THE LARGENESS OF SETS OF POINTS
WITH NON-DENSE ORBIT IN BASIC SETS ON SURFACES

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Abstract. We show that if $f$ is a diffeomorphism of a closed surface and $A$ is a basic set for $f$, then $\text{HD}\{x \in A : the\ orbit\ of\ x\ by\ f\ is\ not\ dense\ in\ A\} = \text{HD}(A)$.

1. Introduction

We consider the largeness of the set of points with non-dense orbit for a given diffeomorphism of a compact manifold.

In general, if a homeomorphism defined on a compact metric space is topologically transitive, then such a set belongs to the first category. This fact means that it is a small set in the sense of general topology.

On the other hand, Urbański [6] has proved the following theorem: For a given transitive $C^2$-Anosov diffeomorphism of a compact manifold $M$, the Hausdorff dimension of the set of points with non-dense orbit is equal to $\dim M$. From this theorem we can say that it has the same ‘fatness’ as the manifold.

Let $A$ be a basic set of a diffeomorphism $f : M \to M$ of a closed manifold $M$. There is no way to calculate the Hausdorff dimension of $A$ in general. However when $M$ is a surface, McCluskey and Manning [5] have shown the Hausdorff dimension of $A$ can be represented by the topological pressures of two continuous functions.

Using this result, we shall estimate a certain probability measure on $A$ and show that the set of points with non-dense orbit for the restriction $f|_A : A \to A$ has the same Hausdorff dimension as $A$.

2. Definitions and the result

Let $M$ be a closed manifold and $f : M \to M$ a diffeomorphism, and let $A \subset M$ be a compact invariant set, $fA = A$. We say that $A$ is hyperbolic for $f$ if:

(a) the tangent bundle of $M$ restricted to $A$ decomposes as a continuous direct sum, $T_AM = E^s \oplus E^u$, which is invariant by the differential of $f$, $Df$;

(b) there exists a Riemannian metric (adapted metric) and a number $0 < \gamma < 1$ such that $\|Df(v)\| \leq \gamma \|v\|$ and $\|Df^{-1}(u)\| \leq \frac{1}{\gamma} \|u\|$ for any $v \in E^s$, $u \in E^u$. Further we say that a hyperbolic set $A$ for $f$ is isolated if there is a neighborhood $U$ of $A$ such that $\bigcap_{n=-\infty}^\infty f^nU = A$. $A \subset M$ is called a basic set for $f$ if $A$ is an isolated hyperbolic...
set for \( f \) and \( f|_A : A \to A \) is topologically transitive. We say that a basic set for \( f \) is **trivial** if it is a periodic orbit for \( f \).

For each subset \( A \) of \( M \) and \( \alpha > 0 \) we define the **\( \alpha \)-measure** of \( A \) by

\[
m_\alpha(A) = \liminf_{\varepsilon \to 0} \inf_{U \in \mathcal{U}} \sum_{U \in \mathcal{U}} (\text{diam}U)^\alpha
\]

where the infimum is taken over all countable covers \( \mathcal{U} \) of \( A \) by sets with diameter less than \( \varepsilon \). The **Hausdorff dimension** of \( A \) is defined by

\[
HD(A) = \inf \{ \alpha : m_\alpha(A) = 0 \}.
\]

This definition is independent of the choice of a Riemannian metric. Remark that for each \( A \subset B \subset M \), \( 0 \leq HD(A) \leq HD(B) \leq HD(M) = \dim M \); also \( HD(A) = 0 \) if \( A \subset M \) is at most countable.

Our purpose is to prove the following.

**Theorem.** Let \( f : M \to M \) be a \( C^1 \)-Hölder diffeomorphism of a closed surface \( M \) and \( A \subset M \) be a basic set for \( f \). For each open set \( V \subset M \) which intersects with \( A \),

\[
HD(\{ x \in V \cap A : \mathcal{O}_f(x) \text{ is not dense in } A \}) = HD(A)
\]

where \( \mathcal{O}_f(x) \) denotes the orbit of \( x \) by \( f \).

**Remark.** When \( A = M \), i.e. \( f : M \to M \) is a transitive Anosov diffeomorphism, this result is the same as Urbański’s theorem [6].

It is unknown that the theorem is extended on manifolds of higher dimension.

In order to prove the above theorem we need a few known results.

