COMPOSITION OF BLOCHS
WITH BOUNDED ANALYTIC FUNCTIONS

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Abstract. If \( f \) is a holomorphic self-map of the open unit disc and \( 1 \leq p < \infty \), then the following are equivalent.

1. \( h \circ f \in H^{2p} \) for all Bloch functions \( h \).

2. \[ \sup_r \int_0^{2\pi} \left( \log \frac{1}{1 - |f(re^{i\theta})|^2} \right)^p \, d\theta < \infty. \]

3. \[ \int_0^{2\pi} \left( \int_0^1 (f^#(re^{i\theta})(1 - r)dr \right)^p \, d\theta < \infty, \]

where \( f^# \) is the hyperbolic derivative of \( f \): \( f^# = |f'|/(1 - |f|^2) \).

1. Introduction

Initiated by works of P. Ahern and W. Rudin ([A], [AR]), there is extensive research on Bloch-to-BMOA pullbacks, that is, research on those holomorphic maps \( f \) of the unit ball of \( \mathbb{C}^n \) into the unit disc of \( \mathbb{C} \) for which the composition operator defined by

\[ C_f(h) = h \circ f \]

takes Bloch functions to functions of BMOA. We refer to [RU] for recent research on Bloch to BMOA pullbacks.

It is known (see [RU], Section 5), when \( n = 1 \), that one of the necessary and sufficient conditions for \( C_f \) to take all Blochs to BMOA is that \( f \) be a function of \( \text{BMOA}_\sigma \), the Yamashita hyperbolic BMOA class (see [G] and [Y3] for BMOA and \( \text{BMOA}_\sigma \)). This note is to resolve a natural parallelism to the phenomenon.

Theorem 1 (Main Result). If \( f \) is a holomorphic self-map of the open unit disc and \( 1 \leq p < \infty \), then the following are equivalent.

1. \( C_f \) takes Blochs to \( H^{2p} \), that is,

\[ h \circ f \in H^{2p} \quad \text{for all Bloch functions } h. \]
(2) $f$ belongs to Yamashita's hyperbolic Hardy class $H^p_\sigma$, that is,
\[
\sup_r \int_0^{2\pi} \left( \log \frac{1}{1 - |f(re^{i\theta})|^2} \right)^p \, d\theta < \infty.
\]

(3) \[
\int_0^{2\pi} \left( \int_0^1 (f^\#(re^{i\theta})(1-r)) \right)^p \, d\theta < \infty,
\]
where $f^\#$ is the hyperbolic derivative of $f$: $f^\# = |f'|/(1 - |f|^2)$.

2. Preliminaries

The Bloch space $B$ consists of those $f$ holomorphic in the open unit disc $D$ of the complex plane for which
\[
\|f\|_B := \sup_{z \in D} |f'(z)|(1 - |z|^2) < \infty.
\]
We let $1 \leq p < \infty$ and set for $f$ subharmonic in $D$
\[
\|f\|_p := \sup_r \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}.
\]

Then $H^p = H^p(D)$ consists of those $f$ holomorphic in $D$ for which $\|f\|_p < \infty$. See [D] and [G] for Bloch and $H^p$ spaces.

The Yamashita hyperbolic Hardy class $H^p_\sigma$ is defined as the set of those holomorphic self-maps $f$ of $D$ for which
\[
\|\sigma(f)\|_p < \infty,
\]
where $\sigma(z)$ denotes the hyperbolic distance of $z$ and $0$ in $D$, namely,
\[
\sigma(z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}.
\]

Though $H^p_\sigma$ is not a linear space, it has, as hyperbolic counterparts, many properties analogous to those of $H^p$. We let $T$ be the boundary of $D$ and set, following Yamashita,
\[
\lambda(f) = \log \frac{1}{1 - |f|^2} \quad \text{and} \quad f^\# = \frac{|f'|}{1 - |f|^2}
\]
for the holomorphic self-map $f$ of $D$. Then $\sigma(f)^p$, $\lambda(f)^p$, and $(f^\#)^p$ are subharmonic functions, so that their integral means over $rT$ are nondecreasing functions of $r$: for example,
\[
\int_0^{2\pi} \lambda(f)^p(re^{i\theta}) \frac{d\theta}{2\pi} \nearrow \|\lambda(f)\|_p^p \quad \text{as} \quad r \nearrow 1.
\]

