

## COMPOSITION OF BLOCHS WITH BOUNDED ANALYTIC FUNCTIONS

E. G. KWON

(Communicated by Theodore Gamelin)

ABSTRACT. If  $f$  is a holomorphic self-map of the open unit disc and  $1 \leq p < \infty$ , then the following are equivalent.

(1)  $h \circ f \in H^{2p}$  for all Bloch functions  $h$ .

$$(2) \quad \sup_r \int_0^{2\pi} \left( \log \frac{1}{1 - |f(re^{i\theta})|^2} \right)^p d\theta < \infty.$$

$$(3) \quad \int_0^{2\pi} \left( \int_0^1 (f^\#)^2(re^{i\theta})(1-r)dr \right)^p d\theta < \infty,$$

where  $f^\#$  is the hyperbolic derivative of  $f$ :  $f^\# = |f'|/(1 - |f|^2)$ .

### 1. INTRODUCTION

Initiated by works of P. Ahern and W. Rudin ([A], [AR]), there is extensive research on Bloch-to-BMOA pullbacks, that is, research on those holomorphic maps  $f$  of the unit ball of  $\mathbf{C}^n$  into the unit disc of  $\mathbf{C}$  for which the composition operator defined by

$$\mathcal{C}_f(h) = h \circ f$$

takes Bloch functions to functions of BMOA. We refer to [RU] for recent research on Bloch to BMOA pullbacks.

It is known (see [RU], Section 5), when  $n = 1$ , that one of the necessary and sufficient conditions for  $\mathcal{C}_f$  to take all Blochs to BMOA is that  $f$  be a function of  $\text{BMOA}_\sigma$ , the Yamashita hyperbolic BMOA class (see [G] and [Y3] for BMOA and  $\text{BMOA}_\sigma$ ). This note is to resolve a natural parallelism to the phenomenon.

**Theorem 1** (Main Result). *If  $f$  is a holomorphic self-map of the open unit disc and  $1 \leq p < \infty$ , then the following are equivalent.*

(1)  $\mathcal{C}_f$  takes Blochs to  $H^{2p}$ , that is,

$$h \circ f \in H^{2p} \quad \text{for all Bloch functions } h.$$

---

Received by the editors January 31, 1994 and, in revised form, October 19, 1994.

1991 *Mathematics Subject Classification*. Primary 30D55, 30D45.

*Key words and phrases*.  $H^p$  space, Bloch space, hyperbolic Hardy class, pullbacks.

This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1993.

(2)  $f$  belongs to Yamashita's hyperbolic Hardy class  $H^p_\sigma$ , that is,

$$\sup_r \int_0^{2\pi} \left( \log \frac{1}{1 - |f(re^{i\theta})|^2} \right)^p d\theta < \infty.$$

(3) 
$$\int_0^{2\pi} \left( \int_0^1 (f^\#)^2(re^{i\theta})(1-r)dr \right)^p d\theta < \infty,$$

where  $f^\#$  is the hyperbolic derivative of  $f$ :  $f^\# = |f'|/(1 - |f|^2)$ .

2. PRELIMINARIES

The Bloch space  $\mathcal{B}$  consists of those  $f$  holomorphic in the open unit disc  $D$  of the complex plane for which

$$\|f\|_{\mathcal{B}} := \sup_{z \in D} |f'(z)|(1 - |z|^2) < \infty.$$

We let  $1 \leq p < \infty$  and set for  $f$  subharmonic in  $D$

$$\|f\|_p := \sup_r \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

Then  $H^p = H^p(D)$  consists of those  $f$  holomorphic in  $D$  for which  $\|f\|_p < \infty$ . See [D] and [G] for Bloch and  $H^p$  spaces.

