PRODUCTS OF $\omega^*$ IMAGES

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Abstract. Let $\omega^*$ be the Čech-Stone remainder $\beta\omega \setminus \omega$. We show that there exists a large class $O$ of images of $\omega^*$ such that whenever $S$ is a subset of $O$ of cardinality at most the continuum, then $\omega^* \times \prod S$ is again an image of $\omega^*$. The class $O$ contains all separable compact spaces, all compact spaces of weight at most $\omega_1$ and all perfectly normal compact spaces.

1. Introduction

For a Tychonoff space $X$, $\beta X$ represents the Čech-Stone compactification of $X$ and for $A \subset X$, $A^*$ represents the subspace $cl_{\beta X}(A) \setminus X$. If $f : X \to Y$, then $\beta f : \beta X \to \beta Y$ is the Čech-Stone extension of $f$ and $f^* = \beta f \upharpoonright X^*$. The first infinite ordinal $\omega$ is given the discrete topology.

W. Just [Ju89] has proven that $\omega^* \times \omega^*$ is consistently not a continuous image of $\omega^*$. So it is consistent that there are two $\omega^*$ images whose product is not an $\omega^*$ image. This creates a limit to naïve product results and a desire to find a general product result that includes many of the $\omega^*$ image results appearing in the literature. This is the theme of this paper. We will define a subclass $O$ of the class of all $\omega^*$ images and show (The Product Theorem) that if $S$ is a subset of $O$ of cardinality at most the continuum, then $\omega^* \times \prod S$ is an $\omega^*$ image. In the succeeding sections, we will show that $O$ contains all separable compact spaces, all compact spaces of weight at most $\omega_1$, all perfectly normal compact spaces, all zero-dimensional, orderable $\omega^*$ images, and all $\omega^*$ images of weight $< b$.

In this paper all spaces are assumed to be Hausdorff and all mappings between spaces are assumed to be continuous surjections. If $X$ is homeomorphic to $Y$, then this is denoted by $X \approx Y$. If $F$ is a family of functions with a common domain $X$ and ranges $Y_f$ for $f \in F$, then $\Delta F$ represents the diagonal function with domain $X$ and range $\prod_{f \in F} Y_f$ defined by $\Delta F(x) = (f(x))_{f \in F}$. A Boolean space is a zero-dimensional, compact space, and $CO(X)$ denotes the algebra of all clopen subsets of a space $X$. Thus, $CO(\omega^*)$ is isomorphic to the quotient algebra $\mathcal{P}(\omega)$ modulo the ideal of finite sets. The cardinality of the continuum is denoted by $c$. For an

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introduction to the theory of $\omega^*$ images we refer the reader to the article by van Mill [vM84].

2. Orthogonality and products

**Definition 2.1.** Two functions $f: X \to Y$ and $g: X \to Z$ are called orthogonal $(f \perp g)$ iff $f \triangle g: X \to Y \times Z$ is a surjection. If the sets are Boolean spaces and the functions are mappings, then the dual notion is as follows: If $A$ and $B$ are two subalgebras of a boolean algebra $C$, then $A$ and $B$ are called independent if for every non-zero $a \in A$ and for every non-zero $b \in B$, $a \land b \neq 0$.

**Definition 2.2.** $X$ is an orthogonal $\omega^*$ image if there exists a finite-to-one surjection $q: \omega \to \omega$ and a mapping $f: \omega^* \to X$ with $f \perp q^*$. If $X$ is a Boolean space, then the dual notion is as follows: A subalgebra $B$ of $CO(\omega^*)$ is called an orthogonal subalgebra if there exists a partition $R = \{R_k : k < \omega\}$ of $\omega$ into finite sets such that for every non-empty $b \in B$, if $b = A^*$ where $A \subset \omega$, then \{ $k < \omega : A \cap R_k \neq \emptyset$ \} is a co-finite subset of $\omega$.

Let us observe that a continuous image of an orthogonal $\omega^*$ image is again an orthogonal $\omega^*$ image.

In this paper, we will find it convenient to view $\omega$ in one of its alternate forms. Let $W = \{(k, j) \in \omega \times \omega : j \leq k\}$ be given the discrete topology (so $W^* \approx \omega^*$), let $p: W \to \omega$ be the projection $p(k, j) = k$, and for each $k < \omega$, let $W_k = p^{-1}(k)$. Let $p$ be a finite-to-one surjection, we have that $p^*: W^* \to \omega^*$ is an open surjection. Thus we have

(\star) $\forall$ open $V \subset W^*$ with $p^*(V) = \omega^*$ $\exists$ clopen $V' \subset V$ with $p^*(V') = \omega^*$.

**Henceforth** $W, W_k, W^*, p$, and $p^*$ will always have the above meanings. **They will be frequently used in this paper.** The following demonstrates the canonical nature of $W$.

