THE STRUCTURE OF MEASURABLE MAPPINGS WITH VALUES IN LOCALLY CONVEX SPACES

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Abstract. The purpose of this paper is to show that a theorem of A. Wisniewski remains valid without the approximation property.

1. Introduction

Let $X$ and $Y$ be Hausdorff topological spaces. Denote by $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ respectively the Borel $\sigma$-algebras on these spaces. Let $\mu$ be a finite Borel measure on $X$. By $\mathcal{B}_\mu(X)$ we denote the completion of the $\sigma$-algebra $\mathcal{B}(X)$ with respect to the measure $\mu$. A mapping $f$ from $X$ into $Y$ is said to be $\mu$-measurable if it is measurable with respect to $(\mathcal{B}_\mu(X), \mathcal{B}(Y))$. Recall that a Suslin space is a Hausdorff topological space which is the continuous image of some Polish space. It is well known but not easy to prove that every finite Borel measure on a Suslin space is Radon, and hence it is tight (see, e.g., [4, Theorem 10 of Chapter II in Part I]).

In [5], Wisniewski extended a theorem of Gihman and Skorohod [2, p. 544] which clarifies the connection between continuous and $\mu$-measurable mappings on Hilbert spaces. Namely, he proved that if $\mu$ is a finite Borel measure on a metric space $X$, and if $Y$ is a separable Banach space with the approximation property, then every $\mu$-measurable mapping $f$ from $X$ into $Y$ is the limit of a sequence of continuous mappings with respect to $\mu$-almost everywhere convergence.

In this paper, we show that the above result remains valid in the case that $Y$ is a Suslin locally convex topological linear space which does not necessarily have the approximation property. Consequently it turns out that Theorem 2 of [5] remains valid without the approximation property.

2. Main results

Theorem 1. Let $\mu$ be a finite Borel measure on a metric space $X$, and let $Y$ be a locally convex Hausdorff topological linear space. Assume that every finite Borel measure on $Y$ is tight (this is satisfied, for instance, if $Y$ is Suslin). If $f$ is a $\mu$-measurable mapping from $X$ into $Y$, then there exists a sequence $\{f_n\}$ of continuous mappings from $X$ into $Y$ such that $f_n \to f$ $\mu$-a.e.
Proof. To prove the theorem it suffices to show that for any $\varepsilon > 0$ and any closed neighborhood $U$ of the origin of $Y$, there exists a continuous mapping $g : X \to Y$ such that

$$
\mu \left\{ x : f(x) - g(x) \notin U \right\} < \varepsilon.
$$

Since $Y$ is locally convex, we may assume that $U$ is absorbent and absolutely convex (see, e.g., [3, p. 12]). Put $\nu(B) = \mu(f^{-1}(B))$ for every $B \in \mathcal{B}(Y)$. Then $\nu$ is a finite Borel measure on $Y$, and hence it is tight by the assumption. Thus we can find a compact subset $K$ of $Y$ such that $\nu(Y - K) < \varepsilon/2$. Put $K' = f^{-1}(K)$. Then we have

$$
(1) \quad \mu(X - K') < \frac{\varepsilon}{2}.
$$

Since $K$ is compact, there exist $y_1, y_2, \cdots, y_m \in K$ such that

$$
K \subset \left( y_1 + \frac{1}{2}U \right) \cup \left( y_2 + \frac{1}{2}U \right) \cup \cdots \cup \left( y_m + \frac{1}{2}U \right).
$$

Consider sets $B_1 = y_1 + \frac{1}{2}U, \cdots, B_i = \left\{ \bigcup_{k=1}^{i-1} \left( y_k + \frac{1}{2}U \right) \right\}^c \cap \left( y_i + \frac{1}{2}U \right), \cdots, B_m = \left\{ \bigcup_{k=1}^{m-1} \left( y_k + \frac{1}{2}U \right) \right\}^c \cap \left( y_m + \frac{1}{2}U \right)$. Then the $B_i$’s are Borel subsets of $Y$ with $K \subset \bigcup_{i=1}^{m} B_i$ and $B_i \cap B_j = \emptyset$ if $i \neq j$. Hence

$$
f(x) - \sum_{i=1}^{m} 1_{B_i}(f(x))y_i \in \frac{1}{2}U \quad \text{for all } x \in K',
$$

where $1_B$ denotes the indicator function of a set $B$. Consequently, using (1), we infer that

$$
(2) \quad \mu \left\{ x : f(x) - \sum_{i=1}^{m} 1_{B_i}(f(x))y_i \notin \frac{1}{2}U \right\} \leq \mu(X - K') \leq \frac{\varepsilon}{2}.
$$

For every $i = 1, 2, \cdots, m$, we put $f_i(x) = 1_{B_i}(f(x))$ for all $x \in X$. Then the $f_i$’s are $\mu$-measurable mapping from $X$ into the real line $R$. On the other hand, since $U$ is absorbent, there exists $\lambda > 0$ such that $y_i \in \lambda U$ for all $i = 1, 2, \cdots, m$. Therefore, in view of Theorem 1 of [5], for every $i = 1, 2, \cdots, m$, we can find continuous mappings $g_i$ from $X$ into $R$ such that

$$
(3) \quad \mu \left\{ x : |f_i(x) - g_i(x)| > \frac{1}{2m\lambda} \right\} < \frac{\varepsilon}{2m}.
$$

Put $g(x) = \sum_{i=1}^{m} g_i(x)y_i$ for every $x \in X$. Then $g$ is a continuous mapping from $X$ into $Y$. Moreover, from (2) and (3), together with the fact that $U$ is convex, we
have

\[
\mu \{ x : f(x) - g(x) \notin U \} = \mu \left\{ x : f(x) - \sum_{i=1}^{m} g_i(x)y_i \notin U \right\}
\]

\[
\leq \mu \left\{ x : f(x) - \sum_{i=1}^{m} f_i(x)y_i \notin \frac{1}{2} U \right\}
\]

\[
+ \mu \left\{ x : \sum_{i=1}^{m} f_i(x)y_i - \sum_{i=1}^{m} g_i(x)y_i \notin \frac{1}{2} U \right\}
\]

\[
\leq \frac{\varepsilon}{2} + \sum_{i=1}^{m} \mu \left\{ x : |f_i(x) - g_i(x)| > \frac{1}{2m\lambda} \right\}
\]

\[
\leq \varepsilon/2 + m \cdot \varepsilon/(2m) = \varepsilon.
\]

This completes the proof of the theorem.

Remark. We can show that Theorem 1 also holds in the case that \( X \) is a normal space and \( \mu \) is a finite Borel measure on \( X \) satisfying the condition that \( \mu(B) = \sup \{ \mu(F) : F \subset B \text{ and } F \text{ is a closed subset of } X \} \) for all \( B \in \mathcal{B}(X) \).

Corollary. In the following cases, Theorem 1 holds:

(a) \( Y \) is a separable Banach space.
(b) \( Y \) is a separable Fréchet space.
(c) \( Y \) is the strict inductive limit of an increasing sequence of separable Fréchet spaces.
(d) \( Y \) is the strong dual of a nuclear Fréchet space.
(e) \( Y \) is the strong dual of the strict inductive limit of an increasing sequence of nuclear Fréchet spaces.

Proof. (a)–(e) are Suslin (in fact, Lusin) locally convex topological linear spaces by [1, Theorem I.5.1].

REFERENCES


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