Let \( (\Sigma_A, \sigma) \) be a topologically mixing Markov subshift and \( \eta : \Sigma_A \to \mathbb{R} \) a Hölder continuous function. Denote for each \( a = (a_i) \in \Sigma_A \) and \( k, l \in \mathbb{Z} \) with \( k < l \),

\[
[ka_lak_{l+1}\ldots a_k] = \{ b = (b_i) \in \Sigma_A : a_j = b_j \text{ for every } j = k, k + 1, \ldots, l \}.
\]

Bowen [2] has proved that there is a unique Borel probability measure \( m \) called Gibbs measure on \( \Sigma_A \) for which one can find a constant \( C \geq 1 \) such that for any \( a = (a_i) \in \Sigma_A \), \( n \in \mathbb{N} \),

\[
C^{-1} \leq \frac{m([a_0a_1\ldots a_{n-1}a_n])}{\exp(-P(\Sigma_A, \sigma, \eta)n + \sum_{j=0}^{n-1} \eta(\sigma^j a))} \leq C
\]

where \( P(\Sigma_A, \sigma, \eta) \) is the topological pressure of \( \eta \) for \( (\Sigma_A, \sigma) \).

Let \( \mu \) be a Borel probability measure on a compact manifold \( M \) with distance \( d \) for which one can find constants \( h > 0 \), \( D \geq 1 \), and \( r_0 > 0 \) such that \( \mu(B_r(x)) \leq Dr^h \) for every \( x \in \text{supp} \mu \) and \( 0 < r < r_0 \) where \( B_r(x) = \{ y \in \mathbb{M} : d(x, y) \leq r \} \). For any integer \( k \geq 1 \) let \( E_k \) denote a finite collection of compact subsets of \( \text{supp} \mu \) with positive measure \( \mu \) and let \( \bigcup E_k \) denote the union of all elements of \( E_k \). We assume that the collection \( E_k \) satisfies the following conditions: \( \bigcup E_1 = \text{supp} \mu \), \( \mu(F \cap G) = 0 \) for \( F, G \in E_k \) with \( F \neq G \), and every set \( H \in E_{k+1} \) is contained in a unique element \( I \in E_k \). We write \( \Delta_k = \inf \{ \text{density}(\bigcup E_{k+1}, F) : F \in E_{k+1} \} \) where

\[
\text{density}(\bigcup E_{k+1}, F) = \frac{\mu(\bigcup E_{k+1} \cap F)}{\mu(F)},
\]

and \( d_k = \sup \{ \text{diam}F : F \in E_k \} \) for every \( k \geq 1 \). It is known that if \( \Delta_k > 0 \), \( d_k < 1 \) for every \( k \geq 1 \) and \( \lim_{k \to \infty} d_k = 0 \), then

\[
HD(\bigcap_{k=1}^{\infty} E_k) \geq h - \limsup_{k \to \infty} \frac{\sum_{j=1}^{k} \log \Delta_j}{\log d_k}.
\]

This is a result by McMullen (see [6]).
In order to state the McCluskey-Manning theorem [5], let \( f : M \to M \) be a diffeomorphism of a closed surface \( M \) and \( A \subset M \) be a basic set for \( f \) with \( \dim E^s = 1 \). Define \( \phi^{(u)}, \phi^{(s)} : \Lambda \to \mathbb{R} \) by \( \phi^{(u)}(x) = -\log \|D_x f|_{E^u}\| \), \( \phi^{(s)}(x) = -\log \|D_x f^{-1}|_{E^s}\| \). Then there exist unique \( 0 \leq \delta^u, \delta^s \leq 1 \) such that
\[
P(A, f|_A, \delta^u \phi^{(u)}) = 0, \quad P(A, f^{-1}|_A, \delta^s \phi^{(s)}) = 0 \text{ and } HD(A) = \delta^u + \delta^s.
\]

3. Proof of the theorem

We shall prove that
\[
HD(\{ x \in V \cap A : O_f(x) \text{ is not dense in } A \}) \geq HD(A).
\]

We can check the case \( \dim E^s = 0 \) or \( \dim E^s = 2 \), \( A \) is trivial. So, \( HD(A) = 0 \) and there is nothing to prove. Thus we consider \( \dim E^s = 1 \).

Taking an adapted metric \( \| \cdot \| \), there is a constant \( \beta > 1 \) such that \( \min_{x \in A} \{ \| D_x f \|, \| D_x f^{-1} \| \} > \beta \). Remark that \( \| D_x f \| = \| D_x f|_{E^u}\| \), \( \| D_x f^{-1} \| = \| D_x f^{-1}|_{E^s}\| \). Let \( d \) be the distance on \( M \) induced by \( \| \cdot \| \) and \( \delta^u, \delta^s \) be as above. If \( \delta^u = 0 \) or \( \delta^s = 0 \), then the topological entropy of \( (A, f|_A) \) is zero. Thus \( A \) is trivial, and \( HD(A) = 0 \).

