

BASIC DIFFERENTIAL FORMS FOR ACTIONS OF LIE GROUPS

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(Communicated by Roe W. Goodman)

ABSTRACT. A section of a Riemannian G -manifold M is a closed submanifold Σ which meets each orbit orthogonally. It is shown that the algebra of G -invariant differential forms on M which are horizontal in the sense that they kill every vector which is tangent to some orbit, is isomorphic to the algebra of those differential forms on Σ which are invariant with respect to the generalized Weyl group of Σ , under some condition.

1. INTRODUCTION

A section of a Riemannian G -manifold M is a closed submanifold Σ which meets each orbit orthogonally. This notion was introduced by Szenthe [26], [27], and in a slightly different form by Palais and Terng in [19], [20]. The case of linear representations was considered by Bott and Samelson [4] and Conlon [9], and then by Dadok [10] who called representations admitting sections polar representations and completely classified all polar representations of connected compact Lie groups. Conlon [8] considered Riemannian manifolds admitting flat sections. We follow here the notion of Palais and Terng.

If M is a Riemannian G -manifold which admits a section Σ , then the trace on Σ of the G -action is a discrete group action by the generalized Weyl group $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$. Palais and Terng [19] showed that then the algebras of invariant smooth functions coincide, $C^\infty(M, \mathbb{R})^G \cong C^\infty(\Sigma, \mathbb{R})^{W(\Sigma)}$.

In this paper we will extend this result to the algebras of differential forms. Our aim is to show that pullback along the embedding $\Sigma \rightarrow M$ induces an isomorphism $\Omega_{\text{hor}}^p(M)^G \cong \Omega^p(\Sigma)^{W(\Sigma)}$ for each p , where a differential form ω on M is called *horizontal* if it kills each vector tangent to some orbit. For each point x in M , the slice representation of the isotropy group G_x on the normal space $T_x(G.x)^\perp$ to the tangent space to the orbit through x is a polar representation. The first step is to show that the result holds for polar representations. This is done in Theorem 3.7 for polar representations whose generalized Weyl group is really a Coxeter group, i.e., is generated by reflections. Every polar representation of a connected compact Lie group has this property. The method used there is inspired by Solomon [25]. Then the general result is proved under the assumption that each slice representation

Received by the editors April 6, 1994 and, in revised form November 9, 1994.

1991 *Mathematics Subject Classification*. Primary 57S15, 20F55.

Key words and phrases. Orbits, sections, basic differential forms.

Supported by Project P 10037-PHY of 'Fonds zur Förderung der wissenschaftlichen Forschung'.

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has a Coxeter group as a generalized Weyl group. The last section gives some perspective to the result.

I want to thank D. Alekseevsky for introducing me to the beautiful results of Palais and Terng. I also thank A. Onishchik and D. Alekseevsky for many discussions about this and related topics, and the editor and the referees for much care and some hints.

2. BASIC DIFFERENTIAL FORMS

2.1. Basic differential forms. Let G be a Lie group with Lie algebra \mathfrak{g} and multiplication $\mu : G \times G \rightarrow G$, and for $g \in G$ let $\mu_g, \mu^g : G \rightarrow G$ denote the left and right translations.

Let $\ell : G \times M \rightarrow M$ be a left action of the Lie group G on a smooth manifold M . We consider the partial mappings $\ell_g : M \rightarrow M$ for $g \in G$ and $\ell^x : G \rightarrow M$ for $x \in M$ and the fundamental vector field mapping $\zeta : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ given by $\zeta_X(x) = T_e(\ell^x)X$. Since ℓ is a left action, the negative $-\zeta$ is a Lie algebra homomorphism.

A differential form $\varphi \in \Omega^p(M)$ is called G -invariant if $(\ell_g)^*\varphi = \varphi$ for all $g \in G$ and horizontal if φ kills each vector tangent to a G -orbit: $i_{\zeta_X}\varphi = 0$ for all $X \in \mathfrak{g}$. We denote by $\Omega_{\text{hor}}^p(M)^G$ the space of all horizontal G -invariant p -forms on M . They are also called *basic forms*.