Also, there are corresponding maximal theorems for these functions: Set
\[
M_\lambda(f, \theta) = \sup \{ \lambda(f)(re^{i\theta}) : 0 \leq r < 1 \}.
\]
then
\begin{equation}
\|M_\lambda(f,.)\|_{L^p} \leq C_p \|\lambda(f)\|_p
\end{equation}
for \( f \in H^p_\sigma \). The left side of (2.1) means usual \( L^p(T) \) norm. The function \( f^\# \) is the hyperbolic counterpart of \( f' \) and it easily follows that
\begin{equation}
\frac{1}{2} \lambda(f) \leq \sigma(f) < \frac{1}{2} \lambda(f) + \log 2
\end{equation}
and
\begin{equation}
\Delta (\lambda(f)^p) = 4p \{(p-1)|f|^2 + \lambda(f)\} \lambda(f)^{p-2}(f^\#)^2,
\end{equation}
where \( \Delta \) denotes the Laplacian:
\[ \Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}. \]
From (2.2) and (2.3), it should be noted that
\( f \in H^p_\sigma \) if and only if \( \|\lambda(f)\|_p < \infty \) and
\begin{equation}
\Delta (\lambda(f)^p) \sim \lambda(f)^{p-1}(f^\#)^2.
\end{equation}
Here and after \( \psi \sim \phi \) means that either both sides are zero or the quotient \( \psi/\phi \) lies between two positive uniform constants. See, for example, [K1], [K2], [Y1], and [Y2] for the theory of \( H^p_\sigma \).

In the remaining sections, we confine \( f \) to being a holomorphic self-map of \( D \) and denote \( f_r, 0 \leq r \leq 1, \) as the function defined by \( f_r(z) = f(rz), z \in D. \) Positive constants depending on \( p \) (or \( q \)) will be denoted by \( C_p \) (or \( C_q \)), whose quantities may vary at each occurrence.

3. Hyperbolic \( g \)-function

For \( h \) holomorphic in \( D, g \)-function of Paley defined by
\begin{equation}
g(\theta) := g(h)(\theta) = \left( \int_0^1 |h'|^2(re^{i\theta})(1-r)dr \right)^{1/2}, \quad 0 \leq \theta < 2\pi,
\end{equation}
satisfies \( \|g(h)\|_{L^p} \sim \|h\|_p \) if \( h(0) = 0 \) ([Z]). Consider Green’s theorem of the form
\[ r \int_0^{2\pi} \frac{\partial \psi}{\partial r} d\theta = \int \int_{|z| \leq r} \Delta \psi \, dx \, dy, \]
valid for \( \psi \in C^2(D) \). If we integrate both sides with respect to \( dr \) after dividing them by \( r \) and applying \( \psi = \lambda(f)^p \), then we obtain, by use of (2.4),
\begin{equation}
\|\lambda(f_\rho)^p - \lambda(f)^p(0) \|
\sim \frac{1}{2\pi} \int_0^\rho \frac{dr}{r} \int \int_{|z| \leq r} \lambda(f)^{p-1}(f^\#)^2(z) \, dx \, dy
\end{equation}
\begin{equation}
= \frac{1}{2\pi} \int \int_{|z| < \rho} \lambda(f)^{p-1}(f^\#)^2(z) \log \frac{\rho}{|z|} \, dx \, dy, \quad 0 \leq \rho \leq 1.
\end{equation}
In particular, we see from (3.2) that \( f \in H_2^1 \) if and only if
\[
\infty > \int \int_D (f^\#)^2(z) \log \frac{1}{|z|} \, dx \, dy.
\]
This suggests we define the hyperbolic version of \( g \)-function using \( f^\# \). We define
\[
(3.3) \quad g_\sigma(\theta) := g_\sigma(f)(\theta) = \int_0^1 (f^\#)^2(re^{i\theta})(1-r) \, dr, \quad 0 \leq \theta < 2\pi.
\]
It is not surprising to see the absence of the square root in the definition of \( g_\sigma \) in (3.3) when we compare it to that of \( g \)-function in (3.1), because there is a known parallelism (see [Y2]) between \( H^2 \) and \( H_2^1 \).