The Yamashita hyperbolic Hardy class  $H^p_\sigma$  is defined as the set of those holomorphic self-maps  $f$  of  $D$  for which

$$\|\sigma(f)\|_p < \infty,$$

where  $\sigma(z)$  denotes the hyperbolic distance of  $z$  and 0 in  $D$ , namely,

$$\sigma(z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}.$$

Though  $H^p_\sigma$  is not a linear space, it has, as hyperbolic counterparts, many properties analogous to those of  $H^p$ . We let  $T$  be the boundary of  $D$  and set, following Yamashita,

$$\lambda(f) = \log \frac{1}{1 - |f|^2} \quad \text{and} \quad f^\# = \frac{|f'|}{1 - |f|^2}$$

for the holomorphic self-map  $f$  of  $D$ . Then  $\sigma(f)^p$ ,  $\lambda(f)^p$ , and  $(f^\#)^p$  are subharmonic functions, so that their integral means over  $rT$  are nondecreasing functions of  $r$ : for example,

$$\int_0^{2\pi} \lambda(f)^p(re^{i\theta}) \frac{d\theta}{2\pi} \nearrow \|\lambda(f)\|_p^p \quad \text{as } r \nearrow 1.$$

Also, there are corresponding maximal theorems for these functions: Set

$$M_\lambda(f, \theta) = \sup \{ \lambda(f)(re^{i\theta}) : 0 \leq r < 1 \};$$

then

$$(2.1) \quad \|M_\lambda(f, \cdot)\|_{L^p} \leq C_p \|\lambda(f)\|_p$$

for  $f \in H_\sigma^p$  ([Y1]). The left side of (2.1) means usual  $L^p(T)$  norm. The function  $f^\#$  is the hyperbolic counterpart of  $f'$  and it easily follows that

$$(2.2) \quad \frac{1}{2}\lambda(f) \leq \sigma(f) < \frac{1}{2}\lambda(f) + \log 2$$

and

$$(2.3) \quad \Delta(\lambda(f)^p) = 4p\{(p-1)|f|^2 + \lambda(f)\}\lambda(f)^{p-2}(f^\#)^2,$$

where  $\Delta$  denotes the Laplacian:

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}.$$

From (2.2) and (2.3), it should be noted that

$$f \in H_\sigma^p \quad \text{if and only if} \quad \|\lambda(f)\|_p < \infty$$

and

$$(2.4) \quad \Delta(\lambda(f)^p) \sim \lambda(f)^{p-1}(f^\#)^2.$$

Here and after  $\psi \sim \phi$  means that either both sides are zero or the quotient  $\psi/\phi$  lies between two positive uniform constants. See, for example, [K1], [K2], [Y1], and [Y2] for the theory of  $H_\sigma^p$ .

In the remaining sections, we confine  $f$  to being a holomorphic self-map of  $D$  and denote  $f_r$ ,  $0 \leq r \leq 1$ , as the function defined by  $f_r(z) = f(rz)$ ,  $z \in D$ . Positive constants depending on  $p$  (or  $q$ ) will be denoted by  $C_p$  (or  $C_q$ ), whose quantities may vary at each occurrence.

### 3. HYPERBOLIC $g$ -FUNCTION

For  $h$  holomorphic in  $D$ ,  $g$ -function of Paley defined by

$$(3.1) \quad g(\theta) := g(h)(\theta) = \left( \int_0^1 |h'|^2(re^{i\theta})(1-r)dr \right)^{1/2}, \quad 0 \leq \theta < 2\pi,$$

satisfies  $\|g(h)\|_{L^p} \sim \|h\|_p$  if  $h(0) = 0$  ([Z]). Consider Green's theorem of the form

$$r \int_0^{2\pi} \frac{\partial \psi}{\partial r} d\theta = \int \int_{|z| \leq r} \Delta \psi \, dx dy,$$

valid for  $\psi \in C^2(D)$ . If we integrate both sides with respect to  $dr$  after dividing them by  $r$  and applying  $\psi = \lambda(f)^p$ , then we obtain, by use of (2.4),

$$(3.2) \quad \begin{aligned} & \|\lambda(f_\rho)\|_p^p - \lambda(f)^p(0) \\ & \sim \frac{1}{2\pi} \int_0^\rho \frac{dr}{r} \int \int_{|z| \leq r} \lambda(f)^{p-1}(f^\#)^2(z) \, dx dy \\ & = \frac{1}{2\pi} \int \int_{|z| < \rho} \lambda(f)^{p-1}(f^\#)^2(z) \log \frac{\rho}{|z|} \, dx dy, \quad 0 \leq \rho \leq 1. \end{aligned}$$

