat{G}_2)$$

for every Borel set $A\subset\hat{G}_1$.

Let λ, m, and δ denote the normalized Haar measures on $\hat{G},\hat{G}_1$, and $\hat{G}_2$, respectively, and let f be a function of the maximal spectral type in $L^2_0(X,\mu)\ominus L^2_0(X,\Pi(\Phi),\mu)$ for U.

From (2) we have $h(T)\geq h(\Phi)>0$. Hence and from the assumption, the action T has a CHS in $L^2_0(X,\mu)\ominus L^2_0(X,\Pi(T),\mu)$, i.e. $\sigma_f\ll m$.

We claim that $\rho_f\ll\lambda$. Indeed, let $\lambda(E)=0$ for a certain Borel set $E\subset\hat{G}$. The uniqueness of the Haar measure implies $\lambda\circ\varphi=m\times\delta$. Therefore we have

$$0=(m\times\delta)(\varphi^{-1}E)=\int_{\hat{G}_2} m((\varphi^{-1}E)_{\chi_2})\delta(d\chi_2)$$

where $(\varphi^{-1}E)_{\chi_2}$ denotes the section of $\varphi^{-1}E$ determined by $\chi_2\in\hat{G}_2$. Since δ is a discrete Haar measure, we get $m((\varphi^{-1}E)_{\chi_2})=0$ and so

$$\begin{aligned} 0 &= \sigma_f((\varphi^{-1}E)_{\chi_2}) = (\rho_f\circ\varphi)((\varphi^{-1}E)_{\chi_2}\times\hat{G}_2) \\ &\geq (\rho_f\circ\varphi)((\varphi^{-1}E)_{\chi_2}\times\{\chi_2\}) \end{aligned}$$

for every $\chi_2\in\hat{G}_2$. Thus $\rho_f(E)=0$, i.e. $\rho_f\ll\lambda$.

Now we shall show that $\rho_f\gg\lambda$, i.e. Φ has a Haar spectrum in $L^2_0(X,\mu)\ominus Ł^2_0(X,\Pi(\Phi),\mu)$, and that the multiplicity of the spectrum is infinite.

Since $h(\Phi)>0$, we also have $h(\Phi|\Pi(\Phi))>0$. In view of the Relative Sinai Theorem there exists a non-trivial finite measurable partition P of X such that $\mathcal{A}=\bigvee_{g\in G}\Phi^g P$ is a Bernoulli factor σ-algebra of Φ independent of $\Pi(\Phi)$. From [Kir] there exists an orthonormal basis $(f_{g,i}, g\in G, i\geq 1)$ in $L^2_0(X,\mathcal{A},\mu)$ such that

$$U^h f_{g,i} = f_{h+g,i}, g,h\in G, i\geq 1.$$

It is clear that $f_{g,i}$ are orthogonal to $L^2_0(X,\Pi(\Phi),\mu), g\in G, i\geq 1$. Since f is a function of the maximal spectral type in $L^2_0(X,\mu)\ominus L^2_0(X,\Pi(\Phi),\mu)$, we have $\rho_f\gg\lambda$, i.e. the maximal spectral type is equal to the type of the Haar measure λ. Moreover, the multiplicity of the spectrum of Φ is infinite. □

Corollary 1. *Every finitely generated abelian (FGA) group is spectrally natural.*

Proof. Let G be an FGA group. It is well known that G is a direct sum of a finite number of cyclic groups of infinite and prime power order. Let G_1 (G_2) denote the subgroup which is the direct sum of cyclic groups of infinite (prime power) order. From [Ka] G_1 is spectrally natural and so, in view of the Theorem, G is also spectrally natural. □

The following corollaries are easy consequences of Corollary 1.

Corollary 2. *For actions of an FGA group the following properties are satisfied:*

(a) *every K-action has CHS,*

(b) *every action with a singular spectrum or a spectrum with finite multiplicity has zero entropy.*

Using arguments similar to those used by Parry in [Pa] one obtains from Corollary 1 the following

Corollary 3. *Every Gaussian action of a FGA group with a singular spectral measure has zero entropy.*

We refer the reader to [Kir] for the definition of general Gaussian actions.

The result given in Corollary 3 for $G = \mathbb{Z}^d$ has been shown in [FK] and [Ru]. By [KL] it can be extended to $G = \mathbb{Z}^\infty$ and then, in view of the Theorem, to every group which is a direct sum of $\mathbb{Z}^\infty$ and an arbitrary finite group.

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