2.2. Lemma. *Under the exterior differential $\Omega_{\text{hor}}(M)^G$ is a subcomplex of $\Omega(M)$.*

Proof. If $\varphi \in \Omega_{\text{hor}}(M)^G$, then the exterior derivative $d\varphi$ is clearly G -invariant. For $X \in \mathfrak{g}$ we have

$$i_{\zeta_X}d\varphi = i_{\zeta_X}d\varphi + di_{\zeta_X}\varphi = \mathcal{L}_{\zeta_X}\varphi = 0,$$

so $d\varphi$ is also horizontal. □

2.3. Sections. Let M be a connected complete Riemannian manifold, and let G be a Lie group which acts isometrically on M from the left. A connected closed smooth submanifold Σ of M is called a *section* for the G -action, if it meets all G -orbits orthogonally.

Equivalently we require that $G.\Sigma = M$ and that for each $x \in \Sigma$ and $X \in \mathfrak{g}$ the fundamental vector field $\zeta_X(x)$ is orthogonal to $T_x\Sigma$.

We only remark here that each section is a totally geodesic submanifold and is given by $\exp(T_x(x.G)^\perp)$ if x lies in a principal orbit.

If we put $N_G(\Sigma) := \{g \in G : g.\Sigma = \Sigma\}$ and $Z_G(\Sigma) := \{g \in G : g.s = s \text{ for all } s \in \Sigma\}$, then the quotient $W(\Sigma) := N_G(\Sigma)/Z_G(\Sigma)$ turns out to be a discrete group acting properly on Σ . It is called the generalized Weyl group of the section Σ .

See [19] or [20] for more information on sections and their generalized Weyl groups.

2.4. Main Theorem. *Let $M \times G \rightarrow M$ be a proper isometric right action of a Lie group G on a smooth Riemannian manifold M , which admits a section Σ . Let us assume that*

- (1) *For each $x \in \Sigma$ the slice representation $G_x \rightarrow O(T_x(G.x)^\perp)$ has a generalized Weyl group which is a reflection group (see section 3).*

Then the restriction of differential forms induces an isomorphism

$$\Omega_{\text{hor}}^p(M)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)}$$

between the space of horizontal G -invariant differential forms on M and the space of all differential forms on Σ which are invariant under the action of the generalized Weyl group $W(\Sigma)$ of the section Σ .

The proof of this theorem will take up the rest of this paper. According to Dadok [10], remark after Proposition 6, for any polar representation of a connected compact Lie group the generalized Weyl group $W(\Sigma)$ is a reflection group, so condition (1) holds if we assume that:

- (2) Each isotropy group G_x is connected.

Proof of injectivity. Let $i : \Sigma \rightarrow M$ be the embedding of the section. We claim that $i^* : \Omega_{\text{hor}}^p(M)^G \rightarrow \Omega^p(\Sigma)^{W(\Sigma)}$ is injective. Let $\omega \in \Omega_{\text{hor}}^p(M)^G$ with $i^*\omega = 0$. For $x \in \Sigma$ we have $i_X\omega_x = 0$ for $X \in T_x\Sigma$ since $i^*\omega = 0$, and also for $X \in T_x(G.x)$ since ω is horizontal. Let $x \in \Sigma \cap M_{\text{reg}}$ be a regular point; then $T_x\Sigma = (T_x(G.x))^\perp$ and so $\omega_x = 0$. This holds along the whole orbit through x since ω is G -invariant. Thus $\omega|_{M_{\text{reg}}} = 0$, and since M_{reg} is dense in M , $\omega = 0$.

So it remains to show that i^* is surjective. This will be done in 4.2 below. □

3. REPRESENTATIONS

3.1. Invariant functions. Let G be a reductive Lie group and let $\rho : G \rightarrow GL(V)$ be a representation in a finite dimensional real vector space V .

According to a classical theorem of Hilbert (as extended by Nagata [15], [16]), the algebra of G -invariant polynomials $\mathbb{R}[V]^G$ on V is finitely generated (in fact finitely presented), so there are G -invariant homogeneous polynomials f_1, \dots, f_m on V such that each invariant polynomial $h \in \mathbb{R}[V]^G$ is of the form $h = q(f_1, \dots, f_m)$ for a polynomial $q \in \mathbb{R}[\mathbb{R}^m]$. Let $f = (f_1, \dots, f_m) : V \rightarrow \mathbb{R}^m$; then this means that the pullback homomorphism $f^* : \mathbb{R}[\mathbb{R}^m] \rightarrow \mathbb{R}[V]^G$ is surjective.

