

ROTATIONAL SYMMETRY OF THE HERMITE PROJECTION OPERATORS

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ABSTRACT. We calculate an integral formula for the Hermite projection operators. We give some applications of our formula. We also give a short proof of a recent theorem of Thangavelu

1. INTRODUCTION

In a recent paper ([T], Thangavelu) it was shown that if $P(x)$ is a solid spherical harmonic of degree m and if $f \in L^2(\mathbb{R}^n)$ is of the form $f(x) = f_0(|x|)P(x)$, then its projection $\mathcal{P}_k f(x)$ onto the space of Hermite functions of degree k is a Laguerre function multiplied by $P(x)$.

More specifically, let L_k^δ denote the Laguerre polynomials of type δ . Define

$$(1.1) \quad R_k^\delta(f_0) = 2 \frac{k!}{\Gamma(k + \delta + 1)} \int_0^\infty f_0(r) L_k^\delta(r^2) e^{-r^2/2} r^{2\delta+1} dr.$$

If $f(x) = f_0(|x|)P(x) \in L^2(\mathbb{R}^n)$, then one has $\mathcal{P}_{2k+m} f(x) = F_k(|x|)P(x)$ where

$$(1.2) \quad F_k(r) = R_k^\delta(f_0) L_k^\delta(r^2) e^{-r^2/2}$$

with $\delta = \frac{n}{2} + m - 1$. For other values of j , $\mathcal{P}_j f = 0$. The proof is based on generating functions.

Since Hermite functions are eigenfunctions of the Fourier transform and Laguerre functions are eigenfunctions of the Fourier-Bessel (Hankel) transform, Thangavelu's formula is closely related to the Hecke-Bochner identity which says that if $f \in L^1 \cap L^2(\mathbb{R}^n)$ such that $f(x) = f_0(|x|)P(x)$, then $\hat{f}(x) = F_0(|x|)P(x)$ where

$$(1.3) \quad \hat{f}(x) = \int_{\mathbb{R}^n} f(y) e^{-ix \cdot y} dy,$$

$$(1.4) \quad F_0(r) = (2\pi)^{n/2} i^{-m} r^{-(n/2+m-1)} \int_0^\infty f_0(s) J_{n/2+m-1}(rs) s^{n/2+m} ds$$

and J_α is the Bessel function of order α . For a proof of this identity see [SW].

One of the important properties of the Fourier transform is its symmetry with respect to rotations. Thangavelu's formula (1.2) suggests the same should be true for the projection operators \mathcal{P}_k . Our main result in this paper is an integral formula

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for the kernels of the projection operators which shows that the projection operators commute with rotations. We will apply our formula to give a simple proof of Thangavelu's formula in the radial case and to give an expression for Laguerre polynomials in terms of Hermite polynomials. We will also give a short proof of Thangavelu's formula which only uses L^2 theory for the Hermite and Laguerre polynomials and the homogeneity of the spherical harmonics.

2. DEFINITIONS

Define \mathcal{H}_m to be the space of all $f \in L^2(\mathbb{R}^n)$ of the form $f_0(|x|)P(x)$ where $P(x)$ is a (solid) spherical harmonic of degree m , i.e., $P(x)$ is a homogeneous harmonic polynomial of degree m . These spaces provide a direct sum decomposition of $L^2(\mathbb{R}^n)$ invariant under the Fourier transform; see [SW].

The Hermite polynomials orthogonal with respect to the weight $e^{-x^2} dx$ on \mathbb{R} are given for $k = 0, 1, \dots$ by

$$(2.1) \quad H_k(x) = (-1)^k \frac{d^k}{dx^k} (e^{-x^2}) e^{x^2} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^j k!}{j!(k-2j)!} (2x)^{k-2j}.$$

They satisfy the orthogonality relation

$$(2.2) \quad \int_{\mathbb{R}} H_j(x) H_k(x) e^{-x^2} dx = 2^k k! \sqrt{\pi} \delta_{j,k}.$$

The Hermite functions $H_k(x)e^{-x^2/2}$ are eigenfunctions of the Fourier transform:

$$(2.3) \quad H_k(x)e^{-x^2/2} = \frac{i^k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_k(y) e^{-y^2/2 - ixy} dy, \quad k = 0, 1, \dots$$