In particular, we see from (3.2) that  $f \in H^1_\sigma$  if and only if

$$\infty > \int \int_D (f^\#)^2(z) \log \frac{1}{|z|} \, dx dy.$$

This suggests we define the hyperbolic version of  $g$ -function using  $f^\#$ . We define

$$(3.3) \quad g_\sigma(\theta) := g_\sigma(f)(\theta) = \int_0^1 (f^\#)^2(re^{i\theta}) (1-r) dr, \quad 0 \leq \theta < 2\pi.$$

It is not surprising to see the absence of the square root in the definition of  $g_\sigma$  in (3.3) when we compare it to that of  $g$ -function in (3.1), because there is a known parallelism (see [Y2]) between  $H^2$  and  $H^1_\sigma$ .

**Theorem 2.** *If  $1 \leq p < \infty$ , then the following are equivalent.*

- (1)  $f \in H^p_\sigma$ .
- (2)  $g_\sigma(f) \in L^p(T)$ .

*In fact,  $\|\lambda(f)\|_p \sim \|g_\sigma(f)\|_{L^p}$  provided  $f(0) = 0$ .*

*Proof.* By (3.2) and (3.3), there is nothing to prove when  $p = 1$ . We assume  $1 < p < \infty$ , and let  $\frac{1}{p} + \frac{1}{q} = 1$ .

(1)  $\implies$  (2) We begin with the identity

$$\|g_\sigma\|_{L^p} = \sup \int_0^{2\pi} g_\sigma(\theta) h(\theta) \frac{d\theta}{2\pi},$$

where the supremum is taken with respect to all nonnegative trigonometric polynomials  $h$  with  $\|h\|_{L^q} \leq 1$ . Since  $(f^\#)^2$  is subharmonic, we have

$$(3.4) \quad (f^\#)^2(r^2 e^{i\theta}) \leq \int_0^{2\pi} P(r, \theta - t) (f^\#)^2(re^{it}) \frac{dt}{2\pi}, \quad 0 \leq r < 1,$$

where  $P(r, \theta)$  is the Poisson kernel:

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}.$$

Let  $u$  be the Poisson integral of  $h$ . Then

$$\begin{aligned} & \int_0^{2\pi} g_\sigma(\theta) h(\theta) d\theta \\ &= \int_0^{2\pi} \int_0^1 (f^\#)^2(re^{i\theta}) h(\theta) (1-r) dr d\theta \\ (3.5) \quad &= \int_0^{2\pi} \int_0^1 (f^\#)^2(r^2 e^{i\theta}) h(\theta) (1-r^2) 2r dr d\theta \\ &\leq 2 \int \int_D (f^\#)^2(z) u(z) (1 - |z|^2) \, dx dy \\ &\leq 4 \int \int_D (f^\#)^2(z) u(z) \log \frac{1}{|z|} \, dx dy, \end{aligned}$$

where we changed the order of integration and used (3.4) in the first inequality.

If we denote  $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$  and  $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ ,  $z = x + iy$ , then it follows from elementary calculation and (2.3) that

$$4(f^\#)^2 u = \Delta(\lambda(f)u) - 4(\partial\lambda(f)\bar{\partial}u + \bar{\partial}\lambda(f)\partial u),$$

so that by (3.5) we have

$$\begin{aligned} & \int_0^{2\pi} g_\sigma(\theta)h(\theta)d\theta \\ (3.6) \quad & \leq \left| \int \int_D \Delta(\lambda(f)u)(z) \log \frac{1}{|z|} dx dy \right| \\ & \quad + 4 \int \int_D |\partial\lambda(f)\bar{\partial}u + \bar{\partial}\lambda(f)\partial u|(z) \log \frac{1}{|z|} dx dy \\ & = (I) + (II). \end{aligned}$$

