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COMPOSITION OF BLOCHS WITH BOUNDED ANALYTIC FUNCTIONS

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ABSTRACT. If f is a holomorphic self-map of the open unit disc and $1 \leq p < \infty$, then the following are equivalent.
(1) $h \circ f \in H^{2p}$ for all Bloch functions h.

$$\sup_r \int_0^{2\pi} \left(\log \frac{1}{1 - |f(re^{i\theta})|^2} \right)^p d\theta \ < \infty. \tag{2}$$

$$\int_0^{2\pi} \left(\int_0^1 (f^{\#})^2 (re^{i\theta})\,(1-r) dr \right)^p d\theta < \infty, \tag{3}$$

where $f^{\#}$ is the hyperbolic derivative of f: $f^{\#} = |f'|/(1-|f|^2)$.

1. INTRODUCTION

Initiated by works of P. Ahern and W. Rudin ([A], [AR]), there is extensive research on Bloch-to-BMOA pullbacks, that is, research on those holomorphic maps f of the unit ball of $\mathbf{C}^n$ into the unit disc of $\mathbf{C}$ for which the composition operator defined by

$$\mathcal{C}_f(h) = h \circ f$$

takes Bloch functions to functions of BMOA. We refer to [RU] for recent research on Bloch to BMOA pullbacks.

It is known (see [RU], Section 5), when $n = 1$, that one of the necessary and sufficient conditions for $\mathcal{C}_f$ to take all Blochs to BMOA is that f be a function of BMOA_σ, the Yamashita hyperbolic BMOA class (see [G] and [Y3] for BMOA and BMOA_σ). This note is to resolve a natural parallelism to the phenomenon.

Theorem 1 (Main Result). *If f is a holomorphic self-map of the open unit disc and $1 \leq p < \infty$, then the following are equivalent.*
(1) *$\mathcal{C}_f$ takes Blochs to H^{2p}, that is,*

$$h \circ f \in H^{2p} \quad \textit{for all Bloch functions } h.$$

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(2) *f belongs to Yamashita's hyperbolic Hardy class H^p_σ, that is,*

$$\sup_r \int_0^{2\pi} \left(log \frac{1}{1-|f(re^{i\theta})|^2} \right)^p d\theta \ < \infty.$$

(3) $$\int_0^{2\pi} \left(\int_0^1 (f^\#)^2 (re^{i\theta})\,(1-r) dr \right)^p d\theta < \infty,$$

where $f^\#$ is the hyperbolic derivative of f: $f^\# = |f'|/(1-|f|^2)$.

2. Preliminaries

The Bloch space $\mathcal{B}$ consists of those f holomorphic in the open unit disc D of the complex plane for which

$$\|f\|_{\mathcal{B}} := \sup_{z\in D} |f'(z)|(1-|z|^2) < \infty.$$

We let $1 \le p < \infty$ and set for f subharmonic in D

$$\|f\|_p := \sup_r \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

Then $H^p = H^p(D)$ consists of those f holomorphic in D for which $\|f\|_p < \infty$. See [D] and [G] for Bloch and H^p spaces.

The Yamashita hyperbolic Hardy class H^p_σ is defined as the set of those holomorphic self-maps f of D for which

$$\|\sigma(f)\|_p \ < \ \infty,$$

where $\sigma(z)$ denotes the hyperbolic distance of z and 0 in D, namely,

$$\sigma(z) = \frac{1}{2}\, log\, \frac{1+|z|}{1-|z|}.$$

Though H^p_σ is not a linear space, it has, as hyperbolic counterparts, many properties analogous to those of H^p. We let T be the boundary of D and set, following Yamashita,

$$\lambda(f) = log \frac{1}{1-|f|^2} \qquad \text{and} \qquad f^\# = \frac{|f'|}{1-|f|^2}$$

for the holomorphic self-map f of D. Then $\sigma(f)^p$, $\lambda(f)^p$, and $(f^\#)^p$ are subharmonic functions, so that their integral means over rT are nondecreasing functions of r: for example,

$$\int_0^{2\pi} \lambda(f)^p (re^{i\theta}) \ \frac{d\theta}{2\pi} \nearrow \ \|\lambda(f)\|_p^p \quad \text{as} \quad r \nearrow 1.$$