The Hermite polynomials on \mathbb{R}^n are defined for multi-indices $\mu = (\mu_1, \dots, \mu_n)$, $\mu_i = 0, 1, \dots$, as products of one-dimensional Hermite polynomials:

$$(2.4) \quad H_\mu(x) = H_{\mu_1}(x_1) H_{\mu_2}(x_2) \cdots H_{\mu_n}(x_n), \quad x = (x_1, \dots, x_n).$$

Note that if $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_n$ is even, then all terms in H_μ are of even degree. The Hermite functions $H_\mu(x)e^{-|x|^2/2}$ are orthogonal on \mathbb{R}^n :

$$(2.5) \quad \int_{\mathbb{R}^n} H_\mu(x) H_\nu(x) e^{-|x|^2} dx = 2^{|\mu|} \mu! \pi^{n/2} \delta_{\mu,\nu}$$

where

$$\mu! = \mu_1! \mu_2! \cdots \mu_n!.$$

They are eigenfunctions of the Fourier transform in \mathbb{R}^n .

A function $f \in L^2(\mathbb{R}^n)$ can be expanded in series of Hermite functions using the formula

$$(2.6) \quad f(x) \sim \sum_{k=0}^{\infty} \mathcal{P}_k f(x)$$

where $\mathcal{P}_k f$ are the Hermite projection operators

$$(2.7) \quad \mathcal{P}_k f(x) = \sum_{|\mu|=k} \hat{f}(\mu) H_\mu(x) e^{-|x|^2/2},$$

$$(2.8) \quad \hat{f}(\mu) = \frac{1}{2^k \mu! \pi^{n/2}} \int_{R^n} f(y) H_\mu(y) e^{-|y|^2/2} dy.$$

In other words,

$$(2.9) \quad \mathcal{P}_k f(x) = \int_{R^n} f(y) \Phi_k(x, y) dy$$

where

$$(2.10) \quad \Phi_k(x, y) = \frac{1}{2^k \pi^{n/2}} \sum_{|\mu|=k} \frac{H_\mu(x) H_\mu(y)}{\mu!} e^{-(|x|^2+|y|^2)/2}.$$

For many interesting theorems concerning expansions in terms of these polynomials, see [T2].

For $\alpha > -1$ the Laguerre polynomials $\{L_k^\alpha(x)\}$ of type α are defined by the formula

$$(2.11) \quad e^{-x} x^\alpha L_k^\alpha(x) = \frac{1}{k!} \frac{d^k}{dx^k} (e^{-x} x^{k+\alpha}) = \sum_{j=0}^k \frac{\Gamma(k + \alpha + 1)}{\Gamma(j + \alpha + 1)} \frac{(-x)^j}{j!(k - j)!}.$$

The Laguerre polynomials satisfy the orthogonality

$$(2.12) \quad \int_0^\infty L_k^\alpha(x) L_j^\alpha(x) e^{-x} x^\alpha dx = \frac{\Gamma(k + \alpha + 1)}{k!} \delta_{k,j}.$$

Note that this implies that $|x|^m L_k^{\frac{n-2}{2}+m}(|x|^2) e^{-|x|^2/2}$, $k = 0, 1, \dots$, are orthogonal on R^n for each m .

The set consisting of the functions $L_k^\alpha(2x) x^{\alpha/2} e^{-x}$ are eigenfunctions of the Hankel transform (see e.g. [Sz]):

$$(2.13) \quad L_k^\alpha(2x) x^{\alpha/2} e^{-x} = \int_0^\infty J_\alpha(2\sqrt{xy}) L_k^\alpha(2y) y^{\alpha/2} e^{-y} dy, \quad k = 0, 1, \dots$$

This implies by the Hecke-Bochner identity that the functions

$$(2.14) \quad L_k^{\frac{n-2}{2}+m}(|x|^2) P(x) e^{-|x|^2/2}, \quad k = 0, 1, \dots,$$

which span the space \mathcal{H}_m are eigenfunctions of the Fourier transform in R^n .

3. A FORMULA FOR THE PROJECTION OPERATOR

Lemma 3.1. *Let $u = (u_1, u_2, \dots, u_n)$ be a unit vector in R^n , and let $x \in R^n$. Then for $k = 0, 1, \dots$*

$$(3.1) \quad H_k(x \cdot u) = \sum_{|\mu|=k} \binom{k}{\mu} H_\mu(x) u^\mu$$

where $u^\mu = u_1^{\mu_1} u_2^{\mu_2} \dots u_n^{\mu_n}$ and $\binom{k}{\mu}$ is the multinomial index $k!/\mu!$.