D. Luna proved in [14] that the pullback homomorphism $f^* : C^\infty(\mathbb{R}^m, \mathbb{R}) \rightarrow C^\infty(V, \mathbb{R})^G$ is also surjective onto the space of all smooth functions on V which are constant on the fibers of f . Note that the polynomial mapping f in this case may not separate the G -orbits.

G. Schwarz proved already in [23] that if G is a compact Lie group, then the pullback homomorphism $f^* : C^\infty(\mathbb{R}^m, \mathbb{R}) \rightarrow C^\infty(V, \mathbb{R})^G$ is actually surjective onto the space of G -invariant smooth functions. This result implies in particular that f separates the G -orbits.

3.2. Lemma. *Let $\ell \in V^*$ be a linear functional on a finite dimensional vector space V , and let $f \in C^\infty(V, \mathbb{R})$ be a smooth function which vanishes on the kernel of ℓ , so that $f|_{\ell^{-1}(0)} = 0$. Then there is a unique smooth function g such that $f = \ell.g$*

Proof. Choose coordinates x^1, \dots, x^n on V with $\ell = x^1$. Then $f(0, x^2, \dots, x^n) = 0$ and we have $f(x^1, \dots, x^n) = \int_0^1 \partial_1 f(tx^1, x^2, \dots, x^n) dt.x^1 = g(x^1, \dots, x^n).x^1$. □

3.3. Lemma. *Let W be a finite reflection group acting on a finite dimensional vector space Σ . Let $f = (f_1, \dots, f_n) : \Sigma \rightarrow \mathbb{R}^n$ be the polynomial map whose components f_1, \dots, f_n are a minimal set of homogeneous generators of the algebra $\mathbb{R}[\Sigma]^W$ of W -invariant polynomials on Σ . Then the pullback homomorphism $f^* : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\Sigma)$ is surjective onto the space $\Omega^p(\Sigma)^W$ of W -invariant differential forms on Σ .*

For polynomial differential forms and more general reflection groups this is the main theorem of Solomon [25]. We adapt his proof to our needs.

Proof. The polynomial generators f_i form a set of algebraically independent polynomials, $n = \dim \Sigma$, and their degrees d_1, \dots, d_n are uniquely determined up to order. We even have (see [12]):

- (1) $d_1 \dots d_n = |W|$, the order of W .
- (2) $d_1 + \dots + d_n = n + N$, where N is the number of reflections in W .

Let us consider the mapping $f = (f_1, \dots, f_n) : \Sigma \rightarrow \mathbb{R}^n$ and its Jacobian $J(x) = \det(df(x))$. Let x^1, \dots, x^n be coordinate functions in Σ . Then for each $\sigma \in W$ we have

$$\begin{aligned}
 J \cdot dx^1 \wedge \dots \wedge dx^n &= df_1 \wedge \dots \wedge df_n = \sigma^*(df_1 \wedge \dots \wedge df_n) \\
 &= (J \circ \sigma) \sigma^*(dx^1 \wedge \dots \wedge dx^n) = (J \circ \sigma) \det(\sigma)(dx^1 \wedge \dots \wedge dx^n), \\
 (3) \quad J \circ \sigma &= \det(\sigma^{-1})J.
 \end{aligned}$$

The generators f_1, \dots, f_n are algebraically independent over \mathbb{R} , thus $J \neq 0$. Since J is a polynomial of degree $(d_1 - 1) + \dots + (d_n - 1) = N$ (see (2)), the W -invariant set $U = \Sigma \setminus J^{-1}(0)$ is open and dense in Σ ; by the inverse function theorem f is a local diffeomorphism on U , thus the 1-forms df_1, \dots, df_n are a coframe on U .

