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THE STRUCTURE OF MEASURABLE MAPPINGS WITH VALUES IN LOCALLY CONVEX SPACES

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ABSTRACT. The purpose of this paper is to show that a theorem of A. Wisniewski remains valid without the approximation property.

1. INTRODUCTION

Let X and Y be Hausdorff topological spaces. Denote by $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ respectively the Borel σ-algebras on these spaces. Let μ be a finite Borel measure on X. By $\mathcal{B}_\mu(X)$ we denote the completion of the σ-algebra $\mathcal{B}(X)$ with respect to the measure μ. A mapping f from X into Y is said to be μ-measurable if it is measurable with respect to $(\mathcal{B}_\mu(X), \mathcal{B}(Y))$. Recall that a Suslin space is a Hausdorff topological space which is the continuous image of some Polish space. It is well known but not easy to prove that every finite Borel measure on a Suslin space is Radon, and hence it is tight (see, e.g., [4, Theorem 10 of Chapter II in Part I]).

In [5], Wisniewski extended a theorem of Gihman and Skorohod [2, p. 544] which clarifies the connection between continuous and μ-measurable mappings on Hilbert spaces. Namely, he proved that if μ is a finite Borel measure on a metric space X, and if Y is a separable Banach space *with the approximation property*, then every μ-measurable mapping f from X into Y is the limit of a sequence of continuous mappings with respect to μ-almost everywhere convergence.

In this paper, we show that the above result remains valid in the case that Y is a Suslin locally convex topological linear space which *does not* necessarily have the approximation property. Consequently it turns out that Theorem 2 of [5] remains valid without the approximation property.

2. MAIN RESULTS

Theorem 1. *Let μ be a finite Borel measure on a metric space X, and let Y be a locally convex Hausdorff topological linear space. Assume that every finite Borel measure on Y is tight (this is satisfied, for instance, if Y is Suslin). If f is a μ-measurable mapping from X into Y, then there exists a sequence $\{f_n\}$ of continuous mappings from X into Y such that $f_n \to f$ μ-a.e.*

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Proof. To prove the theorem it suffices to show that for any $\varepsilon > 0$ and any closed neighborhood U of the origin of Y, there exists a continuous mapping $g : X \to Y$ such that

$$\mu\{x : f(x) - g(x) \notin U\} < \varepsilon.$$

Since Y is locally convex, we may assume that U is absorbent and absolutely convex (see, e.g., [3, p. 12]). Put $\nu(B) = \mu(f^{-1}(B))$ for every $B \in \mathcal{B}(Y)$. Then ν is a finite Borel measure on Y, and hence it is tight by the assumption. Thus we can find a compact subset K of Y such that $\nu(Y - K) < \varepsilon/2$. Put $K' = f^{-1}(K)$. Then we have

$$\mu(X - K') < \frac{\varepsilon}{2}. \tag{1}$$

Since K is compact, there exist $y_1, y_2, \cdots, y_m \in K$ such that

$$K \subset \left(y_1 + \frac{1}{2}U\right) \cup \left(y_2 + \frac{1}{2}U\right) \cup \cdots \cup \left(y_m + \frac{1}{2}U\right).$$

Consider sets $B_1 = y_1 + \frac{1}{2}U, \cdots, B_i = \left\{\bigcup_{k=1}^{i-1}\left(y_k + \frac{1}{2}U\right)\right\}^c \cap \left(y_i + \frac{1}{2}U\right), \cdots,$ $B_m = \left\{\bigcup_{k=1}^{m-1}\left(y_k + \frac{1}{2}U\right)\right\}^c \cap \left(y_m + \frac{1}{2}U\right)$. Then the B_i's are Borel subsets of Y with $K \subset \bigcup_{i=1}^m B_i$ and $B_i \cap B_j = \emptyset$ if $i \neq j$. Hence

$$f(x) - \sum_{i=1}^m 1_{B_i}(f(x))y_i \in \frac{1}{2}U \text{ for all } x \in K',$$

where 1_B denotes the indicator function of a set B. Consequently, using (1), we infer that

$$\mu\left\{x : f(x) - \sum_{i=1}^m 1_{B_i}(f(x))y_i \notin \frac{1}{2}U\right\} \leq \mu(X - K') \leq \frac{\varepsilon}{2}. \tag{2}$$

For every $i = 1, 2, \cdots, m$, we put $f_i(x) = 1_{B_i}(f(x))$ for all $x \in X$. Then the f_i's are μ-measurable mapping from X into the real line R. On the other hand, since U is absorbent, there exists $\lambda > 0$ such that $y_i \in \lambda U$ for all $i = 1, 2, \cdots, m$. Therefore, in view of Theorem 1 of [5], for every $i = 1, 2, \cdots, m$, we can find continuous mappings g_i from X into R such that

$$\mu\left\{x : |f_i(x) - g_i(x)| > \frac{1}{2m\lambda}\right\} < \frac{\varepsilon}{2m}. \tag{3}$$

Put $g(x) = \sum_{i=1}^m g_i(x)y_i$ for every $x \in X$. Then g is a continuous mapping from X into Y. Moreover, from (2) and (3), together with the fact that U is convex, we have

$$\mu\{x: f(x)-g(x)\notin U\} = \mu\left\{x: f(x)-\sum_{i=1}^{m} g_i(x)y_i \notin U\right\}$$
$$\le \mu\left\{x: f(x)-\sum_{i=1}^{m} f_i(x)y_i \notin \frac{1}{2}U\right\}$$
$$+\mu\left\{x: \sum_{i=1}^{m} f_i(x)y_i-\sum_{i=1}^{m} g_i(x)y_i \notin \frac{1}{2}U\right\}$$
$$\le \frac{\varepsilon}{2}+\sum_{i=1}^{m}\mu\left\{x: |f_i(x)-g_i(x)|>\frac{1}{2m\lambda}\right\}$$
$$\le \varepsilon/2+m\cdot\varepsilon/(2m)=\varepsilon.$$

This completes the proof of the theorem. □

Remark. We can show that Theorem 1 also holds in the case that X is a normal space and μ is a finite Borel measure on X satisfying the condition that $\mu(B) = \sup\{\mu(F): F\subset B$ and F is a closed subset of $X\}$ for all $B\in\mathcal{B}(X)$.

Corollary. *In the following cases, Theorem* 1 *holds*:

(a) Y *is a separable Banach space.*
(b) Y *is a separable Fréchet space.*
(c) Y *is the strict inductive limit of an increasing sequence of separable Fréchet spaces.*
(d) Y *is the strong dual of a nuclear Fréchet space.*
(e) Y *is the strong dual of the strict inductive limit of an increasing sequence of nuclear Fréchet spaces.*

Proof. (a)–(e) are Suslin (in fact, Lusin) locally convex topological linear spaces by [1, Theorem I.5.1]. □

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