**Theorem 2.** If \( 1 \leq p < \infty \), then the following are equivalent.

1. \( f \in H_p^\sigma \).
2. \( g_\sigma (f) \in L^p(T) \).

In fact, \( \|\lambda(f)\|_p \approx \|g_\sigma(f)\|_{L^p} \) provided \( f(0) = 0 \).

**Proof.** By (3.2) and (3.3), there is nothing to prove when \( p = 1 \). We assume \( 1 < p < \infty \), and let \( \frac{1}{p} + \frac{1}{q} = 1 \).

(1) \( \implies \) (2) We begin with the identity
\[
\|g_\sigma\|_{L^p} = \sup \int_0^{2\pi} g_\sigma(\theta)h(\theta) \frac{d\theta}{2\pi},
\]
where the supremum is taken with respect to all nonnegative trigonometric polynomials \( h \) with \( \|h\|_{L^q} \leq 1 \). Since \( (f^\#)^2 \) is subharmonic, we have
\[
(3.4) \quad (f^\#)^2(r^2e^{i\theta}) \leq \int_0^{2\pi} P(r, \theta - t) (f^\#)^2(re^{it}) \frac{dt}{2\pi}, \quad 0 \leq r < 1,
\]
where \( P(r, \theta) \) is the Poisson kernel:
\[
P(r, \theta) = \frac{1 - r^2}{1 - 2rcos\theta + r^2}.
\]
Let \( u \) be the Poisson integral of \( h \). Then
\[
\int_0^{2\pi} g_\sigma(\theta)h(\theta) \, d\theta
= \int_0^{2\pi} \int_0^1 (f^\#)^2(re^{i\theta}) \ h(\theta) \ (1-r) \, dr \, d\theta
\]
\[
= \int_0^{2\pi} \int_0^1 (f^\#)^2(r^2e^{i\theta}) \ h(\theta) \ (1-r^2) \ 2r \, dr \, d\theta
\]
\[
\leq 2 \int \int_D (f^\#)^2(z) \ u(z) \ (1 - |z|^2) \, dx \, dy
\]
\[
\leq 4 \int \int_D (f^\#)^2(z) \ u(z) \ log \frac{1}{|z|} \, dx \, dy,
\]
where we changed the order of integration and used (3.4) in the first inequality.

If we denote \( \partial = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}) \) and \( \bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}) \), \( z = x + iy \), then it follows from elementary calculation and (2.3) that

\[
4(f^#)^2 u = \Delta(\lambda(f)u) - 4(\partial\lambda(f)\bar{\partial}u + \bar{\partial}\lambda(f)\partial u),
\]

so that by (3.5) we have

\[
\begin{align*}
(3.6) \quad & \int_0^{2\pi} g_\sigma(\theta)h(\theta) d\theta \\
& \leq \left| \int_D \Delta(\lambda(f)u)(z) \log \frac{1}{|z|} \, dx \, dy \right| \\
& \quad + 4 \int_D |\partial\lambda(f)\bar{\partial}u + \bar{\partial}\lambda(f)\partial u|(z) \log \frac{1}{|z|} \, dx \, dy \\
& = (I) + (II).
\end{align*}
\]

Now, using Green’s theorem (as in (3.2) with \( p = 1 \)) with limiting process and Hölder’s inequality, we obtain

\[
(3.7) \quad (I) = \left| \lim_{\rho \to 1} \int_0^{2\pi} (\lambda(f)u)(\rho e^{i\theta}) \, d\theta - 2\pi(\lambda(f)u)(0) \right| \\
\leq 2\pi \|\lambda(f)\|_p \|u\|_q \leq 2\pi \|\lambda(f)\|_p.
\]

On the other hand, if we let \( \phi \) be a holomorphic function in \( D \) whose real part is \( u \), it then follows from direct differentiation that

\[
|\partial\lambda(f)| = |\bar{\partial}\lambda(f)| = |f|f^#
\]

and

\[
|\partial u| = |\bar{\partial} u| = \frac{1}{2} |\phi'|.
\]