Now let $(\sigma_\alpha)_{\alpha=1, \dots, N}$ be the set of reflections in W , with reflection hyperplanes H_α . Let $\ell_\alpha \in \Sigma^*$ be linear functionals with $H_\alpha = \ell_\alpha^{-1}(0)$. If $x \in H_\alpha$ we have $J(x) = \det(\sigma_\alpha)J(\sigma_\alpha \cdot x) = -J(x)$, so that $J|_{H_\alpha} = 0$ for each α , and by Lemma 3.2 we have

$$(4) \quad J = c \cdot \ell_1 \dots \ell_N.$$

Since J is a polynomial of degree N , c must be a constant. Repeating the last argument for an arbitrary function g and using (4), we get:

$$(5) \quad \begin{aligned} &\text{If } g \in C^\infty(\Sigma, \mathbb{R}) \text{ satisfies } g \circ \sigma = \det(\sigma^{-1})g \text{ for each} \\ &\sigma \in W, \text{ we have } g = J \cdot h \text{ for } h \in C^\infty(\Sigma, \mathbb{R})^W. \end{aligned}$$

After these preparations we turn to the assertion of the lemma. Let $\omega \in \Omega^p(\Sigma)^W$. Since the 1-forms df_j form a coframe on U , we have

$$\omega|_U = \sum_{j_1 < \dots < j_p} g_{j_1 \dots j_p} df_{j_1} \wedge \dots \wedge df_{j_p}|_U$$

for $g_{j_1 \dots j_p} \in C^\infty(U, \mathbb{R})$. Since ω and all df_i are W -invariant, we may replace $g_{j_1 \dots j_p}$ by their averages over W , or assume without loss that $g_{j_1 \dots j_p} \in C^\infty(U, \mathbb{R})^W$.

Let us choose now a form index $i_1 < \dots < i_p$ with $\{i_{p+1} < \dots < i_n\} = \{1, \dots, n\} \setminus \{i_1 < \dots < i_p\}$. Then for some sign $\varepsilon = \pm 1$ we have

$$\begin{aligned}
 \omega|_U \wedge df_{i_{p+1}} \wedge \dots \wedge df_{i_n} &= \varepsilon \cdot g_{i_1 \dots i_p} \cdot df_1 \wedge \dots \wedge df_n = \varepsilon \cdot g_{i_1 \dots i_p} \cdot J \cdot dx^1 \wedge \dots \wedge dx^n, \\
 (6) \quad \omega \wedge df_{i_{p+1}} \wedge \dots \wedge df_{i_n} &= \varepsilon \cdot k_{i_1 \dots i_p} dx^1 \wedge \dots \wedge dx^n
 \end{aligned}$$

for a function $k_{i_1 \dots i_p} \in C^\infty(\Sigma, \mathbb{R})$. Thus

$$(7) \quad k_{i_1 \dots i_p}|_U = g_{i_1 \dots i_p} \cdot J|_U.$$

Since ω and each df_i is W -invariant, from (6) we get $k_{i_1 \dots i_p} \circ \sigma = \det(\sigma^{-1})k_{i_1 \dots i_p}$ for each $\sigma \in W$. But then by (5) we have $k_{i_1 \dots i_p} = \omega_{i_1 \dots i_p} \cdot J$ for unique $\omega_{i_1 \dots i_p} \in C^\infty(\Sigma, \mathbb{R})^W$, and (7) then implies $\omega_{i_1 \dots i_p}|_U = g_{i_1 \dots i_p}$, so that the lemma follows since U is dense. □

3.4. Question. Let $\rho : G \rightarrow GL(V)$ be a representation of a compact Lie group in a finite dimensional vector space V . Let $f = (f_1, \dots, f_m) : V \rightarrow \mathbb{R}^m$ be the polynomial mapping whose components f_i are a minimal set of homogeneous generators for the algebra $\mathbb{R}[V]^G$ of invariant polynomials.

We consider the pullback homomorphism $f^* : \Omega^p(\mathbb{R}^m) \rightarrow \Omega^p(V)$. Is it surjective onto the space $\Omega_{\text{hor}}^p(V)^G$ of G -invariant horizontal smooth p -forms on V ?

The proof of Theorem 3.7 below will show that the answer is yes for polar representations of compact Lie groups if the corresponding generalized Weyl group is a reflection group.

