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ROTATIONAL SYMMETRY OF THE HERMITE PROJECTION OPERATORS

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ABSTRACT. We calculate an integral formula for the Hermite projection operators. We give some applications of our formula. We also give a short proof of a recent theorem of Thangavelu

1. INTRODUCTION

In a recent paper ([T], Thangavelu) it was shown that if $P(x)$ is a solid spherical harmonic of degree m and if $f \in L^2(R^n)$ is of the form $f(x) = f_0(|x|)P(x)$, then its projection $\mathcal{P}_k f(x)$ onto the space of Hermite functions of degree k is a Laguerre function multiplied by $P(x)$.

More specifically, let L_k^δ denote the Laguerre polynomials of type δ. Define

$$R_k^\delta(f_0) = 2\frac{k!}{\Gamma(k+\delta+1)}\int_0^\infty f_0(r)L_k^\delta(r^2)e^{-r^2/2}r^{2\delta+1}\,dr. \tag{1.1}$$

If $f(x) = f_0(|x|)P(x) \in L^2(R^n)$, then one has $\mathcal{P}_{2k+m}f(x) = F_k(|x|)P(x)$ where

$$F_k(r) = R_k^\delta(f_0)L_k^\delta(r^2)e^{-r^2/2} \tag{1.2}$$

with $\delta = \frac{n}{2} + m - 1$. For other values of j, $\mathcal{P}_j f = 0$. The proof is based on generating functions.

Since Hermite functions are eigenfunctions of the Fourier transform and Laguerre functions are eigenfunctions of the Fourier-Bessel (Hankel) transform, Thangavelu's formula is closely related to the Hecke-Bochner identity which says that if $f \in L^1 \cap L^2(R^n)$ such that $f(x) = f_0(|x|)P(x)$, then $\hat{f}(x) = F_0(|x|)P(x)$ where

$$\hat{f}(x) = \int_{R^n} f(y)e^{-ix\cdot y}\,dy, \tag{1.3}$$

$$F_0(r) = (2\pi)^{n/2}i^{-m}r^{-(n/2+m-1)}\int_0^\infty f_0(s)J_{n/2+m-1}(rs)s^{n/2+m}\,ds \tag{1.4}$$

and J_α is the Bessel function of order α. For a proof of this identity see [SW].

One of the important properties of the Fourier transform is its symmetry with respect to rotations. Thangavelu's formula (1.2) suggests the same should be true for the projection operators $\mathcal{P}_k$. Our main result in this paper is an integral formula

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for the kernels of the projection operators which shows that the projection operators commute with rotations. We will apply our formula to give a simple proof of Thangavelu's formula in the radial case and to give an expression for Laguerre polynomials in terms of Hermite polynomials. We will also give a short proof of Thangavelu's formula which only uses L^2 theory for the Hermite and Laguerre polynomials and the homogeneity of the spherical harmonics.

2. Definitions

Define $\mathcal{H}_m$ to be the space of all $f \in L^2(R^n)$ of the form $f_0(|x|)P(x)$ where $P(x)$ is a (solid) spherical harmonic of degree m, i.e., $P(x)$ is a homogeneous harmonic polynomial of degree m. These spaces provide a direct sum decomposition of $L^2(R^n)$ invariant under the Fourier transform; see [SW].

The Hermite polynomials orthogonal with respect to the weight $e^{-x^2}\,dx$ on R are given for $k = 0, 1, \ldots$ by

$$H_k(x) = (-1)^k \frac{d^k}{dx^k}(e^{-x^2})e^{x^2} = \sum_{j=0}^{[\frac{k}{2}]} \frac{(-1)^j k!}{j!(k-2j)!}(2x)^{k-2j}. \tag{2.1}$$

They satisfy the orthogonality relation

$$\int_R H_j(x)H_k(x)e^{-x^2}\,dx = 2^k k!\sqrt{\pi}\delta_{j,k}. \tag{2.2}$$

The Hermite functions $H_k(x)e^{-x^2/2}$ are eigenfunctions of the Fourier transform:

$$H_k(x)e^{-x^2/2} = \frac{i^k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_k(y)e^{-y^2/2-ixy}\,dy, \qquad k = 0, 1, \ldots. \tag{2.3}$$