Therefore it suffices to check with the case \( \delta^u, \delta^s > 0 \). For convenience we may assume that \( f|_A : A \to A \) is topologically mixing. (Use the spectral decomposition [2] if necessary.)

Let \( 0 < c < 1 \) be an expansive constant of \( f|_A \). As \( A \) is a basic set of a \( C^1 \)-Hölder diffeomorphism \( f \), we can choose \( 0 < \varepsilon_0 < c/4 \) so small as to satisfy \( y \mapsto \| D_y f \| \), \( \| D_y f^{-1} \| \) are Hölder continuous functions on \( 2\varepsilon_0 \) neighborhood of \( A \), and there is a \( \lambda \in (0, 1) \) such that if \( x, z \in A \) and \( d(f^k x, f^k z) \leq \varepsilon_0 \) for \( k = -n, \ldots, 0, \ldots, n \), then \( d(x, z) < \lambda^n \) (see [2]).

Let \( \varepsilon_0 > 0 \) be as above and pick a Markov partition \( \mathcal{R} = \{ R_1, \ldots, R_s \} \) of \( A \) such that \( \dim \mathcal{R} < \varepsilon_0 \) and \( \sharp \{ 1 \leq q \leq s : R_p \cap R_q = \emptyset \} > 0 \) for every \( p = 1, 2, \ldots, s \) (see [2]).

Let \( A = (A_{ij})_{1 \leq i, j \leq s} \) be the structure matrix of \( \mathcal{R} \) and \( (\Sigma_A, \sigma) \) be the corresponding Markov subshift. We denote the set of all words of length \( n \) of \( \Sigma_A \) by \( \Sigma(n) \). Since \( (\Sigma_A, \sigma) \) is topologically mixing, there is a \( K_0 \geq 2 \) such that \( A^K > 0 \), i.e. \( (A^K)_{ij} > 0 \) for each \( 1 \leq i, j \leq s \) and \( K \geq K_0 \).

Let \( h : \Sigma_A \to A \) be the continuous surjection which satisfies \( h \circ \sigma = f \circ h \) and \( \bigcap_{i=0}^{k-l} f^{-i}R_{a_i} = h(k[a_k \ldots a_1]) \) for every \( (a_k \ldots a_1) \in \Sigma(k + l + 1) \). We denote \( \mathcal{R}(k, l) = \{ \bigcap_{i=0}^{k-l} f^{-i}R_{a_i} : (a_k \ldots a_1) \in \Sigma(k + l + 1) \} \) for all \( k, l \in \mathbb{Z} \) with \( k \leq l \). Remark that \( h \) is Hölder continuous (see [4]) and that \( h \) is bounded finite to one (this is a Bowen’s result, see [1]), so there is an integer \( \varepsilon_0 > 0 \) such that \( \sharp \{ W \in \mathcal{R}(k, l) : x \in W \} \leq \varepsilon_0 \) for every \( x \in A \) and \( k, l \in \mathbb{Z} \) with \( k \leq l \). (If \( A \) is totally disconnected, then the map \( h \) is bijective.)

By choosing \( \varepsilon_0 > 0 \) small enough and reordering if necessary we can assume that \( R_1 \subset V \cap A \) holds.

Since \( P(\Sigma_A, \sigma, \delta^u \phi^{(u)} \circ h) = P(A, f|_A, \delta^u \phi^{(u)}) = 0 \) and \( \delta^u \phi^{(u)} \circ h \) is Hölder continuous, there is a Borel probability measure (Gibbs measure) \( m_+ \) on \( \Sigma_A \) and a constant \( C_3 \geq 1 \) such that
\[
C_3^{-1} \leq \frac{m_+(a_0 \ldots a_n)}{\exp\{ \sum_{j=0}^{n} \delta^u \phi^{(u)}(h(\sigma^j a)) \}} \leq C_3
\]
for every \( a = (a_i) \in \Sigma_A \), \( n \in \mathbb{Z}^+ \).
Similarly, we can find a Borel probability measure $m_-$ on $\Sigma_A$ and a constant $C_4 \geq 1$ such that
\[ C_4^{-1} \leq \frac{m_-(-n[a_{-n} \ldots a_0]_0)}{\exp\{\sum_{j=0}^n \delta^j \phi(s) \circ h_j^{-1} \}} \leq C_4 \]
for every $a = (a_i) \in \Sigma_A$, $n \in \mathbb{Z}^+$. 