Hence

\[
(II) \leq 4 \int_D |\phi'(z)| \, |f(z)| \, f^#(z) \log \frac{1}{|z|} \, dx \, dy \\
\leq 4 \int_0^{2\pi} \int_0^1 |\phi'(re^{i\theta})| \, |f(re^{i\theta})| \, f^#(re^{i\theta})(1 - r) \, dr \, d\theta.
\]

Since \( |f(re^{i\theta})| \leq \sqrt{\lambda(f)(re^{i\theta})} \leq M_\lambda^{1/2}(f, \theta) \), we have, by the Schwarz inequality,

\[
(II) \leq 4 \int_0^{2\pi} M_\lambda^{1/2}(f, \theta) \sqrt{g_\sigma(\theta)} \, g_\phi(\theta) \, d\theta,
\]

where \( g_\phi(\theta) \) is Paley \( g \)-function of \( \phi \). Applying Hölder’s inequality with indices \( 2p, 2p, q \) to the right side of (3.8) and using maximal theorem (2.1), we arrive at

\[
(II) \leq C_p \|\lambda(f)\|_p^{1/2} \|g_\sigma(f)\|_{L^{2p}}^{1/2} \|g_\phi\|_{L^q}.
\]
From the theory of \( g \)-function, we know \( \| g_0 \|_{L^q} \leq C_q \| \phi \|_q \), and it follows from the theorem of M. Riesz ([Z]) that \( \| \phi \|_q \leq C_q \| u \|_q \leq C_q \). Thus
\[
(3.9)
\]

Hence, combining estimates (3.6), (3.7), and (3.9), we have
\[
\int_0^{2\pi} g_\sigma(\theta) h(\theta) d\theta \leq (I) + (II) \leq 2\pi \| \lambda(f) \|_p + C_p \| \lambda(f) \|_p^{1/2} \| g_\sigma(f) \|_{L^p}^{1/2},
\]
for all positive trigonometric polynomials \( h \) with \( \| h \|_q \leq 1 \). Therefore we obtain
\[
(3.10) \quad \| g_\sigma(f) \|_{L^p} \leq \| \lambda(f) \|_p + C_p \| \lambda(f) \|_p^{1/2} \| g_\sigma(f) \|_{L^p}^{1/2}, \quad f \in H^p.
\]
Now we conclude from (3.10) that
\[
(3.11) \quad \| g_\sigma(f) \|_{L^p} \leq C_p \| \lambda(f) \|_p.
\]
In fact, if \( f \equiv 0 \), then there is nothing to prove; otherwise, setting \( X(r) = X(f, p, r) = (\| g_\sigma(f) \|_{L^p})/\| \lambda(f) \|_p \) \), \( 0 < r < 1 \), (3.10) with \( f_r \) in place of \( f \) becomes
\[
X^2(r) \leq 1 + C_p X(r),
\]
and this means, by comparing the order of \( X(r) \), that \( X(r), 0 < r < 1 \), does not exceed the larger root of the equation \( X^2 = 1 + C_p X \). This proves (3.11) with \( f_r \), \( 0 \leq r < 1 \), in place of \( f \), and so (3.11) follows by the monotonicity of both sides.

(2) \( \implies \) (1) It follows from (3.2) that
\[
\| \lambda(f_r) \|_p - \lambda(f)^p(0) \sim C_p \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^r \lambda(f)^{p-1}(pe^{i\theta}) \log \rho \rho d\rho
\]

Hölder’s inequality and the maximal theorem show the last quantity to be bounded by
\[
C_p \| \lambda(f_r) \|_p^{p-1} \| g_\sigma(f) \|_{L^p},
\]
so that we have
\[
(3.12) \quad \| \lambda(f_r) \|_p - \lambda(f)^p(0) \leq C_p \| \lambda(f_r) \|_p^{p-1} \| g_\sigma(f) \|_{L^p}.
\]
If \( g_\sigma \equiv 0 \), there remains nothing to prove. Otherwise, under the assumption \( 0 < \| g_\sigma \|_{L^p} < \infty \), by considering the order of
\[
Y(r) = Y(f, p, r) = (\| \lambda(f_r) \|_p/\| g_\sigma(f_r) \|_{L^p}), \quad 0 < r < 1,
\]
we conclude that \( Y(r), 0 < r < 1 \), does not exceed the largest root of the equation
\[
Y^p - \frac{\lambda(f)^p(0)}{\| g_\sigma(f) \|_{L^p}^{p-1}} = C_p Y^{p-1},
\]
and from this follows
\[
\| \lambda(f) \|_p \leq C_p \| g_\sigma(f) \|_{L^p}.
\]
Here, \( C_p(f) \) denotes a constant depending on \( p \) and \( f \).