The Hermite polynomials on R^n are defined for multi-indices $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_i = 0, 1, \ldots$, as products of one-dimensional Hermite polynomials:

$$H_\mu(x) = H_{\mu_1}(x_1)H_{\mu_2}(x_2)\cdots H_{\mu_n}(x_n), \qquad x = (x_1, \ldots, x_n). \tag{2.4}$$

Note that if $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_n$ is even, then all terms in H_μ are of even degree. The Hermite functions $H_\mu(x)e^{-|x|^2/2}$ are orthogonal on R^n:

$$\int_{R^n} H_\mu(x)H_\nu(x)e^{-|x|^2}\,dx = 2^{|\mu|}\mu!\pi^{n/2}\delta_{\mu,\nu} \tag{2.5}$$

where

$$\mu! = \mu_1!\mu_2!\cdots\mu_n!.$$

They are eigenfunctions of the Fourier transform in R^n.

A function $f \in L^2(R^n)$ can be expanded in series of Hermite functions using the formula

$$f(x) \sim \sum_{k=0}^{\infty} \mathcal{P}_k f(x) \tag{2.6}$$

where $\mathcal{P}_k f$ are the Hermite projection operators

$$\mathcal{P}_k f(x) = \sum_{|\mu|=k} \hat{f}(\mu)H_\mu(x)e^{-|x|^2/2}, \tag{2.7}$$

$$\hat{f}(\mu) = \frac{1}{2^k \mu! \pi^{n/2}} \int_{R^n} f(y) H_\mu(y) e^{-|y|^2/2}\, dy. \tag{2.8}$$

In other words,

$$\mathcal{P}_k f(x) = \int_{R^n} f(y) \Phi_k(x, y)\, dy \tag{2.9}$$

where

$$\Phi_k(x, y) = \frac{1}{2^k \pi^{n/2}} \sum_{|\mu|=k} \frac{H_\mu(x) H_\mu(y)}{\mu!} e^{-(|x|^2+|y|^2)/2}. \tag{2.10}$$

For many interesting theorems concerning expansions in terms of these polynomials, see [T2].

For $\alpha > -1$ the Laguerre polynomials $\{L_k^\alpha(x)\}$ of type α are defined by the formula

$$e^{-x} x^\alpha L_k^\alpha(x) = \frac{1}{k!} \frac{d^k}{dx^k}(e^{-x} x^{k+\alpha}) = \sum_{j=0}^{k} \frac{\Gamma(k+\alpha+1)}{\Gamma(j+\alpha+1)} \frac{(-x)^j}{j!(k-j)!}. \tag{2.11}$$

The Laguerre polynomials satisfy the orthogonality

$$\int_0^\infty L_k^\alpha(x) L_j^\alpha(x) e^{-x} x^\alpha\, dx = \frac{\Gamma(k+\alpha+1)}{k!} \delta_{k,j}. \tag{2.12}$$

Note that this implies that $|x|^m L_k^{\frac{n-2}{2}+m}(|x|^2) e^{-|x|^2/2}, k = 0, 1, \dots,$ are orthogonal on R^n for each m.

The set consisting of the functions $L_k^\alpha(2x) x^{\alpha/2} e^{-x}$ are eigenfunctions of the Hankel transform (see e.g. [Sz]):

$$L_k^\alpha(2x) x^{\alpha/2} e^{-x} = \int_0^\infty J_\alpha(2\sqrt{xy}) L_k^\alpha(2y) y^{\alpha/2} e^{-y} dy, \qquad k = 0, 1, \dots. \tag{2.13}$$

This implies by the Hecke-Bochner identity that the functions

$$L_k^{\frac{n-2}{2}+m}(|x|^2) P(x) e^{-|x|^2/2}, \qquad k = 0, 1, \dots, \tag{2.14}$$

which span the space $\mathcal{H}_m$ are eigenfunctions of the Fourier transform in R^n.

3. A formula for the projection operator

Lemma 3.1. *Let $u = (u_1, u_2, \dots, u_n)$ be a unit vector in R^n, and let $x \in R^n$. Then for $k = 0, 1, \dots$*

$$H_k(x \cdot u) = \sum_{|\mu|=k} \binom{k}{\mu} H_\mu(x) u^\mu \tag{3.1}$$

where $u^\mu = u_1^{\mu_1} u_2^{\mu_2} \dots u_n^{\mu_n}$ and $\binom{k}{\mu}$ is the multinomial index $k!/\mu!$.