In general the answer is no. A counterexample is the following: Let the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of order n , viewed as the group of n -th roots of unity, act on $\mathbb{C} = \mathbb{R}^2$ by complex multiplication. A generating system of polynomials consists of $f_1 = |z|^2, f_2 = \text{Re}(z^n), f_3 = \text{Im}(z^n)$. But then each df_i vanishes at 0 and there is no chance to have the horizontal invariant volume form $dx \wedge dy$ in $f^*\Omega(\mathbb{R}^3)$.

3.5. Polar representations. Let G be a compact Lie group and let $\rho : G \rightarrow GL(V)$ be an orthogonal representation in a finite dimensional real vector space V which admits a section Σ . Then the section turns out to be a linear subspace and the representation is called a *polar representation*, following Dadok [10], who gave a complete classification of all polar representations of connected Lie groups. They were called variationally complete representations by Conlon [9] before.

3.6. Theorem (Terng [28], Theorem D or [19], 4.12). Let $\rho : G \rightarrow GL(V)$ be a polar representation of a compact Lie group G , with section Σ and generalized Weyl group $W = W(\Sigma)$. Then the algebra $\mathbb{R}[V]^G$ of G -invariant polynomials on V is isomorphic to the algebra $\mathbb{R}[\Sigma]^W$ of W -invariant polynomials on the section Σ , via the restriction mapping $f \mapsto f|_{\Sigma}$.

3.7. Theorem. Let $\rho : G \rightarrow GL(V)$ be a polar representation of a compact Lie group G , with section Σ and generalized Weyl group $W = W(\Sigma)$. Let us suppose that $W = W(\Sigma)$ is generated by reflections (a reflection group or Coxeter group). Then the pullback to Σ of differential forms induces an isomorphism

$$\Omega_{\text{hor}}^p(V)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)}.$$

According to Dadok [10], remark after Proposition 6, for any polar representation of a connected compact Lie group the generalized Weyl group $W(\Sigma)$ is a reflection group. This theorem is true for polynomial differential forms, and also for real analytic differential forms, by essentially the same proof.

Proof. Let $i : \Sigma \rightarrow V$ be the embedding. By the first part of the proof of Theorem 2.4 the pullback mapping $i^* : \Omega_{\text{hor}}^p(V)^G \rightarrow \Omega_{\text{hor}}^p(\Sigma)^W$ is injective, and we shall show that it is also surjective. Let f_1, \dots, f_n be a minimal set of homogeneous generators of the algebra $\mathbb{R}[\Sigma]^W$ of W -invariant polynomials on Σ . Then by Lemma 3.3 each $\omega \in \Omega^p(\Sigma)^W$ is of the form

$$\omega = \sum_{j_1 < \dots < j_p} \omega_{j_1 \dots j_p} df_{j_1} \wedge \dots \wedge df_{j_p},$$

where $\omega_{j_1 \dots j_p} \in C^\infty(\Sigma, \mathbb{R})^W$. By Theorem 3.6 the algebra $\mathbb{R}[V]^G$ of G -invariant polynomials on V is isomorphic to the algebra $\mathbb{R}[\Sigma]^W$ of W -invariant polynomials on

the section Σ , via the restriction mapping i^* . Choose polynomials $\tilde{f}_1, \dots, \tilde{f}_n \in \mathbb{R}[V]^G$ with $\tilde{f}_i \circ i = f_i$ for all i . Put $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n) : V \rightarrow \mathbb{R}^n$. Then we use the theorem of G. Schwarz (see 3.1) to find $h_{i_1, \dots, i_p} \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with $h_{i_1, \dots, i_p} \circ f = \omega_{i_1, \dots, i_p}$ and consider

$$\tilde{\omega} = \sum_{j_1 < \dots < j_p} (h_{j_1 \dots j_p} \circ \tilde{f}) d\tilde{f}_{j_1} \wedge \dots \wedge d\tilde{f}_{j_p},$$

which is in $\Omega_{\text{hor}}^p(V)^G$ and satisfies $i^*\tilde{\omega} = \omega$. □