Proof. For $n = 2$ this formula is well known:

$$(3.2) \quad H_k(x \cos \theta + y \sin \theta) = \sum_{j=0}^k \binom{k}{j} H_j(x) H_{k-j}(y) \cos^j \theta \sin^{k-j} \theta.$$

The more general formula is proved inductively. Let u be a unit vector in R^{n+1} . Then we can write

$$u = (\cos \theta, (\sin \theta)v_1, (\sin \theta)v_2, \dots, (\sin \theta)v_n)$$

where $v = (v_1, v_2, \dots, v_n)$ is a unit vector in R^n . Let $x = (x_1, x_2, \dots, x_{n+1}) \in R^{n+1}$. Let $y = (x_2, x_3, \dots, x_{n+1}) \in R^n$. Then

$$x \cdot u = x_1 \cos \theta + y \cdot v \sin \theta$$

so that we have

$$\begin{aligned} H_k(x \cdot u) &= \sum_{j=0}^k \binom{k}{j} H_j(x_1) H_{k-j}(y \cdot v) \cos^j \theta \sin^{k-j} \theta \\ &= \sum_{j=0}^k \binom{k}{j} H_j(x_1) \sum_{|\nu|=k-j} \binom{k-j}{\nu} H_\nu(y) v^\nu \cos^j \theta \sin^{k-j} \theta \\ &= \sum_{|\mu|=k} \binom{k}{\mu} H_\mu(x) u^\mu. \end{aligned}$$

Theorem 3.2. For $k = 0, 1, \dots$

$$(3.3) \quad \Phi_k(x, y) = \frac{(-i)^k e^{(|x|^2 - |y|^2)/2}}{\pi^n k!} \int_{R^n} |s|^k H_k\left(y \cdot \frac{s}{|s|}\right) e^{-|s|^2 + 2is \cdot x} ds.$$

Proof. It is well known (see e.g. [Sz]) that

$$(3.4) \quad H_k(x) = \frac{2^k (-i)^k e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} s^n e^{-s^2 + 2isx} ds, \quad k = 0, 1, \dots$$

It easily follows for $|\mu| = k$ that

$$(3.5) \quad H_\mu(x) = \frac{2^k (-i)^k e^{|x|^2}}{\pi^{n/2}} \int_{R^n} s^\mu e^{-|s|^2 + 2is \cdot x} ds.$$

Therefore we have

$$\begin{aligned} \sum_{|\mu|=k} \binom{k}{\mu} H_\mu(x) H_\mu(y) &= \sum_{|\mu|=k} \binom{k}{\mu} \left(\frac{2^k (-i)^k e^{|x|^2}}{\pi^{n/2}} \int_{R^n} s^\mu e^{-|s|^2 + 2is \cdot x} ds \right) H_\mu(y) \\ &= \frac{2^k (-i)^k e^{|x|^2}}{\pi^{n/2}} \int_{R^n} \left\{ \sum_{|\mu|=k} \binom{k}{\mu} H_\mu(y) s^\mu \right\} e^{-|s|^2 + 2is \cdot x} ds \\ (3.6) \quad &= \frac{2^k (-i)^k e^{|x|^2}}{\pi^{n/2}} \int_{R^n} |s|^k H_k\left(y \cdot \frac{s}{|s|}\right) e^{-|s|^2 + 2is \cdot x} ds. \end{aligned}$$

Finally since

$$(3.7) \quad \Phi_k(x, y) = \frac{e^{-(|x|^2 + |y|^2)/2}}{2^k k! \pi^{n/2}} \sum_{|\mu|=k} \binom{k}{\mu} H_\mu(x) H_\mu(y),$$

we obtain the result.

Note that for $n = 1$ (3.3) reduces to $H_k(x)H_k(y)e^{-(x^2+y^2)/2}/2^k\sqrt{\pi}k!$ as expected.