Also, there are corresponding maximal theorems for these functions: Set

$$M_\lambda(f,\theta) = sup \left\{ \lambda(f)(re^{i\theta}) : 0 \le r < 1 \right\};$$

then

$$\|M_\lambda(f,.)\|_{L^p} \le C_p \|\lambda(f)\|_p \tag{2.1}$$

for $f \in H^p_\sigma$ ([Y1]). The left side of (2.1) means usual $L^p(T)$ norm. The function $f^\#$ is the hyperbolic counterpart of f' and it easily follows that

$$\frac{1}{2}\lambda(f) \le \sigma(f) < \frac{1}{2}\lambda(f) + log2 \tag{2.2}$$

and

$$\Delta\left(\lambda(f)^p\right) \;=\; 4p\left\{(p-1)|f|^2 + \lambda(f)\right\}\lambda(f)^{p-2}(f^\#)^2, \tag{2.3}$$

where Δ denotes the Laplacian:

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

From (2.2) and (2.3), it should be noted that

$$f \in H^p_\sigma \qquad \text{if and only if} \qquad \|\lambda(f)\|_p < \infty$$

and

$$\Delta\left(\lambda(f)^p\right) \sim \lambda(f)^{p-1}(f^\#)^2. \tag{2.4}$$

Here and after $\psi \sim \phi$ means that either both sides are zero or the quotient ψ/ϕ lies between two positive uniform constants. See, for example, [K1], [K2], [Y1], and [Y2] for the theory of H^p_σ.

In the remaining sections, we confine f to being a holomorphic self-map of D and denote f_r, $0 \le r \le 1$, as the function defined by $f_r(z) = f(rz)$, $z \in D$. Positive constants depending on p (or q) will be denoted by C_p (or C_q), whose quantities may vary at each occurrence.

3. Hyperbolic g-function

For h holomorphic in D, g-function of Paley defined by

$$g(\theta) := g(h)(\theta) = \left(\int_0^1 |h'|^2(re^{i\theta})(1-r)dr\right)^{1/2}, \quad 0 \le \theta < 2\pi, \tag{3.1}$$

satisfies $\|g(h)\|_{L^p} \sim \|h\|_p$ if $h(0) = 0$ ([Z]). Consider Green's theorem of the form

$$r\int_0^{2\pi} \frac{\partial \psi}{\partial r} d\theta = \int\int_{|z|\le r} \Delta\psi \, dxdy,$$

valid for $\psi \in C^2(D)$. If we integrate both sides with respect to dr after dividing them by r and applying $\psi = \lambda(f)^p$, then we obtain, by use of (2.4),

$$\begin{aligned} &\|\lambda(f_\rho)\|_p^p - \lambda(f)^p(0) \\ &\sim \frac{1}{2\pi}\int_0^\rho \frac{dr}{r}\int\int_{|z|\le r} \lambda(f)^{p-1}(f^\#)^2(z)\; dxdy \\ &= \frac{1}{2\pi}\int\int_{|z|<\rho} \lambda(f)^{p-1}(f^\#)^2(z)\; log\frac{\rho}{|z|}\, dxdy, \qquad 0 \le \rho \le 1. \end{aligned} \tag{3.2}$$

In particular, we see from (3.2) that $f \in H^1_\sigma$ if and only if

$$\infty > \int\int_D (f^\#)^2(z)\ log\frac{1}{|z|}\ dxdy.$$

This suggests we define the hyperbolic version of g-function using $f^\#$. We define

$$g_\sigma(\theta) := g_\sigma(f)(\theta) = \int_0^1 (f^\#)^2(re^{i\theta})\ (1-r)dr,\ \ 0 \le \theta < 2\pi. \tag{3.3}$$

It is not surprising to see the absence of the square root in the definition of g_σ in (3.3) when we compare it to that of g- function in (3.1), because there is a known parallelism (see [Y2]) between H^2 and H^1_σ.

Theorem 2. *If $1 \le p < \infty$, then the following are equivalent.*
(1) $f \in H^p_\sigma$.
(2) $g_\sigma(f) \in L^p(T)$.
In fact, $\|\lambda(f)\|_p \sim \|g_\sigma(f)\|_{L^p}$ provided $f(0) = 0$.

Proof. By (3.2) and (3.3), there is nothing to prove when $p = 1$. We assume $1 < p < \infty$, and let $\frac{1}{p} + \frac{1}{q} = 1$.