Let $m$ be a Borel probability measure on $\Sigma_A$ such that
\[
m(-k[a_{-k} \ldots a_0]_l) = \begin{cases} C_5 m_+(0[a_0 \ldots a_l]_0) m_-(-k[a_{-k} \ldots a_0]_0) & (a_0 = 1), \\ 0 & (a_0 \neq 1) \end{cases}
\]
for each $a = (a_i) \in \Sigma_A$, $k, l \in \mathbb{Z}^+$, and $C_5 = m_+(0[1]_0)^{-1} m_-(-0[1]_0)^{-1}$.

Define a Borel probability measure $\mu$ on $M$ by
\[
\mu(A) = m(h^{-1}(A \cap \Lambda))
\]
for each Borel set $A$ of $M$. Clearly $\text{supp} \mu = R_1$ and $\mu(\bigcup_{n=-\infty}^\infty f^{-i}(\partial \mathcal{R})) = 0$ where $\partial \mathcal{R} = \{x \in \Lambda : x \in R_p \cap R_q \text{ for some } 1 \leq p < q \leq s\}$.

If $a_0 = 1$ and $(a_{-k} \ldots a_0 \ldots a_l) \in \Sigma(k + l + 1)$, $k, l \in \mathbb{Z}^+$, then we have that
\[
C_1^{-1} = \frac{\mu(\bigcap_{l=-k}^l f^{-i} R_{a_l})}{\exp\{\sum_{i=0}^l \delta^i \phi(a_i)(f^i x) + \sum_{j=0}^k \delta^j \phi(s)(f^{-j} x)\}} \leq C_1
\]
where $C_1 = C_3 C_4 C_5$. (Such a measure $\mu$ has been constructed by Mañe in [4].)

**Lemma 1.** There are constants $C_2 \geq 1$ and $r_0 > 0$ such that
\[
\mu(B_r(x)) \leq C_2 r^{\text{HD}(\Lambda)}
\]
for all $x \in R_1$ and $0 < r < r_0$.

**Remark.** The measure $\mu$ is true that for some constant $C_7 \leq 1$,
\[
\mu(B_r(x)) \geq C_7 r^{\text{HD}(\Lambda)} \quad (x \in R_1, \ 0 < r < r_0).
\]

Applying this $\mu$ to McMullen’s result, our result is obtained as follows.

Take an integer $l_0 \geq 1$ so large as to satisfy $C_7 l \beta^{-\delta l_0}$, $C_7 l \beta^{-\delta l} \leq 1/4$ for all $l \geq l_0$.

As $(\Sigma_A, \sigma)$ is topologically mixing, for some $1 \leq p_0, p_1, p_2, q_0, q_1, q_2 \leq s$,

\[ p_1 \neq p_2, \ q_1 \neq q_2, \ \text{and} \ A_{p_0 p_1} = A_{p_0 p_2} = A_{q_0 q_1} = A_{q_0 q_2} = 1. \]

Fix an integer $l > \max\{l_0, K_0\}$, and pick $(v_{-l} \ldots v_0 \ldots v_l) \in \Sigma(2l + 1)$ with $v_0 \neq 1$ and so that $(p_0 p_1), (q_0 q_1)$ do not appear as any substring of $(v_0 \ldots v_l)$, $(v_{-l} \ldots v_0)$, respectively. Set $Y = \bigcap_{j=-l}^l f^{-j} R_{v_j}$ and define
\[
E_1 = \{R_1\},
\]
\[
E_{k+1} = E_{k+1}(l)
\]

where $Y$ is the interior of $Y$ in $\Lambda$, $k \in \mathbb{N}$.

Then we can check that for each $k \geq 1$, $d_k < \lambda^{(k-1)l}$. 

Lemma 2. (1) \( \Delta_1 > 0 \), (2) \( \Delta_{k+1} \geq 1/2 \) for every \( k \geq 1 \).

Therefore,
\[
\sum_{j=1}^{k} \frac{\log \Delta_j}{\log d_k} \leq \frac{\log \Delta_1 + (k-1) \log 1/2}{(k-1)\log \lambda}
\]
for every \( k \geq 2 \). Other conditions of McMullen’s result are obviously satisfied.

On the other hand, since
\[
\bigcap_{k=1}^{\infty} \cup E_k \subset \{ x \in V \cap A : \mathcal{O}_f(x) \text{ is not dense in } A \},
\]
we have
\[
\text{HD}(\{ x \in V \cap A : \mathcal{O}_f(x) \text{ is not dense in } A \}) \geq \text{HD}(\bigcap_{k=1}^{\infty} \cup E_k) \geq \text{HD}(A) - \lim_{k \to \infty} \frac{\log \Delta_1 + (k-1) \log 1/2}{(k-1)\log \lambda} = \text{HD}(A) - \frac{\log 1/2}{\log \lambda}.
\]

Since \( l > \max\{l_0, K_0\} \) is arbitrary, our requirement is obtained.