Using $H_{2k}(0) = (-1)^k(2k)!/k!$ and the Hecke-Bochner formula we obtain

$$\begin{aligned}
 \Phi_{2k}(x, 0) &= \frac{e^{|x|^2/2}}{\pi^n k!} \int_{R^n} e^{2is \cdot x} |s|^{2k} e^{-|s|^2} ds \\
 &= \frac{2e^{|x|^2/2}}{\pi^{n/2} k! |x|^{\frac{n-2}{2}}} \int_0^\infty J_{\frac{n-2}{2}}(2r|x|) r^{2k+\frac{n}{2}} e^{-r^2} dr \\
 (3.8) \qquad &= \frac{e^{-|x|^2/2}}{\pi^{n/2}} L_k^{\frac{n-2}{2}}(|x|^2)
 \end{aligned}$$

where we have used

$$(3.9) \qquad L_m^\nu(x) = \frac{e^x x^{-\nu/2}}{m!} \int_0^\infty s^{m+\nu/2} J_\nu(2\sqrt{xs}) e^{-s} ds,$$

see e.g. [Sz]. On the other hand,

$$(3.10) \qquad \Phi_{2k}(x, 0) = \frac{e^{-|x|^2/2}}{2^{2k} \pi^{n/2}} \sum_{|\mu|=2k} \frac{H_\mu(0)}{\mu!} H_\mu(x).$$

Therefore we obtain

Corollary 3.3. *For $k = 0, 1, \dots$*

$$(3.11) \qquad L_k^{\frac{n-2}{2}}(|x|^2) = \frac{1}{2^{2k}} \sum_{|\mu|=2k} \frac{H_\mu(0)}{\mu!} H_\mu(x).$$

Note that $\Phi_{2k+1}(x, 0) = 0$ for all x and $k = 0, 1, \dots$

Theorem 3.4. *Let $R_\rho f(x) = f(\rho x)$ where ρ is a rotation. Then for $k = 0, 1, \dots$*

$$(3.12) \qquad \mathcal{P}_k R_\rho f(x) = R_\rho \mathcal{P}_k f(x).$$

Proof. This follows from the observation that

$$(3.13) \qquad \Phi_k(\rho x, \rho y) = \Phi_k(x, y), \qquad k = 0, 1, \dots,$$

which follows easily from (3.3).

4. A PROOF OF THANGAVELU'S FORMULA IN THE RADIAL CASE

Theorem 3.2 leads to a proof of Thangavelu's formula in the radial case. One can evaluate the integral

$$(4.1) \qquad \int_{\Sigma_{n-1}} H_{2k}(x \cdot u') du'$$

by first integrating over the parallel

$$L_\theta = \{u' \in \Sigma_{n-1} : x' \cdot u' = \cos\theta\}$$

and then integrating over θ , $0 \leq \theta \leq \pi$. Then since $\omega_{n-2} \sin^{n-2} \theta$ is the measure of L_θ , we have

$$\begin{aligned}
 \int_{\Sigma_{n-1}} H_{2k}(x \cdot u') \, du' &= \omega_{n-2} \int_0^\pi H_{2k}(r \cos \theta) \sin^{n-2} \theta \, d\theta \\
 &= \omega_{n-2} \int_{-1}^1 H_{2k}(rt)(1-t^2)^{\frac{n-3}{2}} \, dt \\
 (4.2) \qquad &= \frac{2\pi^{n/2}(-1)^k(2k)!}{\Gamma(k + \frac{n}{2})} L_k^{\frac{n-2}{2}}(r^2)
 \end{aligned}$$

where in the last step we have used the well-known formula of Uspenski (see [Sz]):

$$(4.3) \qquad L_k^\alpha(x) = \frac{(-1)^k \pi^{-1/2} \Gamma(k + \alpha + 1)}{\Gamma(\alpha + \frac{1}{2})(2k)!} \int_{-1}^1 H_{2k}(\sqrt{x}t)(1-t^2)^{\alpha-\frac{1}{2}} \, dt, \quad \alpha > -\frac{1}{2}.$$

Letting $|y| = r$ and $|x| = \gamma$ we obtain

$$\begin{aligned}
 \int_{\Sigma_{n-1}} \Phi_{2k}(x, y) \, dy' &= \frac{(-1)^k e^{(r^2-\gamma^2)/2}}{\pi^n(2k)!} \\
 &\quad \cdot \int_{R^n} |s|^{2k} \left\{ \int_{\Sigma_{n-1}} H_{2k}(ry' \cdot s') \, dy' \right\} e^{-|s|^2+2is \cdot x} \, ds \\
 &= \frac{2k!}{\Gamma(k + \frac{n}{2})} L_k^{\frac{n-2}{2}}(r^2) L_k^{\frac{n-2}{2}}(\gamma^2) e^{-(r^2+\gamma^2)/2}.
 \end{aligned}$$

Therefore if $f(x) = f_0(|x|) \in L^2(R^n)$ we obtain Thangavelu’s formula:

$$\begin{aligned}
 \mathcal{P}_{2k}f(x) &= \int_{R^n} f(y)\Phi_{2k}(x, y) \, dy \\
 &= \int_0^\infty f_0(r) \left\{ \int_{\Sigma_{n-1}} \Phi_{2k}(x, ry') \, dy' \right\} r^{n-1} \, dr \\
 (4.4) \qquad &= R_k^{\frac{n-2}{2}}(f_0) L_k^{\frac{n-2}{2}}(\gamma^2) e^{-\gamma^2/2}.
 \end{aligned}$$

It is easy to see that $\mathcal{P}_{2k+1}f(x) = 0$.