Now, using Green's theorem (as in (3.2) with  $p = 1$ ) with limiting process and Hölder's inequality, we obtain

$$\begin{aligned} (3.7) \quad (I) & = \left| \lim_{\rho \rightarrow 1} \int_0^{2\pi} (\lambda(f)u)(\rho e^{i\theta}) d\theta - 2\pi(\lambda(f)u)(0) \right| \\ & \leq 2\pi \|\lambda(f)\|_p \|u\|_q \leq 2\pi \|\lambda(f)\|_p. \end{aligned}$$

On the other hand, if we let  $\phi$  be a holomorphic function in  $D$  whose real part is  $u$ , it then follows from direct differentiation that

$$|\partial\lambda(f)| = |\bar{\partial}\lambda(f)| = |f|f^\#$$

and

$$|\partial u| = |\bar{\partial}u| = \frac{1}{2}|\phi'|.$$

Hence

$$\begin{aligned} (II) & \leq 4 \int \int_D |\phi'(z)| |f(z)| f^\#(z) \log \frac{1}{|z|} dx dy \\ & \leq 4 \int_0^{2\pi} \int_0^1 |\phi'(re^{i\theta})| |f(re^{i\theta})| f^\#(re^{i\theta}) (1-r) dr d\theta. \end{aligned}$$

Since  $|f(re^{i\theta})| \leq \sqrt{\lambda(f)(re^{i\theta})} \leq M_\lambda^{1/2}(f, \theta)$ , we have, by the Schwarz inequality,

$$(3.8) \quad (II) \leq 4 \int_0^{2\pi} M_\lambda^{1/2}(f, \theta) \sqrt{g_\sigma(\theta)} g_\phi(\theta) d\theta,$$

where  $g_\phi(\theta)$  is Paley  $g$ -function of  $\phi$ . Applying Hölder's inequality with indices  $2p$ ,  $2p$ ,  $q$  to the right side of (3.8) and using maximal theorem (2.1), we arrive at

$$(II) \leq C_p \|\lambda(f)\|_p^{1/2} \|g_\sigma(f)\|_{L^p}^{1/2} \|g_\phi\|_{L^q}.$$

From the theory of  $g$ -function, we know  $\|g_\phi\|_{L^q} \leq C_q \|\phi\|_q$ , and it follows from the theorem of M. Riesz ([Z]) that  $\|\phi\|_q \leq C_q \|u\|_q \leq C_q$ . Thus

$$(3.9) \quad (II) \leq C_p \|\lambda(f)\|_p^{1/2} \|g_\sigma(f)\|_{L^p}^{1/2}.$$

Hence, combining estimates (3.6), (3.7), and (3.9), we have

$$\begin{aligned} \int_0^{2\pi} g_\sigma(\theta)h(\theta)d\theta &\leq (I) + (II) \\ &\leq 2\pi\|\lambda(f)\|_p + C_p \|\lambda(f)\|_p^{1/2} \|g_\sigma(f)\|_{L^p}^{1/2}, \end{aligned}$$

for all positive trigonometric polynomials  $h$  with  $\|h\|_q \leq 1$ . Therefore we obtain

$$(3.10) \quad \|g_\sigma(f)\|_{L^p} \leq \|\lambda(f)\|_p + C_p \|\lambda(f)\|_p^{1/2} \|g_\sigma(f)\|_{L^p}^{1/2}, \quad f \in H_\sigma^p.$$

Now we conclude from (3.10) that

$$(3.11) \quad \|g_\sigma(f)\|_{L^p} \leq C_p \|\lambda(f)\|_p.$$

In fact, if  $f \equiv 0$ , then there is nothing to prove; otherwise, setting  $X(r) = X(f, p, r) = (\|g_\sigma(f_r)\|_{L^p} / \|\lambda(f_r)\|_p)^{\frac{1}{2}}$ ,  $0 < r < 1$ , (3.10) with  $f_r$  in place of  $f$  becomes

$$X^2(r) \leq 1 + C_p X(r),$$

and this means, by comparing the order of  $X(r)$ , that  $X(r)$ ,  $0 < r < 1$ , does not exceed the larger root of the equation  $X^2 = 1 + C_p X$ . This proves (3.11) with  $f_r$ ,  $0 \leq r < 1$ , in place of  $f$ , and so (3.11) follows by the monotonicity of both sides.