Sketch of another proof avoiding 3.3 (suggested by a referee). Let $R = C^\infty(V)^G = C^\infty(\Sigma)^W$ and let Ω_R^p be its module of Kähler p -forms (see Kunz [13] for the notion of Kähler forms). Also let $S = \mathbb{R}[V]^G = \mathbb{R}[\Sigma]^W$ (using 3.6). Then the canonical mapping $\Omega_R^p \rightarrow \Omega^p(\Sigma)^W$ is surjective. This follows for the canonical mapping from Ω_S^p into the space of forms with polynomial coefficients from the result of Solomon [25] by using 3.6 again as in the proof of 3.7; and it can be extended to smooth coefficients by Theorem 1.4 of Ronga [22], which says that equivariant stability and infinitesimal equivariant stability are equivalent, in a way which is similar to the argument of Proposition 6.8 of Schwarz [24]. So we see that the composition $\Omega_R^p \rightarrow \Omega^p(V)^G \rightarrow \Omega^p(\Sigma)^W$ is surjective, thus also the right hand side mapping has to be surjective. □

3.8. Corollary. *Let $\rho : G \rightarrow O(V, \langle \cdot, \cdot \rangle)$ be an orthogonal polar representation of a compact Lie group G , with section Σ and generalized Weyl group $W = W(\Sigma)$. Let us suppose that $W = W(\Sigma)$ is generated by reflections (a reflection group or Coxeter group). Let $B \subset V$ be an open ball centered at 0.*

Then the restriction of differential forms induces an isomorphism

$$\Omega_{\text{hor}}^p(B)^G \xrightarrow{\cong} \Omega^p(\Sigma \cap B)^{W(\Sigma)}.$$

Proof. Check the proof of 3.7 or use the following argument. Suppose that $B = \{v \in V : |v| < 1\}$ and consider a smooth diffeomorphism $f : [0, 1) \rightarrow [0, \infty)$ with $f(t) = t$ near 0. Then $g(v) := \frac{f(|v|)}{|v|}v$ is a G -equivariant diffeomorphism $B \rightarrow V$ and by 3.7 we get:

$$\Omega_{\text{hor}}^p(B)^G \xrightarrow{(g^{-1})^*} \Omega_{\text{hor}}^p(V)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)} \xrightarrow{g^*} \Omega^p(\Sigma \cap B)^{W(\Sigma)}. \quad \square$$

4. PROOF OF THE MAIN THEOREM

Let us assume that we are in the situation of the main theorem 2.4, for the rest of this section.

4.1. For $x \in M$ let S_x be a (normal) slice and G_x the isotropy group, which acts on the slice. Then $G.S_x$ is open in M and G -equivariantly diffeomorphic to the associated bundle $G \rightarrow G/G_x$ via

$$\begin{array}{ccccc} G \times S_x & \xrightarrow{q} & G \times_{G_x} S_x & \xrightarrow{\cong} & G.S_x \\ & & \downarrow & & \downarrow r \\ & & G/G_x & \xrightarrow{\cong} & G.x \end{array}$$

where r is the projection of a tubular neighborhood. Since $q : G \times S_x \rightarrow G \times_{G_x} S_x$ is a principal G_x -bundle with principal right action $(g, s).h = (gh, h^{-1}.s)$, we have an isomorphism

$$q^* : \Omega(G \times_{G_x} S_x) \rightarrow \Omega_{G_x\text{-hor}}(G \times S_x)^{G_x}.$$

Since q is also G -equivariant for the left G -actions, the isomorphism q^* maps the subalgebra $\Omega_{\text{hor}}^p(G.S_x)^G \cong \Omega_{\text{hor}}^p(G \times_{G_x} S_x)^G$ of $\Omega(G \times_{G_x} S_x)$ to the subalgebra $\Omega_{G_x\text{-hor}}^p(S_x)^{G_x}$ of $\Omega_{G_x\text{-hor}}(G \times S_x)^{G_x}$. So we have proved:

Lemma. *In this situation there is a canonical isomorphism*

$$\Omega_{\text{hor}}^p(G.S_x)^G \xrightarrow{\cong} \Omega_{G_x\text{-hor}}^p(S_x)^{G_x}$$

which is given by pullback along the embedding $S_x \rightarrow G.S_x$.

4.2. Rest of the proof of Theorem 2.4. Now let us consider $\omega \in \Omega^p(\Sigma)^{W(\Sigma)}$. We want to construct a form $\tilde{\omega} \in \Omega_{\text{hor}}^p(M)^G$ with $i^*\tilde{\omega} = \omega$. This will finish the proof of Theorem 2.4.