Proof. For $n = 2$ this formula is well known:

$$H_k(x\cos\theta + y\sin\theta) = \sum_{j=0}^{k} \binom{k}{j} H_j(x) H_{k-j}(y) \cos^j\theta \sin^{k-j}\theta. \tag{3.2}$$

The more general formula is proved inductively. Let u be a unit vector in R^{n+1}. Then we can write

$$u = (\cos\theta, (\sin\theta)v_1, (\sin\theta)v_2, \dots, (\sin\theta)v_n)$$

where $v=(v_1, v_2, \ldots, v_n)$ is a unit vector in R^n. Let $x = (x_1, x_2, \ldots, x_{n+1}) \in R^{n+1}$. Let $y=(x_2, x_3, \ldots, x_{n+1}) \in R^n$. Then

$$x \cdot u = x_1 \cos\theta + y \cdot v \sin\theta$$

so that we have

$$\begin{aligned} H_k(x \cdot u) &= \sum_{j=0}^{k} \binom{k}{j} H_j(x_1) H_{k-j}(y \cdot v) \cos^j\theta \sin^{k-j}\theta \\ &= \sum_{j=0}^{k} \binom{k}{j} H_j(x_1) \sum_{|\nu|=k-j} \binom{k-j}{\nu} H_\nu(y) v^\nu \cos^j\theta \sin^{k-j}\theta \\ &= \sum_{|\mu|=k} \binom{k}{\mu} H_\mu(x) u^\mu. \end{aligned}$$

Theorem 3.2. *For* $k = 0, 1, \ldots$

$$\Phi_k(x,y) = \frac{(-i)^k e^{(|x|^2-|y|^2)/2}}{\pi^n k!} \int_{R^n} |s|^k H_k\left(y \cdot \frac{s}{|s|}\right) e^{-|s|^2+2is\cdot x}\, ds. \tag{3.3}$$

Proof. It is well known (see e.g. [Sz]) that

$$H_k(x) = \frac{2^k(-i)^k e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} s^n e^{-s^2+2isx}\, ds, \qquad k = 0, 1, \ldots. \tag{3.4}$$

It easily follows for $|\mu| = k$ that

$$H_\mu(x) = \frac{2^k(-i)^k e^{|x|^2}}{\pi^{n/2}} \int_{R^n} s^\mu e^{-|s|^2+2is\cdot x}\, ds. \tag{3.5}$$

Therefore we have

$$\begin{aligned} \sum_{|\mu|=k} \binom{k}{\mu} H_\mu(x) H_\mu(y) &= \sum_{|\mu|=k} \binom{k}{\mu} \left(\frac{2^k(-i)^k e^{|x|^2}}{\pi^{n/2}} \int_{R^n} s^\mu e^{-|s|^2+2is\cdot x}\, ds \right) H_\mu(y) \\ &= \frac{2^k(-i)^k e^{|x|^2}}{\pi^{n/2}} \int_{R^n} \left\{ \sum_{|\mu|=k} \binom{k}{\mu} H_\mu(y) s^\mu \right\} e^{-|s|^2+2is\cdot x}\, ds \\ &= \frac{2^k(-i)^k e^{|x|^2}}{\pi^{n/2}} \int_{R^n} |s|^k H_k\left(y \cdot \frac{s}{|s|}\right) e^{-|s|^2+2is\cdot x}\, ds. \end{aligned} \tag{3.6}$$

Finally since

$$\Phi_k(x,y) = \frac{e^{-(|x|^2+|y|^2)/2}}{2^k k! \pi^{n/2}} \sum_{|\mu|=k} \binom{k}{\mu} H_\mu(x) H_\mu(y), \tag{3.7}$$

we obtain the result.

Note that for $n = 1$ (3.3) reduces to $H_k(x)H_k(y)e^{-(x^2+y^2)/2}/2^k\sqrt{\pi}k!$ as expected.