(2)  $\implies$  (1) It follows from (3.2) that

$$\begin{aligned} \|\lambda(f_r)\|_p^p - \lambda(f)^p(0) &\sim C_p \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^r (f^\#)^2 \lambda(f)^{p-1} (\rho e^{i\theta}) \log \frac{r}{\rho} \rho d\rho \\ &\leq C_p \int_0^{2\pi} M_\lambda^{p-1}(f_r, \theta) g_\sigma(f)(\theta) \frac{d\theta}{2\pi}. \end{aligned}$$

Hölder’s inequality and the maximal theorem show the last quantity to be bounded by

$$C_p \|\lambda(f_r)\|_p^{p-1} \|g_\sigma(f)\|_{L^p},$$

so that we have

$$(3.12) \quad \|\lambda(f_r)\|_p^p - \lambda(f)^p(0) \leq C_p \|\lambda(f_r)\|_p^{p-1} \|g_\sigma(f)\|_{L^p}.$$

If  $g_\sigma \equiv 0$ , there remains nothing to prove. Otherwise, under the assumption  $0 < \|g_\sigma\|_{L^p} < \infty$ , by considering the order of

$$Y(r) = Y(f, p, r) = (\|\lambda(f_r)\|_p / \|g_\sigma(f_r)\|_{L^p}), \quad 0 < r < 1,$$

we conclude that  $Y(r)$ ,  $0 < r < 1$ , does not exceed the largest root of the equation

$$Y^p - \frac{\lambda(f)^p(0)}{\|g_\sigma(f)\|_{L^p}^p} = C_p Y^{p-1},$$

and from this follows

$$\|\lambda(f)\|_p \leq C_p(f) \|g_\sigma(f)\|_{L^p}.$$

Here,  $C_p(f)$  denotes a constant depending on  $p$  and  $f$ .

The last assertion of Theorem 2 follows from (3.11) and (3.12).

4. BLOCH TO  $H^p$  PULLBACKS

Now we prove our main theorem, Theorem 1. By the aid of Theorem 2, it suffices to prove the following.

**Theorem 3.** *If  $1 \leq p < \infty$ , then the following are equivalent.*

- (1)  $g_\sigma(f) \in L^p(T)$ .
- (2)  $\mathcal{C}_f$  takes Blochs to  $H^{2p}$ .

*Proof.* (1)  $\implies$  (2) Let  $h \in \mathcal{B}$ . Then

$$(4.1) \quad |(\mathcal{C}_f h)'| = |(h \circ f)'| \leq \|h\|_{\mathcal{B}} f^\#.$$

Hence

$$(4.2) \quad \begin{aligned} & \|g_{h \circ f}\|_{L^{2p}}^{2p} \\ &= \int_0^{2\pi} \left( \int_0^1 |(h \circ f)'(re^{i\theta})|^2 (1-r) dr \right)^p \frac{d\theta}{2\pi} \\ &\leq \|h\|_{\mathcal{B}}^{2p} \int_0^{2\pi} \left( \int_0^1 (f^\#)^2(re^{i\theta}) (1-r) dr \right)^p \frac{d\theta}{2\pi} \\ &= \|h\|_{\mathcal{B}}^{2p} \|g_\sigma(f)\|_{L^p}^p. \end{aligned}$$

Therefore  $\|h \circ f\|_{2p} < \infty$  if  $g_\sigma(f) \in L^p(T)$ .

(2)  $\implies$  (1) Using  $g$ -function, (2) says that

$$(4.3) \quad \int_0^{2\pi} \left( \int_0^1 |(h \circ f)'|^2(re^{i\theta}) (1-r) dr \right)^p d\theta < \infty \quad \text{if} \quad h \in \mathcal{B}.$$