Choose $x \in \Sigma$ and an open ball B_x with center 0 in $T_x M$ such that the Riemannian exponential mapping $\exp_x : T_x M \rightarrow M$ is a diffeomorphism on B_x . We consider now the compact isotropy group G_x and the slice representation $\rho_x : G_x \rightarrow O(V_x)$, where $V_x = \text{Nor}_x(G.x) = (T_x(G.x))^\perp \subset T_x M$ is the normal space to the orbit. This is a polar representation with section $T_x \Sigma$, and its generalized Weyl group is given by $W(T_x \Sigma) \cong N_G(\Sigma) \cap G_x / Z_G(\Sigma) = W(\Sigma)_x$ (see [19]); it is a Coxeter group by assumption (1) in 2.4. Then $\exp_x : B_x \cap V_x \rightarrow S_x$ is a diffeomorphism onto a slice and $\exp_x : B_x \cap T_x \Sigma \rightarrow \Sigma_x \subset \Sigma$ is a diffeomorphism onto an open neighborhood Σ_x of x in the section Σ .

Let us now consider the pullback $(\exp|_{B_x \cap T_x \Sigma})^* \omega \in \Omega^p(B_x \cap T_x \Sigma)^{W(T_x \Sigma)}$. By Corollary 3.8 there exists a unique form $\varphi^x \in \Omega_{G_x\text{-hor}}^p(B_x \cap V_x)^{G_x}$ such that $i^* \varphi^x = (\exp|_{B_x \cap T_x \Sigma})^* \omega$, where i_x is the embedding. Then we have

$$((\exp|_{B_x \cap V_x})^{-1})^* \varphi^x \in \Omega_{G_x\text{-hor}}^p(S_x)^{G_x}$$

and by Lemma 4.1 this form corresponds uniquely to a differential form $\omega^x \in \Omega_{\text{hor}}^p(G.S_x)^G$ which satisfies $(i|\Sigma_x)^* \omega^x = \omega|_{\Sigma_x}$, since the exponential mapping commutes with the respective restriction mappings. Now the intersection $G.S_x \cap \Sigma$ is the disjoint union of all the open sets $w_j(\Sigma_x)$ where we pick one w_j in each left coset of the subgroup $W(\Sigma)_x$ in $W(\Sigma)$. If we choose $g_j \in N_G(\Sigma)$ projecting on w_j for all j , then

$$\begin{aligned} (i|_{w_j(\Sigma_x)})^* \omega^x &= (\ell_{g_j} \circ i|\Sigma_x \circ w_j^{-1})^* \omega^x = (w_j^{-1})^* (i|\Sigma_x)^* \ell_{g_j}^* \omega^x \\ &= (w_j^{-1})^* (i|\Sigma_x)^* \omega^x = (w_j^{-1})^* (\omega|_{\Sigma_x}) = \omega|_{w_j(\Sigma_x)}, \end{aligned}$$

so that $(i|_{G.S_x \cap \Sigma})^* \omega^x = \omega|_{G.S_x \cap \Sigma}$. We can do this for each point $x \in \Sigma$.

Using the method of Palais ([18], proof of 4.3.1) we may find a sequence of points $(x_n)_{n \in \mathbb{N}}$ in Σ such that the $\pi(\Sigma_{x_n})$ form a locally finite open cover of the orbit space $M/G \cong \Sigma/W(\Sigma)$, and a smooth partition of unity f_n consisting of G -invariant functions with $\text{supp}(f_n) \subset G.S_{x_n}$. Then $\tilde{\omega} := \sum_n f_n \omega^{x_n} \in \Omega_{\text{hor}}^p(M)^G$ has the required property $i^* \tilde{\omega} = \omega$. \square

5. BASIC VERSUS EQUIVARIANT COHOMOLOGY

5.1. Basic cohomology. For a Lie group G and a smooth G -manifold M , by 2.2 we may consider the basic cohomology $H_{G\text{-basic}}^p(M) = H^p(\Omega_{\text{hor}}^*(M)^G, d)$.