Using $H_{2k}(0) = (-1)^k (2k)!/k!$ and the Hecke-Bochner formula we obtain

$$\begin{aligned}\Phi_{2k}(x,0) &= \frac{e^{|x|^2/2}}{\pi^n k!}\int_{R^n} e^{2is\cdot x}|s|^{2k}e^{-|s|^2}\,ds \\ &= \frac{2e^{|x|^2/2}}{\pi^{n/2}k!|x|^{\frac{n-2}{2}}}\int_0^\infty J_{\frac{n-2}{2}}(2r|x|)r^{2k+\frac{n}{2}}e^{-r^2}\,dr \\ &= \frac{e^{-|x|^2/2}}{\pi^{n/2}}L_k^{\frac{n-2}{2}}(|x|^2) \qquad (3.8)\end{aligned}$$

where we have used

$$L_m^\nu(x) = \frac{e^x x^{-\nu/2}}{m!}\int_0^\infty s^{m+\nu/2}J_\nu(2\sqrt{xs})e^{-s}\,ds, \qquad (3.9)$$

see e.g. [Sz]. On the other hand,

$$\Phi_{2k}(x,0) = \frac{e^{-|x|^2/2}}{2^{2k}\pi^{n/2}}\sum_{|\mu|=2k}\frac{H_\mu(0)}{\mu!}H_\mu(x). \qquad (3.10)$$

Therefore we obtain

Corollary 3.3. *For $k = 0, 1, \ldots$*

$$L_k^{\frac{n-2}{2}}(|x|^2) = \frac{1}{2^{2k}}\sum_{|\mu|=2k}\frac{H_\mu(0)}{\mu!}H_\mu(x). \qquad (3.11)$$

Note that $\Phi_{2k+1}(x,0) = 0$ for all x and $k = 0, 1, \ldots$.

Theorem 3.4. *Let $R_\rho f(x) = f(\rho x)$ where ρ is a rotation. Then for $k = 0, 1, \ldots$*

$$\mathcal{P}_k R_\rho f(x) = R_\rho \mathcal{P}_k f(x). \qquad (3.12)$$

Proof. This follows from the observation that

$$\Phi_k(\rho x, \rho y) = \Phi_k(x,y), \qquad k = 0, 1, \ldots, \qquad (3.13)$$

which follows easily from (3.3).

4. A proof of Thangavelu's formula in the radial case

Theorem 3.2 leads to a proof of Thangavelu's formula in the radial case. One can evaluate the integral

$$\int_{\Sigma_{n-1}} H_{2k}(x\cdot u')\,du' \qquad (4.1)$$

by first integrating over the parallel

$$L_\theta = \{u' \in \Sigma_{n-1}:\ x'\cdot u' = \cos\theta\}$$

and then integrating over θ, $0 \le \theta \le \pi$. Then since $\omega_{n-2}\sin^{n-2}\theta$ is the measure of L_θ, we have

$$\begin{aligned}\int_{\Sigma_{n-1}} H_{2k}(x\cdot u')\,du' &= \omega_{n-2}\int_0^\pi H_{2k}(r\cos\theta)\sin^{n-2}\theta\,d\theta\\ &= \omega_{n-2}\int_{-1}^1 H_{2k}(rt)(1-t^2)^{\frac{n-3}{2}}\,dt\\ &= \frac{2\pi^{n/2}(-1)^k(2k)!}{\Gamma(k+\frac{n}{2})}L_k^{\frac{n-2}{2}}(r^2) \end{aligned}\tag{4.2}$$

where in the last step we have used the well-known formula of Uspenski (see [Sz]):

$$L_k^\alpha(x) = \frac{(-1)^k\pi^{-1/2}\Gamma(k+\alpha+1)}{\Gamma(\alpha+\frac12)(2k)!}\int_{-1}^1 H_{2k}(\sqrt{x}t)(1-t^2)^{\alpha-\frac12}\,dt, \qquad \alpha > -\frac12. \tag{4.3}$$

Letting $|y| = r$ and $|x| = \gamma$ we obtain

$$\begin{aligned}\int_{\Sigma_{n-1}}\Phi_{2k}(x,y)\,dy' &= \frac{(-1)^k e^{(r^2-\gamma^2)/2}}{\pi^n(2k)!}\\ &\quad\cdot\int_{R^n}|s|^{2k}\left\{\int_{\Sigma_{n-1}}H_{2k}(ry'\cdot s')\,dy'\right\}e^{-|s|^2+2is\cdot x}\,ds\\ &= \frac{2k!}{\Gamma(k+\frac{n}{2})}L_k^{\frac{n-2}{2}}(r^2)L_k^{\frac{n-2}{2}}(\gamma^2)e^{-(r^2+\gamma^2)/2}.\end{aligned}$$

Therefore if $f(x) = f_0(|x|) \in L^2(R^n)$ we obtain Thangavelu's formula:

$$\begin{aligned}\mathcal{P}_{2k}f(x) &= \int_{R^n} f(y)\Phi_{2k}(x,y)\,dy\\ &= \int_0^\infty f_0(r)\left\{\int_{\Sigma_{n-1}}\Phi_{2k}(x,ry')dy'\right\}r^{n-1}\,dr\\ &= R_k^{\frac{n-2}{2}}(f_0)L_k^{\frac{n-2}{2}}(\gamma^2)e^{-\gamma^2/2}.\end{aligned}\tag{4.4}$$

It is easy to see that $\mathcal{P}_{2k+1}f(x) = 0$.