**Proposition 2.3.** If $X$ is an orthogonal $\omega^*$ image, then there exists a mapping $f: W^* \to X$ such that $f \perp p^*$.

**Proof.** Let $q$ be a finite-to-one surjection $q: \omega \to \omega$ and let $g$ be a mapping $g: \omega^* \to X$ such that $g \perp q^*$. Since $g \perp q^*$, for every $x \in X$ we have that $q^*(g^{-1}(x)) = \omega^*$. This means that for every $x \in X$ and for every $A \subset \omega$ such that $g^{-1}(x) \subset A^*$, $\{ n < \omega : A \cap q^{-1}(n) \neq \emptyset \}$ is a co-finite subset of $\omega$. We will produce a finite-to-one surjection $\varphi: W \to \omega$ such that whenever $A \subset \omega$ satisfies that $\{ n < \omega : A \cap q^{-1}(n) \neq \emptyset \}$ is co-finite in $\omega$, then $\{ n < \omega : \varphi^{-1}(A) \cap W_n \neq \emptyset \}$ is co-finite in $\omega$. Putting $f = g \circ \varphi^*$, we see that $f \perp p^*$. To construct $\varphi$, first get an increasing function $\psi: \omega \to \omega$ such that for every $n < \omega$, $\psi(n) \geq |q^{-1}(n)|$. For every $i \leq 0$ let $\varphi_i: W_i \to \omega$ be arbitrary. For every $n \geq 0$ and for every $i$ with $\psi(n) \leq i < \psi(n+1)$ let $\varphi_i$ be a surjection from $W_i$ onto $q^{-1}(n)$. Define $\varphi: W \to \omega$ so that for every $i < \omega$, $\varphi \mid W_i = \varphi_i$.\]

**Proposition 2.4.** The product of an $\omega^*$ image and an orthogonal $\omega^*$ image is an $\omega^*$ image.

**Proof.** This is an immediate consequence of the definition of orthogonal $\omega^*$ image.\]

It follows from Just’s result that $\omega^*$ need not be an orthogonal $\omega^*$ image.
Theorem 2.5 (The Product Theorem). The product of at most \( \epsilon \) many orthogonal \( \omega^* \) images is an orthogonal \( \omega^* \) image.

Proof. Invoking Proposition 2.3, for every \( s \in S \subset 2^\omega \) let \( f_s : W^* \to X_s \) satisfy \( f_s \perp p^* \). For every \( k < \omega \) let \( U_k = \{ t : t \) is a function with domain \( 2^k \) and range \( W_k \} \). Let \( U = \bigcup_{k<\omega} U_k \) have the discrete topology (so \( U^* \approx \omega^* \)). If \( t \in U_k \)
then define \( r(t) = k \). As \( r : U \to \omega \) is a finite-to-one surjection, \( r^* : U^* \to \omega^* \)
is a surjection. For every \( s \in S \) define \( p_s : U \to W \) by \( p_s(t) = t(s \upharpoonright k) \) where \( t \in U_k \), i.e., \( p_s \upharpoonright U_k \) is the projection from \( U_k \) onto the factor indexed by \( s \upharpoonright k \). As \( p_s \) is a finite-to-one surjection, \( p^*_s : U^* \to \omega^* \)
is a surjection. We claim that \( r^* \perp \Delta \{ f_s \circ p^*_s : s \in S \} \). By compactness, it suffices to show that for any finite set \( T \subset S \), \( r^* \perp \Delta \{ f_s \circ p^*_s : s \in T \} \); for which, in turn, it suffices to show that for any collection of non-empty sets \( \{ V, V_s : s \in T \} \) where \( V \) is clopen in \( \omega^* \) and \( V_s \) is open in \( X_s \) for \( s \in T \), that \( r^* \perp \bigcap_{s \in T} (f_s \circ p^*_s)^{-1}(V_s) \neq \emptyset \). Let us check this. Since
\[
f_t \perp p^*, \text{ we have } p^*((f_t^{-1}(V_t))) = \omega^*. \text{ So by } (\star), \text{ we can choose clopen } V'_t \subset W^* \text{ with } V'_t \subset f_t^{-1}(V_t) \text{ and } p^*(V'_t) = \omega^*. \text{ We will prove that } r^* \perp \bigcap_{s \in T} (f_s \circ p^*_s)^{-1}(V_s) \neq \emptyset. \]
Let \( A \subset \omega \), \( A_s \subset W \) for \( s \in T \) be subsets with \( V = A^* \), \( V'_t = A^*_t \). \( A \) is infinite and \( p(A_s) \) is co-finite for every \( s \in T \) since \( p^*(A_s^*) = \omega^* \). So \( A \cap \bigcap_{s \in T} p(A_s) \) is infinite.
Let \( k \in A \cap \bigcap_{s \in T} p(A_s) \) be big enough to distinguish all members of \( T \), i.e., if \( s \neq s', s, s' \in T \), then \( s \upharpoonright k \neq s' \upharpoonright k \). For every such \( k \), there is a mapping \( t \in U_k \) such that for every \( s \in T \), \( t(s \upharpoonright k) \in A_s \). Since any \( t \) like this is in \( r^{-1}(A) \cap \bigcap_{s \in T} p_s^{-1}(A_s) \), the set \( r^{-1}(A) \cap \bigcap_{s \in T} p_s^{-1}(A_s) \) is infinite. So \( r^* \perp \bigcap_{s \in T} (f_s \circ p^*_s)^{-1}(V_s) \neq \emptyset. \)