(1) $\Longrightarrow$ (2) We begin with the identity

$$\|g_\sigma\|_{L^p} = sup \int_0^{2\pi} g_\sigma(\theta)h(\theta)\frac{d\theta}{2\pi},$$

where the supremum is taken with respect to all nonnegative trigonometric polynomials h with $\|h\|_{L^q} \le 1$. Since $(f^\#)^2$ is subharmonic, we have

$$(f^\#)^2(r^2e^{i\theta})\ \le\ \int_0^{2\pi} P(r,\theta-t)\ (f^\#)^2(re^{it})\ \frac{dt}{2\pi},\quad 0 \le r < 1, \tag{3.4}$$

where $P(r,\theta)$ is the Poisson kernel:

$$P(r,\theta) = \frac{1-r^2}{1-2rcos\theta + r^2}.$$

Let u be the Poisson integral of h. Then

$$\begin{aligned}
&\int_0^{2\pi} g_\sigma(\theta)h(\theta)d\theta \\
=&\int_0^{2\pi}\int_0^1 (f^\#)^2(re^{i\theta})\ h(\theta)\ (1-r)drd\theta \\
=&\int_0^{2\pi}\int_0^1 (f^\#)^2(r^2e^{i\theta})\ h(\theta)\ (1-r^2)\,2rdrd\theta \\
\le&2\int\int_D (f^\#)^2(z)\ u(z)\ (1-|z|^2)\ dxdy \\
\le&4\int\int_D (f^\#)^2(z)\ u(z)\ log\,\frac{1}{|z|}\ dxdy,
\end{aligned} \tag{3.5}$$

where we changed the order of integration and used (3.4) in the first inequality.

If we denote $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, $z = x + iy$, then it follows from elementary calculation and (2.3) that

$$4(f^{\#})^2 u = \Delta(\lambda(f)u) - 4\left(\partial\lambda(f)\bar{\partial}u + \bar{\partial}\lambda(f)\partial u\right),$$

so that by (3.5) we have

$$\begin{aligned} &\int_0^{2\pi} g_\sigma(\theta)h(\theta)d\theta \\ &\leq \left|\int\int_D \Delta(\lambda(f)u)(z)\, log\, \frac{1}{|z|}\, dxdy\right| \\ &\quad + 4\int\int_D |\partial\lambda(f)\bar{\partial}u + \bar{\partial}\lambda(f)\partial u|(z)\, log\, \frac{1}{|z|}\, dxdy \\ &= (I) + (II). \end{aligned} \tag{3.6}$$

Now, using Green's theorem (as in (3.2) with $p = 1$) with limiting process and Hölder's inequality, we obtain

$$\begin{aligned} (I) \;=\; &\left|\lim_{\rho\to 1}\int_0^{2\pi} (\lambda(f)u)(\rho e^{i\theta})\; d\theta - 2\pi(\lambda(f)u)(0)\right| \\ &\leq 2\pi\|\lambda(f)\|_p\|u\|_q \;\leq\; 2\pi\|\lambda(f)\|_p. \end{aligned} \tag{3.7}$$

On the other hand, if we let ϕ be a holomorphic function in D whose real part is u, it then follows from direct differentiation that

$$|\partial\lambda(f)| = |\bar{\partial}\lambda(f)| = |f|f^{\#}$$

and

$$|\partial u| = |\bar{\partial}u| = \frac{1}{2}|\phi'|.$$

Hence

$$\begin{aligned} (II) \;&\leq 4\int\int_D |\phi'(z)|\; |f(z)|\; f^{\#}(z)\; log\, \frac{1}{|z|}\, dxdy \\ &\leq 4\int_0^{2\pi}\int_0^1 |\phi'(re^{i\theta})|\, |f(re^{i\theta})|\; f^{\#}(re^{i\theta})\,(1-r)\, drd\theta. \end{aligned}$$