Uspensky's formula can be generalized to give a proof of Thangavalu's formula in the general case. However, the constants become very unwieldy. At any rate, the generalization is given in Section 6.

5. A short proof of Thangavelu's formula

Lemma 5.1. *If $|\nu| < m$ or $|\nu| - m$ is odd and $P(x)$ is a spherical harmonic of degree m, then*

$$\int_{\Sigma_{n-1}}(x')^\nu P(x')\,dx' = 0. \tag{5.1}$$

Proof. Let $|\nu| = k$ so that x^ν is a homogeneous polynomial of degree k. Thus we can write

$$x^\nu = P_0(x) + |x|^2P_1(x) + \cdots + |x|^{2l}P_l(x) \tag{5.2}$$

where P_j is a spherical harmonic of degree $k-2j$, $j=0,1,\ldots,l$; see [SW]. Therefore, since spherical harmonics are orthogonal on the unit sphere, we have the result.

Let $H_{\mu,m}(x)$ be a "cut" Hermite polynomial. That is, all terms of degree $\leq m-1$ are dropped. Then Lemma 5.1 shows that

$$\int_{\Sigma_{n-1}} H_\mu(rx')P(x')\,dx' = \int_{\Sigma_{n-1}} H_{\mu,m}(rx')P(x')\,dx'. \tag{5.3}$$

Theorem 5.2. *Let $P(x)$ be a spherical harmonic of degree m. Define for $k=0,1,\ldots$.*

$$F_k(x) = L_k^{\frac{n-2}{2}+m}(|x|^2)P(x)e^{-|x|^2/2}. \tag{5.4}$$

Then

$$\mathcal{P}_j F_k(x) = \begin{cases} F_k(x) & \text{if } j = 2k+m, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Proof. (For $m=0$ this follows from Corollary 3.3.) Note that each $F_k(x)$ can be written as a linear combination of Hermite functions of degree $\leq 2k+m$. Therefore the theorem will follow if we show that $\widehat{F}_k(\mu)=0$, $|\mu|<2k+m$. Consider the function $\psi_\mu(r)$ defined for $r>0$ by

$$\psi_\mu(r^2) = \frac{1}{r^m}\int_{\Sigma_{n-1}} H_\mu(ry')P_m(y')\,dy'. \tag{5.6}$$

By Lemma 5.1, if $|\mu|+m$ is odd or $|\mu|<m$, then $\psi_\mu(r^2)=0$. If $|\mu|=2j+m$, $j=0,1,\ldots$, then

$$\psi_\mu(r^2) = \frac{1}{r^m}\int_{\Sigma_{n-1}} H_{\mu,m}(ry')P_m(y')\,dy'. \tag{5.7}$$

In this case $\psi_\mu(r)$ is a polynomial of degree j so that if $j<k$ we have

$$\int_0^\infty \psi_\mu(r)L_k^{\frac{n-2}{2}+m}(r)e^{-r}r^{\frac{n-2}{2}+m}\,dr = 0. \tag{5.8}$$

Therefore, if $|\mu|<2k+m$ by switching to polar coordinates we have

$$\begin{aligned}\widehat{F}_k(\mu) &= \frac{1}{2^k\mu!\pi^{n/2}}\int_{R^n} L_k^{\frac{n-2}{2}+m}(|y|^2)P_m(y)H_\mu(y)e^{-|y|^2}\,dy \\ &= \frac{1}{2^k\mu!\pi^{n/2}}\int_0^\infty \psi_\mu(r^2)L_k^{\frac{n-2}{2}+m}(r^2)e^{-r^2}r^{n+2m-1}\,dr \\ &= 0. \end{aligned} \tag{5.9}$$

This completes the proof.