The best known application of basic cohomology is the case of a compact connected Lie group G acting on itself by left translations; see e.g. [11] and papers cited therein: By homotopy invariance and integration we get $H(G) = H_{G\text{-basic}}(G) = H(\Lambda(\mathfrak{g}^*))$, and the latter space turns out as the space $\Lambda(\mathfrak{g}^*)^{\mathfrak{g}}$ of $\text{ad}(\mathfrak{g})$ -invariant forms, using the inversion. This is the theorem of Chevalley and Eilenberg. Moreover, $\Lambda(\mathfrak{g}^*)^{\mathfrak{g}} = \Lambda(P)$, where P is the graded subspace of primitive elements, using the Weil map and transgression, whose determination in all concrete cases by Borel and Hirzebruch is a beautiful part of modern mathematics.

In more general cases the determination of basic cohomology was more difficult. A replacement for it is equivariant cohomology, which comes in two guises:

5.2. Equivariant cohomology, Borel model. For a topological group and a topological G -space the equivariant cohomology was defined as follows; see [3]: Let $EG \rightarrow BG$ be the classifying G -bundle, and consider the associated bundle $EG \times_G M$ with standard fiber the G -space M . Then the equivariant cohomology is given by $H^p(EG \times_G M; \mathbb{R})$.

5.3. Equivariant cohomology, Cartan model. For a Lie group G and a smooth G -manifold M we consider the space

$$(S^k \mathfrak{g}^* \otimes \Omega^p(M))^G$$

of all homogeneous polynomial mappings $\alpha : \mathfrak{g} \rightarrow \Omega^p(M)$ of degree k from the Lie algebra \mathfrak{g} of G to the space of p -forms, which are G -equivariant: $\alpha(\text{Ad}(g^{-1})X) = \ell_g^* \alpha(X)$ for all $g \in G$. The mapping

$$\begin{aligned} d_{\mathfrak{g}} : A_G^q(M) &\rightarrow A_G^{q+1}(M), \\ A_G^q(M) &:= \bigoplus_{2k+p=q} (S^k \mathfrak{g}^* \otimes \Omega^p(M))^G, \\ (d_{\mathfrak{g}} \alpha)(X) &:= d(\alpha(X)) - i_{\zeta_X} \alpha(X) \end{aligned}$$

satisfies $d_{\mathfrak{g}} \circ d_{\mathfrak{g}} = 0$ and the following result holds.

Theorem. *Let G be a compact connected Lie group and let M be a smooth G -manifold. Then*

$$H^p(EG \times_G M; \mathbb{R}) = H^p(A_G^*(M), d_{\mathfrak{g}}).$$

This result is stated in [1] together with some arguments, and it is attributed to [5], [6] in chapter 7 of [2]. I was unable to find a satisfactory published proof.

5.4. Let M be a smooth G -manifold. Then the obvious embedding $j(\omega) = 1 \otimes \omega$ gives a mapping of graded differential algebras

$$j : \Omega_{\text{hor}}^p(M)^G \rightarrow (S^0 \mathfrak{g}^* \otimes \Omega^p(M))^G \rightarrow \bigoplus_k (S^k \mathfrak{g}^* \otimes \Omega^{p-2k}(M))^G = A_G^p(M).$$

On the other hand evaluation at $0 \in \mathfrak{g}$ defines a homomorphism of graded differential algebras $\text{ev}_0 : A_G^*(M) \rightarrow \Omega^*(M)^G$, and $\text{ev}_0 \circ j$ is the embedding $\Omega_{\text{hor}}^*(M)^G \rightarrow \Omega^*(M)^G$. Thus we get canonical homomorphisms in cohomology

$$\begin{array}{ccccc} H^p(\Omega_{\text{hor}}^*(M)^G) & \xrightarrow{J^*} & H^p(A_G^*(M), d_{\mathfrak{g}}) & \longrightarrow & H^p(\Omega^*(M)^G, d) \\ \parallel & & \parallel & & \parallel \\ H_{G\text{-basic}}^p(M) & \longrightarrow & H_G^p(M) & \longrightarrow & H^p(M)^G. \end{array}$$

If G is compact and connected we have $H^p(M)^G = H^p(M)$, by integration and homotopy invariance.

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