On the other hand, W. Ramey and D. Ullrich ([RU], Proposition 5.4) constructed two Bloch functions  $h_j, j = 1, 2$ , such that

$$(4.4) \quad (1 - |z|^2)(|h_1'(z)| + |h_2'(z)|) \geq 1, \quad z \in D.$$

From (4.4) it follows that  $(|h_1' \circ f| + |h_2' \circ f|) \geq (1 - |f|^2)^{-1}$ , so that

$$(4.5) \quad \begin{aligned} & \left( \int_0^1 |(h_1 \circ f)'|^2 (1-r) dr \right)^p + \left( \int_0^1 |(h_2 \circ f)'|^2 (1-r) dr \right)^p \\ &\geq 2^{-2p} \left( \int_0^1 |f'|^2 (|h_1' \circ f| + |h_2' \circ f|)^2 (1-r) dr \right)^p \\ &\geq 2^{-2p} \left( \int_0^1 \frac{|f'|^2}{(1-|f|^2)^2} (1-r) dr \right)^p \\ &= 2^{-2p} g_\sigma^p(f). \end{aligned}$$

Now, integrating (4.5) with respect to  $d\theta$  and applying (4.3) with  $h_j, j = 1, 2$ , in place of  $h$ , we obtain

$$\|g_\sigma(f)\|_p \leq C_p (\|h_1 \circ f\|_{2p} + \|h_2 \circ f\|_{2p}).$$

This completes the proof.

5. EXTREME POINTS OF  $H^\infty$ 

$H^\infty$  denotes, as usual, the space of bounded holomorphic functions on  $D$ . A well-known theorem of deLeeuw and Rudin ([D], Theorem 7.9) says that

$$\int_0^{2\pi} \log \frac{1}{1 - |f(e^{i\theta})|^2} d\theta = \infty$$

is necessary and sufficient for a holomorphic  $f$  with  $\|f\|_\infty = \sup_{z \in D} |f(z)| = 1$  to be an extreme point of the closed unit ball of  $H^\infty$ . The following is a direct corollary of Theorem 1.

**Corollary 4.** *Let  $f \in H^\infty$ ,  $\|f\|_\infty = 1$ . Then the following are equivalent.*

- (1)  $f$  is an extreme point of the closed unit ball of  $H^\infty$ .
- (2)  $h \circ f \notin H^2$  for some  $h \in \mathcal{B}$ .

## ACKNOWLEDGMENT

The author wishes to express his sincere gratitude to the referee for his helpful suggestion.

## REFERENCES

- [A] Patrick R. Ahern, *On the behavior near torus of functions holomorphic in the ball*, Pacific J. Math. **107** (1983), 267–278. MR **84i**:32023
- [AR] Patrick R. Ahern and Walter Rudin, *Bloch functions, BMO, and boundary zeros*, Indiana Univ. Math. J. **36** (1987), 131–148. MR **88d**:42036
- [D] Peter. L. Duren, *The theory of  $H^p$  functions*, Academic Press, New York, 1970. MR **42**:3552
- [G] John. B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981. MR **83g**:30037
- [K1] E. G. Kwon, *Fractional integration and the hyperbolic derivative*, Bull. Austral. Math. Soc. **38** (1988), 357–364. MR **90a**:30096
- [K2] ———, *Mean growth of the hyperbolic Hardy class functions*, Math. Japonica **35** (1990), 451–460. MR **91e**:30064
- [RU] Wade Ramey and David Ullrich, *Bounded mean oscillations of Bloch pullbacks*, Math. Ann. **291** (1991), 591–606. MR **92i**:32004
- [Y1] Shinji Yamashita, *Functions with  $H^p$  hyperbolic derivative*, Math. Scand. **13** (1983), 238–244. MR **85f**:30055
- [Y2] ———, *Hyperbolic Hardy classes and hyperbolically Dirichlet finite functions*, Hokkaido Math. J. **10** (1981), 709–722, Special Issue.
- [Y3] ———, *Holomorphic functions of hyperbolically bounded mean oscillations*, Bollentino U.M.I. **5-B** (1986), no. 6, 983–1000. MR **88e**:30092
- [Z] A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, London, 1959. MR **21**:6498

DEPARTMENT OF MATHEMATICS-EDUCATION, ANDONG NATIONAL UNIVERSITY, ANDONG 760-749, KOREA

*E-mail address:* egkwon@anu.andong.ac.kr