Theorem 5.3 (Thangavelu's formula)**.** *Define*

$$R_k^\delta(f_0) = 2\frac{k!}{\Gamma(k+\delta+1)}\int_0^\infty f_0(r)L_k^\delta(r^2)e^{-r^2/2}r^{2\delta+1}\,dr. \tag{5.10}$$

Then if $f(x)=f_0(|x|)P(x)\in L^2(R^n)$, then one has $\mathcal{P}_{2k+m}f(x)=F_k(|x|)P(x)$

where

$$F_k(r) = R_k^\delta(f_0) L_k^\delta(r^2) e^{-r^2/2} \tag{5.11}$$

with $\delta = \frac{n}{2} + m - 1$*. For other values of* j, $\mathcal{P}_j f = 0$.

Proof. Since $f \in L^2(R^n)$, we have $f_0(r) \in L^2((0,\infty); r^{2\delta+1}\,dr)$. Thus

$$f_0(r) = \sum_{i=0}^{\infty} R_i^\delta(f_0) L_i^\delta(r^2) e^{-r^2/2} \tag{5.12}$$

converging in the $L^2((0,\infty);\, r^{2\delta+1}\,dr)$ norm. This implies that

$$f(x) = \sum_{i=0}^{\infty} R_i^\delta(f_0) L_i^\delta(|x|^2) e^{-|x|^2/2} P(x) \tag{5.13}$$

converging in $L^2(R^n)$. But $\mathcal{P}_j$ are bounded operators in $L^2(R^n)$; therefore the result follows from Theorem 5.2.

6. A generalization of Uspensky's formula

Let $P_m^\lambda(t)$, $m = 0, 1, \ldots$, denote the ultraspherical polynomials of order λ. Then since for $u, x' \in \Sigma_{n-1}$

$$Z_u^{(m)}(x') = c_{m,n} P_m^{\frac{n-2}{2}+m}(u \cdot x') \tag{6.1}$$

where $Z_u^{(m)}(x')$ denotes the zonal harmonic of degree m and pole u (see [SW]), Thangavelu's formula could be proved from the following generalization of Uspensky's formula.

Theorem 6.1. *For* k, $m = 0, 1, \ldots, \lambda > -1$

$$\int_{-1}^{1} H_{2k+m}(rt) P_m^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}}\,dt = c_{m,k} r^m L_k^{\lambda+m}(r^2) \tag{6.2}$$

where

$$c_{m,k} = \frac{H_{2k+m}^{(m)}(0)\lambda_m}{m!\gamma_m L_k^{\lambda+m}(0)}, \tag{6.3}$$

$$\gamma_m = 2^m \frac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)} \tag{6.4}$$

is the coefficient of t^m *in* $P_m^\lambda(t)$ *and*

$$\lambda_m = \int_{-1}^{1} [P_m^\lambda(t)]^2 (1-t^2)^{\lambda-\frac{1}{2}}\,dt. \tag{6.5}$$

Proof. We use the Rodriguez formulas; see e.g. [Sz]. For simplicity, let $c_{m,k}$ below be a constant which may change with each usage. We have

$$\begin{aligned}
&\int_{-1}^{1} H_{2k+m}(rt)P_m^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}}\,dt \\
&\quad = c_{m,k}\int_{-1}^{1} H_{2k+m}(rt)\left(\left(\frac{d}{dt}\right)^m (1-t^2)^{\lambda+m-\frac{1}{2}}\right)dt \\
&\quad = c_{m,k}\int_{-1}^{1} \left(\left(\frac{d}{dt}\right)^m H_{2k+m}(rt)\right)(1-t^2)^{\lambda+m-\frac{1}{2}}\,dt \\
&\quad = c_{m,k}r^m\int_{-1}^{1} H_{2k}(rt)(1-t^2)^{\lambda+m-\frac{1}{2}}\,dt \qquad (6.6)\\
&\quad = c_{m,k}r^m L_k^{\lambda+m}(r^2).
\end{aligned}$$

Let $H_{j,m}(x)$ denote the jth Hermite polynomial with all terms of degree $\leq m-1$ dropped. Then

$$\int_{-1}^{1} H_{2k+m}(rt)P_m^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}}\,dt = \int_{-1}^{1} H_{2k+m,m}(rt)P_m^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}}\,dt \qquad (6.7)$$

so that $c_{m,k}$ can be calculated by dividing both sides in (6.6) by r^m and setting $r=0$.

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