The last assertion of Theorem 2 follows from (3.11) and (3.12).
4. Bloch to $H^p$ pullbacks

Now we prove our main theorem, Theorem 1. By the aid of Theorem 2, it suffices to prove the following.

**Theorem 3.** If $1 \leq p < \infty$, then the following are equivalent.

1. $g_\sigma(f) \in L^p(T)$.
2. $C_f$ takes Blochs to $H^{2p}$.

**Proof.** (1) $\implies$ (2) Let $h \in \mathcal{B}$. Then

\begin{equation}
\|g_{h \circ f}\|_{L^{2p}}^2 = \int_0^{2\pi} \left( \int_0^1 |(h \circ f)'(re^{i\theta})|^2 (1 - r)dr \right)^p \frac{d\theta}{2\pi}
\end{equation}

Hence

\begin{equation}
\|g_{h \circ f}\|_{L^{2p}}^2 \leq \|h\|_{2p}^{2p} \int_0^{2\pi} \left( \int_0^1 (f')^2(re^{i\theta}) (1 - r)dr \right)^p \frac{d\theta}{2\pi}
\end{equation}

Therefore $\|h \circ f\|_{2p} < \infty$ if $g_\sigma(f) \in L^p(T)$.

(2) $\implies$ (1) Using $g$-function, (2) says that

\begin{equation}
\int_0^{2\pi} \left( \int_0^1 |(h \circ f)'|^2(re^{i\theta}) (1 - r)dr \right)^p d\theta < \infty \quad \text{if} \quad h \in \mathcal{B}.
\end{equation}

On the other hand, W. Ramey and D. Ullrich ([RU], Proposition 5.4) constructed two Bloch functions $h_j, j = 1, 2$, such that

\begin{equation}
(1 - |z|^2)(|h_1'(z)| + |h_2'(z)|) \geq 1, \quad z \in D.
\end{equation}

From (4.4) it follows that $|(h_1' \circ f) + (h_2' \circ f)| \geq (1 - |f|^2)^{-1}$, so that

\begin{equation}
\left( \int_0^1 |(h_1 \circ f)|^2 (1 - r)dr \right)^p + \left( \int_0^1 |(h_2 \circ f)|^2 (1 - r)dr \right)^p \\
\geq 2^{-2p} \left( \int_0^1 |f'|^2 \left(|h_1' \circ f| + |h_2' \circ f|\right)^2 (1 - r)dr \right)^p \\
\geq 2^{-2p} \int_0^1 \frac{|f|^2}{(1 - |f|^2)^2} (1 - r)dr
\end{equation}

Now, integrating (4.5) with respect to $d\theta$ and applying (4.3) with $h_j, j = 1, 2$, in place of $h$, we obtain

\begin{equation}
\|g_\sigma(f)\|_p \leq C_p \left( \|h_1 \circ f\|_{2p} + \|h_2 \circ f\|_{2p} \right).
\end{equation}

This completes the proof.
5. Extreme points of $H^\infty$

$H^\infty$ denotes, as usual, the space of bounded holomorphic functions on $D$. A well-known theorem of deLeeuw and Rudin ([D], Theorem 7.9) says that

$$
\int_0^{2\pi} \log \frac{1}{1 - |f(e^{i\theta})|^2} \, d\theta = \infty
$$

is necessary and sufficient for a holomorphic $f$ with $\|f\|_\infty = \sup_{z \in D}|f(z)| = 1$ to be an extreme point of the closed unit ball of $H^\infty$. The following is a direct corollary of Theorem 1.

**Corollary 4.** Let $f \in H^\infty$, $\|f\|_\infty = 1$. Then the following are equivalent.

1. $f$ is an extreme point of the closed unit ball of $H^\infty$.
2. $h \circ f \notin H^2$ for some $h \in B$.

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