We now strengthen a classical result.

Proposition 2.6. Every separable, compact space is an orthogonal \( \omega^* \) image.

Proof. It suffices to show that \( \beta \omega \) is an orthogonal \( \omega^* \) image. Let \( q : W \to \omega \) be the second projection \( q(k, j) = j \). As \( q \) is an infinite-to-one surjection, \( q^* : W^* \to \beta \omega \)
is a surjection. Since for every \( j < \omega \), \( \{ k < \omega : q^{-1}(j) \cap W_k \neq \emptyset \} \) is a co-finite subset of \( \omega \), we see that \( q^* \perp p^* \).

3. Weight \( \omega_1 \)

Definition 3.1. A thick clopen subset \( V \subset W^* \) is a clopen \( V \subset W^* \) with \( V = A^* \) where \( A \subset W \) satisfies \( \lim_{k \to \infty} |A \cap W_k| = \infty \). A thick zero set \( Z \subset W^* \) is a zero set \( Z \subset W^* \) with \( Z = \bigcap_{k<\omega} V_k \) where \( \{ V_k \} \) is a decreasing sequence of thick clopen subsets. If \( G \) is a clopen subset of \( W^* \), then a thick mapping \( f : G \to X \)
is a mapping \( f : G \to X \) with \( f^{-1}(x) \) a thick zero set for every \( x \in X \). We now list some basic properties of thick sets.

T1. If \( V \subset V' \) are clopen sets and \( V \) is thick, then \( V' \) is thick. If \( Z \subset Z' \) are zero sets and \( Z \) is thick, then \( Z' \) is thick.

T2. If \( \{ Z_k \} \) is a decreasing sequence of thick zero sets, then \( \bigcap_{k<\omega} Z_k \) is a thick zero set.
T3. A thick zeroset contains a thick clopen set.
   A thick clopen set contains two disjoint thick clopen sets.

T4. A thick mapping \( f : W^* \to X \) satisfies \( f \perp p^* \).

Let us prove the first part of T3. Let \( Z \subset W^* \) with \( Z = \bigcap_{j<\omega} V_j \) where \( \{V_j\} \) is a decreasing sequence of thick clopen sets. For \( j < \omega \) let \( A_j \subset W^* \) such that \( V_j = A_j^* \). Thus we have (a) \( j < k \) implies \( A_k \setminus A_j \) is finite and (b) for every \( j \), \( \lim_{k \to \infty} |A_j \cap W_k| = 0 \). By induction, using (a) and (b), define an increasing sequence \( \{r_n\} \subset \omega \) such that for every \( k \geq r_n \), \( A_n \cap W_k \subset \bigcap_{i<n} A_i \cap W_k \) and \( |A_n \cap W_k| \geq n \). For each \( n > 0 \) and for each \( i \) with \( r_n \leq i < r_{n+1} \) let \( F_i \) be a subset of \( A_n \cap W_i \) of cardinality \( n \). If \( F = \bigcup_{i<n} F_i \), then \( F^* \) is a thick clopen set with \( F^* \subset Z \).

We now give a topological proof of a strengthening of a result of Parovičenko [Pa63] using some ideas in Blaszczyk and Szymański [BS80].

**Proposition 3.2.** Let \( X \) be a compact metric space and let \( f : W^* \to X \) be a thick mapping. Let \( E \) and \( F \) be closed in \( X \) with \( X = E \cup F \). Then, there exists a clopen \( G \subset W^* \) such that \( f \restriction G \) is a thick mapping onto \( E \) and \( f \restriction (W^* \setminus G) \) is a thick mapping onto \( F \).