4. Proof of Lemma 1

Mañé [4] has proved Lemma 1 when \( \Lambda \) is totally disconnected. We give a proof of Lemma 1, for the general case, as follows.

Choose \( C_6 \geq 1 \) such that for any \( n, m \in \mathbb{N} \),
\[
C_6^{-1} \leq \prod_{j=0}^{n-1} \frac{\| Df_{jz}f \|}{\| Df_{jx}f \|} \leq C_6
\]
for all \( x \in A \), \( z \in M \) satisfying \( \max_{0 \leq k \leq n-1} d(f^kx, f^kz) \leq \varepsilon_0 \), and
\[
C_6^{-1} \leq \prod_{j=0}^{m-1} \frac{\| Df^{-jw}f^{-1} \|}{\| Df^{-jy}f^{-1} \|} \leq C_6
\]
for all \( y \in A \), \( w \in M \) satisfying \( \max_{0 \leq k \leq m-1} d(f^{-k}y, f^{-k}w) \leq \varepsilon_0 \).

Put \( \delta_0 = \inf \{ d(x, y) : x \in R_p, y \in R_q, R_p \cap R_q = \emptyset, 1 \leq p < q \leq s \} > 0 \) and \( r_0 = \min\{\delta_0/2, \varepsilon_0/2\} > 0 \).

Step 1. For each \( x \in A \), \( 0 < r < r_0 \), and \( l, n \in \mathbb{N} \), if \( C_6r \exp\{-\sum_{i=0}^{l-1} \phi^{(n)}(f^ix)\} < r_0 \) and \( C_6r \exp\{-\sum_{i=0}^{l-1} \phi^{(s)}(f^{-i}x)\} < r_0 \), then
\[
B_r(x) \subset B_{r_0}^l(x, -l, n)
\]
where \( B_{r_0}^l(x, -l, n) = \{ z \in M : d(f^jx, f^jz) \leq r_0 \text{ for } j = -l, \ldots, 0, \ldots, n \} \).
Proof. We first show that for \( y \in B_1(x) \), \( d(f^j x, f^j y) \leq r_0 \) for every \( j = 1, \ldots, n \) if \( C_6 r \exp \{ -\sum_{i=0}^{n-1} \phi^{(i)}(f^i x) \} < r_0 \).

To see this, we choose \( \varepsilon > 0 \) so small that \( C_6 (r + \varepsilon) \exp \{ -\sum_{i=0}^{n-1} \phi^{(i)}(f^i x) \} < r_0 \) and take a smooth curve \( \xi : [0, 1] \rightarrow M \) such that \( \xi(0) = x \), \( \xi(1) = y \), and \( \int_0^1 \| \dot{\xi}(s) \| \, ds \leq r + \varepsilon \).

Since
\[
d(x, \xi(t)) \leq \int_0^t \| \dot{\xi}(s) \| \, ds
\]
we have
\[
d(f x, f \xi(t)) \leq \int_0^t \| Df(\dot{\xi}(s)) \| \, ds
\]
\[
\leq \int_0^t \| D\xi(s) f \| \cdot \| \dot{\xi}(s) \| \, ds
\]
\[
\leq C_6 \exp \{ -\phi^{(n)}(x) \} \int_0^t \| \dot{\xi}(s) \| \, ds \quad (0 \leq t \leq 1).
\]

Assume that for each \( 1 \leq k - 1 < n \), \( 0 \leq t \leq 1 \), and \( j = 1, 2, \ldots, k - 1 \),
\[
d(f^j x, f^j \xi(t)) \leq C_6 \exp \{ -\sum_{i=0}^{j-1} \phi^{(i)}(f^i x) \} \int_0^t \| \dot{\xi}(s) \| \, ds.
\]

Then we have
\[
d(f^j x, f^j \xi(t)) \leq C_6 (r + \varepsilon) \exp \{ -\sum_{i=0}^{n-1} \phi^{(i)}(f^i x) \}
\]
\[
< r_0 \quad < \varepsilon_0.
\]

This implies that
\[
d(f^k x, f^k \xi(t)) \leq \int_0^t \| Df^k(\dot{\xi}(s)) \| \, ds
\]
\[
\leq \int_0^t \prod_{i=0}^{k-1} \| Df^i \xi(s) f \| \cdot \| \dot{\xi}(s) \| \, ds
\]
\[
\leq C_6 \exp \{ -\sum_{i=0}^{k-1} \phi^{(i)}(f^i x) \} \int_0^t \| \dot{\xi}(s) \| \, ds
\]
for each \( 0 \leq t \leq 1 \).