Uspensky’s formula can be generalized to give a proof of Thangavelu’s formula in the general case. However, the constants become very unwieldy. At any rate, the generalization is given in Section 6.

5. A SHORT PROOF OF THANGAVELU’S FORMULA

Lemma 5.1. *If $|\nu| < m$ or $|\nu| - m$ is odd and $P(x)$ is a spherical harmonic of degree m , then*

$$(5.1) \qquad \int_{\Sigma_{n-1}} (x')^\nu P(x') \, dx' = 0.$$

Proof. Let $|\nu| = k$ so that x^ν is a homogeneous polynomial of degree k . Thus we can write

$$(5.2) \qquad x^\nu = P_0(x) + |x|^2 P_1(x) + \dots + |x|^{2l} P_l(x)$$

where P_j is a spherical harmonic of degree $k - 2j$, $j = 0, 1, \dots, l$; see [SW]. Therefore, since spherical harmonics are orthogonal on the unit sphere, we have the result.

Let $H_{\mu,m}(x)$ be a “cut” Hermite polynomial. That is, all terms of degree $\leq m - 1$ are dropped. Then Lemma 5.1 shows that

$$(5.3) \quad \int_{\Sigma_{n-1}} H_{\mu}(rx')P(x') dx' = \int_{\Sigma_{n-1}} H_{\mu,m}(rx')P(x') dx'.$$

Theorem 5.2. *Let $P(x)$ be a spherical harmonic of degree m . Define for $k = 0, 1, \dots$*

$$(5.4) \quad F_k(x) = L_k^{\frac{n-2}{2}+m}(|x|^2)P(x)e^{-|x|^2/2}.$$

Then

$$(5.5) \quad \mathcal{P}_j F_k(x) = \begin{cases} F_k(x) & \text{if } j = 2k + m, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (For $m = 0$ this follows from Corollary 3.3.) Note that each $F_k(x)$ can be written as a linear combination of Hermite functions of degree $\leq 2k + m$. Therefore the theorem will follow if we show that $\widehat{F}_k(\mu) = 0$, $|\mu| < 2k + m$. Consider the function $\psi_{\mu}(r)$ defined for $r > 0$ by

$$(5.6) \quad \psi_{\mu}(r^2) = \frac{1}{r^m} \int_{\Sigma_{n-1}} H_{\mu}(ry')P_m(y') dy'.$$

By Lemma 5.1, if $|\mu| + m$ is odd or $|\mu| < m$, then $\psi_{\mu}(r^2) = 0$. If $|\mu| = 2j + m$, $j = 0, 1, \dots$, then

$$(5.7) \quad \psi_{\mu}(r^2) = \frac{1}{r^m} \int_{\Sigma_{n-1}} H_{\mu,m}(ry')P_m(y') dy'.$$

In this case $\psi_{\mu}(r)$ is a polynomial of degree j so that if $j < k$ we have

$$(5.8) \quad \int_0^{\infty} \psi_{\mu}(r)L_k^{\frac{n-2}{2}+m}(r)e^{-r}r^{\frac{n-2}{2}+m} dr = 0.$$

Therefore, if $|\mu| < 2k + m$ by switching to polar coordinates we have

$$(5.9) \quad \begin{aligned} \widehat{F}_k(\mu) &= \frac{1}{2^k \mu! \pi^{n/2}} \int_{R^n} L_k^{\frac{n-2}{2}+m}(|y|^2)P_m(y)H_{\mu}(y)e^{-|y|^2} dy \\ &= \frac{1}{2^k \mu! \pi^{n/2}} \int_0^{\infty} \psi_{\mu}(r^2)L_k^{\frac{n-2}{2}+m}(r^2)e^{-r^2}r^{n+2m-1} dr \\ &= 0. \end{aligned}$$

This completes the proof.