Since $|f(re^{i\theta})| \leq \sqrt{\lambda(f)(re^{i\theta})} \leq M_\lambda^{1/2}(f,\theta)$, we have, by the Schwarz inequality,

$$(II) \;\leq\; 4\int_0^{2\pi} M_\lambda^{1/2}(f,\theta)\,\sqrt{g_\sigma(\theta)}\; g_\phi(\theta)\, d\theta, \tag{3.8}$$

where $g_\phi(\theta)$ is Paley g-function of ϕ. Applying Hölder's inequality with indices $2p$, $2p$, q to the right side of (3.8) and using maximal theorem (2.1), we arrive at

$$(II) \;\leq\; C_p\; \|\lambda(f)\|_p^{1/2}\; \|g_\sigma(f)\|_{L^p}^{1/2}\; \|g_\phi\|_{L^q}.$$

From the theory of g-function, we know $\|g_\phi\|_{L^q} \le C_q\|\phi\|_q$, and it follows from the theorem of M. Riesz ([Z]) that $\|\phi\|_q \le C_q\|u\|_q \le C_q$. Thus

$$(II) \;\le\; C_p\, \|\lambda(f)\|_p^{1/2}\, \|g_\sigma(f)\|_{L^p}^{1/2}. \tag{3.9}$$

Hence, combining estimates (3.6), (3.7), and (3.9), we have

$$\begin{aligned}\int_0^{2\pi} g_\sigma(\theta)h(\theta)d\theta &\le (I) + (II)\\ &\le 2\pi\|\lambda(f)\|_p \;+\; C_p\, \|\lambda(f)\|_p^{1/2}\, \|g_\sigma(f)\|_{L^p}^{1/2},\end{aligned}$$

for all positive trigonometric polynomials h with $\|h\|_q \le 1$. Therefore we obtain

$$\|g_\sigma(f)\|_{L^p} \;\le\; \|\lambda(f)\|_p \;+\; C_p\, \|\lambda(f)\|_p^{1/2}\, \|g_\sigma(f)\|_{L^p}^{1/2}, \quad f \in H^p_\sigma. \tag{3.10}$$

Now we conclude from (3.10) that

$$\|g_\sigma(f)\|_{L^p} \le C_p\, \|\lambda(f)\|_p. \tag{3.11}$$

In fact, if $f \equiv 0$, then there is nothing to prove; otherwise, setting $X(r) = X(f,p,r)$ $=(\|g_\sigma(f_r)\|_{L^p}/\|\lambda(f_r)\|_p)^{\frac{1}{2}}$, $0 < r < 1$, (3.10) with f_r in place of f becomes

$$X^2(r) \le 1 + C_p X(r),$$

and this means, by comparing the order of $X(r)$, that $X(r)$, $0 < r < 1$, does not exceed the larger root of the equation $X^2 = 1 + C_p X$. This proves (3.11) with f_r, $0 \le r < 1$, in place of f, and so (3.11) follows by the monotonicity of both sides.

(2) $\Longrightarrow$ (1) It follows from (3.2) that

$$\begin{aligned}\|\lambda(f_r)\|_p^p - \lambda(f)^p(0) \;&\sim C_p \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^r (f^\#)^2\; \lambda(f)^{p-1}(\rho e^{i\theta})\; log\frac{r}{\rho}\, \rho d\rho\\ &\le C_p \int_0^{2\pi} M_\lambda^{p-1}(f_r,\theta)\; g_\sigma(f)(\theta)\; \frac{d\theta}{2\pi}.\end{aligned}$$

Hölder's inequality and the maximal theorem show the last quantity to be bounded by

$$C_p\|\lambda(f_r)\|_p^{p-1}\|g_\sigma(f)\|_{L^p},$$

so that we have

$$\|\lambda(f_r)\|_p^p - \lambda(f)^p(0) \;\le\; C_p\|\lambda(f_r)\|_p^{p-1}\|g_\sigma(f)\|_{L^p}. \tag{3.12}$$

If $g_\sigma \equiv 0$, there remains nothing to prove. Otherwise, under the assumption $0 < \|g_\sigma\|_{L^p} < \infty$, by considering the order of

$$Y(r) = Y(f,p,r) = \left(\|\lambda(f_r)\|_p/\|g_\sigma(f_r)\|_{L^p}\right),\; 0 < r < 1,$$

we conclude that $Y(r), 0 < r < 1$, does not exceed the largest root of the equation

$$Y^p \;-\; \frac{\lambda(f)^p(0)}{\|g_\sigma(f)\|_{L^p}^p} \;=\; C_p\, Y^{p-1}\,,$$

and from this follows

$$\|\lambda(f)\|_p \le C_p(f)\, \|g_\sigma(f)\|_{L^p}.$$

Here, $C_p(f)$ denotes a constant depending on p and f.