Using the induction, for each \( 0 \leq t \leq 1 \) and \( j = 1, 2, \ldots, n \)
\[
d(f^j x, f^j \xi(t)) \leq C_6 \exp \{ -\sum_{i=0}^{j-1} \phi^{(i)}(f^i x) \} \int_0^t \| \dot{\xi}(s) \| \, ds
\]
\[
\leq C_6 (r + \varepsilon) \exp \{ -\sum_{i=0}^{n-1} \phi^{(i)}(f^i x) \} < r_0.
\]

Therefore, \( d(f^j x, f^j y) < r_0 \) \((1 \leq j \leq n)\) if \( t = 1 \).
Similarly, we can check that $d(f^{-j}x, f^{-j}y) < r_0$ for $j = 1, \ldots, l$ if $C_\delta r \exp\{-\sum_{i=0}^{n} \phi(s)(f^{-i}x)\} < r_0.
$

**Step 2.** For each $x \in A$, $0 < r < r_0$, let

$$n_1 = \min \left\{ n \in \mathbb{Z}^+ : C_\delta r \exp\{-\sum_{i=0}^{n} \phi(s)(f^i x)\} \geq r_0 \right\},$$

$$n_2 = \min \left\{ n \in \mathbb{Z}^+ : C_\delta r \exp\{-\sum_{i=0}^{n} \phi(s)(f^{-i} x)\} \geq r_0 \right\}.$$

Then there exist $m = m(x,r_0,-n_2,n_1) \leq s^2 e_0$ and $W_1, \ldots, W_m \in \mathcal{R}(-n_2,n_1)$ such that

$$W_k \cap B_{r_0}^l(x,-n_2,n_1) \neq \emptyset \quad \text{for every} \quad k = 1, \ldots, m.$$

**Proof.** For $y \in A$, define $\mathcal{R}_y = \{Q \in \mathcal{R} : R \cap Q \neq \emptyset \text{ for some } y \in R \in \mathcal{R}\}$ and $P_y = \bigcup_{Q \in \mathcal{R}_y} Q$. Remark that $P_y \subset B_{2e_0}(y) \cap A$, and if $z \in A, d(y,z) < \delta_0$, then for any $Q \in \mathcal{R}$ containing $z$, $Q \in \mathcal{R}_y$.

Put

$$\mathcal{P}(x,-n_2,n_1) = \left\{ W = \bigcap_{j=-n_2}^{n_1} f^{-j} R_{a_j} \in \mathcal{R}(-n_2,n_1) : R_{a_j} \subset P_{f^j x} \text{ for every } j = -n_2, 0, \ldots, n_1 \right\}.$$  

By Step 1, we have

$$B_r(x) \cap A \subset B_{r_0}^l(x,-n_2,n_1) \cap A \subset \bigcup_{W \in \mathcal{P}(x,-n_2,n_1)} W \subset \bigcap_{j=-n_2}^{n_1} f^{-j} P_{f^j x}.$$  

Let $\mathcal{P}(x,r_0,-n_2,n_1) = \{W \in \mathcal{P}(x,-n_2,n_1) : B_{r_0}^l(x,-n_2,n_1) \cap W \neq \emptyset\}$. Then we have that $m = \sharp \mathcal{P}(x,r_0,-n_2,n_1) \leq s^2 e_0$. To prove this, take $l_1 = \sharp \mathcal{R}_{f^{n_1} x}$, $l_2 = \sharp \mathcal{R}_{f^{-n_2} x}$, and $\alpha_1, \ldots, \alpha_{l_1}, \beta_1, \ldots, \beta_{l_2} \in \{1, 2, \ldots, s\}$ such that $P_{f^j x} = \bigcup_{\alpha=1}^{l_1} R_{\alpha_x}$, $P_{f^{-n_2} x} = \bigcup_{\beta=1}^{l_2} R_{\beta_x}$.

Then, $\mathcal{P}(x,r_0,-n_2,n_1) = \bigcup_{p=1}^{l_1} \bigcup_{q=1}^{l_2} Q_{p,q}$ where $Q_{p,q} = \{W \in \mathcal{P}(x,r_0,-n_2,n_1) : W \subset f^{n_2} R_{\beta_x} \cap f^{-n_1} R_{\alpha_x}\}$, $Q_{p,q} \neq \emptyset$, and put $n = n(p,q) = \sharp Q_{p,q}$.