Theorem 5.3 (Thangavelu’s formula). *Define*

$$(5.10) \quad R_k^{\delta}(f_0) = 2 \frac{k!}{\Gamma(k + \delta + 1)} \int_0^{\infty} f_0(r)L_k^{\delta}(r^2)e^{-r^2/2}r^{2\delta+1} dr.$$

Then if $f(x) = f_0(|x|)P(x) \in L^2(R^n)$, then one has $\mathcal{P}_{2k+m}f(x) = F_k(|x|)P(x)$

where

$$(5.11) \quad F_k(r) = R_k^\delta(f_0)L_k^\delta(r^2)e^{-r^2/2}$$

with $\delta = \frac{n}{2} + m - 1$. For other values of j , $\mathcal{P}_j f = 0$.

Proof. Since $f \in L^2(\mathbb{R}^n)$, we have $f_0(r) \in L^2((0, \infty); r^{2\delta+1} dr)$. Thus

$$(5.12) \quad f_0(r) = \sum_{i=0}^{\infty} R_i^\delta(f_0)L_i^\delta(r^2)e^{-r^2/2}$$

converging in the $L^2((0, \infty); r^{2\delta+1} dr)$ norm. This implies that

$$(5.13) \quad f(x) = \sum_{i=0}^{\infty} R_i^\delta(f_0)L_i^\delta(|x|^2)e^{-|x|^2/2}P(x)$$

converging in $L^2(\mathbb{R}^n)$. But \mathcal{P}_j are bounded operators in $L^2(\mathbb{R}^n)$; therefore the result follows from Theorem 5.2.

6. A GENERALIZATION OF USPENSKY'S FORMULA

Let $P_m^\lambda(t)$, $m = 0, 1, \dots$, denote the ultraspherical polynomials of order λ . Then since for $u, x' \in \Sigma_{n-1}$

$$(6.1) \quad Z_u^{(m)}(x') = c_{m,n}P_m^{\frac{n-2}{2}+m}(u \cdot x')$$

where $Z_u^{(m)}(x')$ denotes the zonal harmonic of degree m and pole u (see [SW]), Thangavelu's formula could be proved from the following generalization of Uspensky's formula.

Theorem 6.1. For $k, m = 0, 1, \dots, \lambda > -1$

$$(6.2) \quad \int_{-1}^1 H_{2k+m}(rt)P_m^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}} dt = c_{m,k}r^m L_k^{\lambda+m}(r^2)$$

where

$$(6.3) \quad c_{m,k} = \frac{H_{2k+m}^{(m)}(0)\lambda_m}{m!\gamma_m L_k^{\lambda+m}(0)},$$

$$(6.4) \quad \gamma_m = 2^m \frac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)}$$

is the coefficient of t^m in $P_m^\lambda(t)$ and

$$(6.5) \quad \lambda_m = \int_{-1}^1 [P_m^\lambda(t)]^2(1-t^2)^{\lambda-\frac{1}{2}} dt.$$

Proof. We use the Rodriguez formulas; see e.g. [Sz]. For simplicity, let $c_{m,k}$ below be a constant which may change with each usage. We have

$$\begin{aligned}
 & \int_{-1}^1 H_{2k+m}(rt) P_m^\lambda(t) (1-t^2)^{\lambda-\frac{1}{2}} dt \\
 &= c_{m,k} \int_{-1}^1 H_{2k+m}(rt) \left(\left(\frac{d}{dt} \right)^m (1-t^2)^{\lambda+m-\frac{1}{2}} \right) dt \\
 &= c_{m,k} \int_{-1}^1 \left(\left(\frac{d}{dt} \right)^m H_{2k+m}(rt) \right) (1-t^2)^{\lambda+m-\frac{1}{2}} dt \\
 (6.6) \quad &= c_{m,k} r^m \int_{-1}^1 H_{2k}(rt) (1-t^2)^{\lambda+m-\frac{1}{2}} dt \\
 &= c_{m,k} r^m L_k^{\lambda+m}(r^2).
 \end{aligned}$$

Let $H_{j,m}(x)$ denote the j th Hermite polynomial with all terms of degree $\leq m-1$ dropped. Then

$$(6.7) \quad \int_{-1}^1 H_{2k+m}(rt) P_m^\lambda(t) (1-t^2)^{\lambda-\frac{1}{2}} dt = \int_{-1}^1 H_{2k+m,m}(rt) P_m^\lambda(t) (1-t^2)^{\lambda-\frac{1}{2}} dt$$

so that $c_{m,k}$ can be calculated by dividing both sides in (6.6) by r^m and setting $r=0$.

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