The last assertion of Theorem 2 follows from (3.11) and (3.12).

4. Bloch to H^p pullbacks

Now we prove our main theorem, Theorem 1. By the aid of Theorem 2, it suffices to prove the following.

Theorem 3. *If $1 \le p < \infty$, then the following are equivalent.*
(1) $g_\sigma(f) \in L^p(T)$.
(2) $\mathcal{C}_f$ *takes Blochs to* H^{2p}.

Proof. (1) $\Longrightarrow$ (2) Let $h \in \mathcal{B}$. Then

$$|(\mathcal{C}_f h)'| = |(h \circ f)'| \le \|h\|_{\mathcal{B}}\, f^{\#}. \tag{4.1}$$

Hence

$$\begin{aligned} &\|g_{h\circ f}\|_{L^{2p}}^{2p} \\ &= \int_0^{2\pi} \left(\int_0^1 |(h\circ f)'(re^{i\theta})|^2 \,(1-r)dr \right)^p \frac{d\theta}{2\pi} \\ &\le \|h\|_{\mathcal{B}}^{2p} \int_0^{2\pi} \left(\int_0^1 (f^{\#})^2(re^{i\theta})\,(1-r)dr \right)^p \frac{d\theta}{2\pi} \\ &= \|h\|_{\mathcal{B}}^{2p}\ \|g_\sigma(f)\|_{L^p}^p. \end{aligned} \tag{4.2}$$

Therefore $\|h\circ f\|_{2p} < \infty$ if $g_\sigma(f) \in L^p(T)$.

(2) $\Longrightarrow$ (1) Using g-function, (2) says that

$$\int_0^{2\pi} \left(\int_0^1 |(h\circ f)'|^2(re^{i\theta})\ (1-r)dr \right)^p d\theta < \infty \qquad \text{if} \qquad h \in \mathcal{B}. \tag{4.3}$$

On the other hand, W. Ramey and D. Ullrich ([RU], Proposition 5.4) constructed two Bloch functions $h_j, j = 1, 2$, such that

$$(1-|z|^2)(|h_1'(z)| + |h_2'(z)|) \ge 1, \quad z \in D. \tag{4.4}$$

From (4.4) it follows that $(|h_1' \circ f| + |h_2' \circ f|) \ge (1-|f|^2)^{-1}$, so that

$$\begin{aligned} &\left(\int_0^1 |(h_1\circ f)'|^2\,(1-r)dr \right)^p + \left(\int_0^1 |(h_2\circ f)'|^2\,(1-r)dr \right)^p \\ &\ge 2^{-2p} \left(\int_0^1 |f'|^2\ (|h_1'\circ f| + |h_2'\circ f|)^2\ (1-r)dr \right)^p \\ &\ge 2^{-2p} \left(\int_0^1 \frac{|f'|^2}{(1-|f|^2)^2}\,(1-r)dr \right)^p \\ &= 2^{-2p} g_\sigma^p(f). \end{aligned} \tag{4.5}$$

Now, integrating (4.5) with respect to $d\theta$ and applying (4.3) with $h_j, j = 1, 2$, in place of h, we obtain

$$\|g_\sigma(f)\|_p \le C_p\ (\|h_1\circ f\|_{2p} + \|h_2\circ f\|_{2p}).$$

This completes the proof.

5. Extreme points of H^∞

H^∞ denotes, as usual, the space of bounded holomorphic functions on D. A well-known theorem of deLeeuw and Rudin ([D], Theorem 7.9) says that

$$\int_0^{2\pi} log\frac{1}{1-|f(e^{i\theta})|^2}\,d\theta \ = \infty$$

is necessary and sufficient for a holomorphic f with $\|f\|_\infty = sup_{z\in D}|f(z)| = 1$ to be an extreme point of the closed unit ball of H^∞. The following is a direct corollary of Theorem 1.

Corollary 4. *Let $f \in H^\infty, \|f\|_\infty = 1$. Then the following are equivalent.*
(1) *f is an extreme point of the closed unit ball of H^∞.*
(2) *$h \circ f \notin H^2$ for some $h \in \mathcal{B}$.*

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