Let $W_{p,q}^n, W_{p,q}^2, \ldots, W_{p,q}^n$ denote all elements belonging to $Q_{p,q}$.
We can choose \((\alpha, w_1 \ldots w_{K_0-1} \beta) \in \Sigma(K_0 + 1)\) and take
\[
z_{i_{p,q}}^i \in W_{i_{p,q}} \cap ( \bigcap_{k=1}^{K_0-1} f^{-n_1-k} R_{w_k} ) \quad \text{such that} \quad f^{n_1+n_2+K_0} z_{i_{p,q}}^i = z_{i_{p,q}}^i
\]
for each \(i = 1, 2, \ldots, n\).

Then we have that for \(1 \leq i, j \leq n\),
\[
d(f^k z_{i_{p,q}}^i, f^k z_{i_{p,q}}^j) \leq d(f^k z_{i_{p,q}}^i, f^k x) + d(f^k x, f^k z_{i_{p,q}}^j) \\
\leq 2\varepsilon_0 + 2\varepsilon_0 < c
\]
for \(k = -n_2, \ldots, 0, \ldots, n_1\),
\[
d(f^{n_1+k} z_{i_{p,q}}^i, f^{n_1+k} z_{i_{p,q}}^j) \leq \text{diam} R_{w_k} < c
\]
for \(k = 1, \ldots, K_0 - 1\).

Thus, \(z_{i_{p,q}}^i = z_{i_{p,q}}^j\) for each \(1 \leq i, j \leq n\), and therefore
\[
z_{1_{p,q}}^1 \in \bigcap_{i=1}^n W_{i_{p,q}} = \bigcap_{w \in \mathcal{Q}_{p,q}} W.
\]
Since \(\sharp\{W \in \mathcal{R}(-n_2, n_1) : z_{1_{p,q}}^1 \in W\} \leq e_0\), we have \(\sharp\mathcal{Q}_{p,q} \leq e_0\).

Therefore,
\[
m = \sharp\mathcal{P}(x, r_0, -n_2, n_1) \\
= \sum_{p=1}^{l_1} \sum_{q=1}^{l_2} \sharp\mathcal{Q}_{p,q} \leq s^2 e_0. \quad \Box
\]

**Step 3.** There exists \(C_2 \geq 1\) such that
\[
\mu(B_r(x)) \leq C_2 r^{HD(\Lambda)}
\]
for all \(x \in R_1\) and \(0 < r < r_0\).

Proof. Fix \(x \in R_1\) and \(0 < r < r_0\). Take \(n_1, n_2 \in \mathbb{Z}^+\), \(m \leq s^2 e_0\), and \(W_1, \ldots, W_m \in \mathcal{R}(-n_2, n_1)\) as Step 2. For \(k = 1, \ldots, m\), pick \(y_k \in W_k \cap B_{r_0}^i(x, -n_2, n_1)\). Then we have
\[
\exp \sum_{i=0}^{n_1} \phi^{(u)}(f^i y_k) \leq C_6 \exp \sum_{i=0}^{n_1} \phi^{(u)}(f^i x),
\]
\[
\exp \sum_{j=0}^{n_2} \phi^{(s)}(f^{-j} y_k) \leq C_6 \exp \sum_{j=0}^{n_2} \phi^{(s)}(f^{-j} x),
\]
from which
\[
\mu(W_k) \leq C_1 \exp \{ \sum_{i=0}^{n_1} \delta_u \phi^{(u)}(f^i y_k) + \sum_{j=0}^{n_2} \delta_s \phi^{(s)}(f^{-j} y_k) \}
\]
\[
\leq C_1 C_6^{\delta_u + \delta_s} \exp \{ \sum_{i=0}^{n_1} \delta_u \phi^{(u)}(f^i x) + \sum_{j=0}^{n_2} \delta_s \phi^{(s)}(f^{-j} x) \}
\]
\[
\leq C_1 C_6^{HD(\Lambda)} \left( \frac{C_6 r}{r_0} \right)^{\delta_u + \delta_s}.
\]
Therefore,

$$\mu(B_r(x)) = \mu(B_r(x) \cap A)$$

$$\leq \sum_{k=1}^{m} \mu(W_k) \leq C_2 r^{HD(A)}$$

where \( C_2 = s^2 e_0 r_0^{-HD(A)} C_1 C_6^{2HD(A)} \). \( \square \)

5. Proof of Lemma 2

Proof of (1). As \( l - 1 \geq K_0 \), we pick \((a_{l-1} \ldots a_0) \in \Sigma(2l + 1)\) with \( a_{-l} = q_1, a_{-l+1} = q_0, a_0 = 1, a_{-1} = p_0 \), and \( a_l = p_1 \).

Put \( W_0 = \bigcap_{j=-l}^{l} f^{-j}R_{a_j} \); then \( W_0 \in \bigcup E_2 \) and so \( \Delta_1 = \frac{\mu(R_1 \cap \bigcup E_2)}{\mu(R_1)} \geq \mu(W_0) > 0. \) \( \square \)

Proof of (2). Fix \( k \geq 1 \) and \( W = \bigcap_{j=-kl}^{kl} f^{-j}R_{a_j} \in E_{k+1}, (a_{-kl} \ldots a_0 \ldots a_{kl}) \in \Sigma(2kl + 1). \) By the definition of \( E_{k+1} \), we have that \( f^jW \cap \emptyset Y = \emptyset \) for \( j = -kl + 1, \ldots, 0, \ldots, kl - 1 \) and that

\[
\begin{align*}
f^{-kl}W \cap \emptyset Y \neq \emptyset & \quad \text{iff} \quad (a_{(k-1)l} \ldots a_{kl}) = (v_{-l} \ldots v_0), \\
f^{-kl-1}W \cap \emptyset Y \neq \emptyset & \quad \text{iff} \quad (a_{(k-1)l+1} \ldots a_{kl}) = (v_{-l} \ldots v_{-1}), \\
& \quad \ldots \\
f^{-(k+1)l}W \cap \emptyset Y \neq \emptyset & \quad \text{iff} \quad (a_{-kl}a_{kl}) = (v_{-l}v_{-l+1}), \\
f^{kl}W \cap \emptyset Y \neq \emptyset & \quad \text{iff} \quad (a_{-kl} \ldots a_{-(k-1)l}) = (v_0 \ldots v_l), \\
f^{kl+1}W \cap \emptyset Y \neq \emptyset & \quad \text{iff} \quad (a_{-kl} \ldots a_{-(k-1)l-1}) = (v_1 \ldots v_l), \\
& \quad \ldots \\
f^{(k+1)l}W \cap \emptyset Y \neq \emptyset & \quad \text{iff} \quad (a_{-kl}a_{-kl+1}) = (v_{l-1}v_l).
\end{align*}
\]

For \( j = 0, 1, \ldots, l - 1 \), put

\[
\begin{align*}
W_j^- = & \quad \begin{cases} 
  h(-kl[a_{-kl} \ldots a_{kl}v_{-j} \ldots v_{-j+l}](k+1)l)] 
  & (f^{-kl-j}W \cap \emptyset Y \neq \emptyset), \\
  \emptyset & \quad \text{(otherwise)},
\end{cases} \\
W_j^+ = & \quad \begin{cases} 
  h(-(k+1)l[v_{j} \ldots v_{j-1}a_{-kl} \ldots a_{kl}](k+1)l)] 
  & (f^{kl+j}W \cap \emptyset Y \neq \emptyset), \\
  \emptyset & \quad \text{(otherwise)}.
\end{cases}
\end{align*}
\]

Then we have \( W \setminus \bigcup E_{k+2} = \bigcup_{j=0}^{l-1}(W_j^- \cup W_j^+) \), and take \( x \in W_j^- \subset W \) if \( W_j^- \neq \emptyset \) for \( j = 0, 1, \ldots, l - 1 \). Then,

\[
\begin{align*}
\frac{\mu(W_j^-)}{\mu(W)} & \leq C_1 \exp\left\{ \sum_{i=0}^{(k+1)l} \delta^u \phi^{(u)}(f^i x) + \sum_{j=0}^{kl} \delta^s \phi^{(s)}(f^{-j} x) \right\} \\
& \quad \times \frac{C_1}{C_1} \exp\left\{ \sum_{i=0}^{(k+1)l} \delta^u \phi^{(u)}(f^i x) + \sum_{j=0}^{kl} \delta^s \phi^{(s)}(f^{-j} x) \right\} \\
& \leq C_2^2 \beta^{-l} \leq \frac{1}{4l}.
\end{align*}
\]
Similarly, for every \( j = 0, 1, \ldots, l - 1 \)
\[
\frac{\mu(W_j^+)}{\mu(W)} \leq C_1 \beta^{-s \ell} \leq \frac{1}{4l}.
\]

Thus,
\[
\frac{\mu(\bigcup E_{k+2} \cap W)}{\mu(W)} = 1 - \frac{\mu(W \setminus \bigcup E_{k+2})}{\mu(W)} \geq 1 - \sum_{j=0}^{l-1} \left( \frac{\mu(W_j^-)}{\mu(W)} + \frac{\mu(W_j^+)}{\mu(W)} \right) \geq 1/2.
\]

Therefore, \( \Delta_{k+1} \geq 1/2 \) for any \( k \geq 1 \). \( \square \)

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