**Proof.** Let \( D \) be a countable, dense subset of \( E \cap F \). For every \( d \in D \) (by T3) choose disjoint, thick clopen subsets \( U_d \) and \( V_d \) of \( f^{-1}(d) \). Thus \( G_0 = f^{-1}(E \setminus F) \cup \bigcup_{d \in D} U_d \) and \( G_1 = f^{-1}(F \setminus E) \cup \bigcup_{d \in D} V_d \) are disjoint cozerosets of \( W^* \). Let \( G \) be clopen in \( W^* \) with \( G_0 \subset G \) and \( G_1 \subset (W^* \setminus G) \). Then \( f \restriction G \) is a mapping onto \( E \). Let \( x \in E \) and let \( Z \) be a zero-dimensional, compact metric space of weight at most \( \omega_1 \). Via an embedding into the Cantor cube \( 2^{\omega_1} \), we may express \( X \) as an inverse limit space. Let \( X = \lim \{X_\alpha, p^{\alpha}_{\alpha}, \alpha < \beta < \omega_1 \} \) such that:

(i) \( X_\beta \) limit implies \( X_\beta = \lim \{X_\alpha, p^{\alpha}_{\alpha}, \alpha < \gamma < \beta \} \) (the spectrum is continuous).

(ii) \( X_\alpha \) is a zero-dimensional, compact metric space.

(iii) \( X_{\alpha+1} = X_\alpha^0 \cup X_\alpha^1 \) where \( X_0 \cup X_1 \) and \( X_0 \cup X_1 \) are disjoint, clopen subsets of \( X_0 \cup X_1 \) and for \( i = 0, 1 \), \( p^\alpha_{\alpha+1} \restriction X^i_{\alpha+1} \) is one-to-one (short projections are “simple”).

(iv) \( |X_0| = 1 \).

We will construct a system of thick mappings \( \{f_\alpha : W^* \to X_\alpha : \alpha < \omega_1 \} \) such that \( f_\alpha = p^\beta_\alpha \circ f_\beta \), \( \alpha < \beta < \omega_1 \). Since for every \( \alpha < \omega_1 \), \( f_\alpha \perp p^* \), by compactness we
will get that $f_{\omega_1} = \lim \{f_\alpha : \alpha < \omega_1\}$ satisfies $f_{\omega_1} \perp p^*$.

**Limit Step.** If $\beta$ is a limit, then let $f_\beta = \lim \{f_\alpha : \alpha < \beta\}$. Property T2 implies that $f_\beta$ is a thick mapping.

**Successor Step.** If $\beta = \alpha + 1$, then let $E = p^{\alpha+1}_\alpha(X^0_{\alpha+1})$ and $F = p^{\alpha+1}_\alpha(X^1_{\alpha+1})$. Apply Proposition 3.2 to get a clopen $G \subset W^*$ with $f_\alpha \restriction G$ a thick mapping onto $E$ and $f_\alpha \restriction (W^* \setminus G)$ a thick mapping onto $F$. Define $f_{\alpha+1} : W^* \to X_{\alpha+1}$ by

$$f_{\alpha+1}(z) = \begin{cases} (p^{\alpha+1}_\alpha \restriction X^0_{\alpha+1})^{-1}(f_\alpha(z)) & \text{if } z \in G, \\ (p^{\alpha+1}_\alpha \restriction X^1_{\alpha+1})^{-1}(f_\alpha(z)) & \text{if } z \notin G. \end{cases}$$

$f_{\alpha+1}$ is a thick mapping because

$$f^{-1}_{\alpha+1}(x) = \begin{cases} (f_\alpha \restriction G)^{-1}(p^{\alpha+1}_\alpha(x)) & \text{if } x \in X^0_{\alpha+1}, \\ (f_\alpha \restriction (W^* \setminus G))^{-1}(p^{\alpha+1}_\alpha(x)) & \text{if } x \in X^1_{\alpha+1}, \end{cases}$$

and both $f_\alpha \restriction G$ and $f_\alpha \restriction (W^* \setminus G)$ are thick mappings. $\square$

4. Perfectly normal

**Proposition 4.1.** Let $X$, $Y$, $\overline{V}$ and $Z$ be compact spaces and let $f : X \to Y$, $g : X \to Z$, $h : \overline{Y} \to Y$, and $k : X \to \overline{Y}$ be mappings such that $f \perp g$, $f = h \circ k$, and $h$ is irreducible. Then, $k \perp g$.

**Proof.** Let $y \in \overline{Y}$ and let $O_y$ be a neighbourhood of $y$. Irreducibility of $h$ implies that there exists a non-empty, open $V \subset Y$ with $h^{-1}(V) \subset O_y$. Therefore, $k^{-1}(h^{-1}(V)) = f^{-1}(V) \subset k^{-1}(O_y)$. But $g(f^{-1}(V)) = Z$ since $f \perp g$. So $g(k^{-1}(O_y)) = Z$. Thus $g(k^{-1}(y)) = Z$. $\square$

We now strengthen a result of Przymusiński [Pr82]. For the reader’s convenience, we sketch his arguments and indicate how we apply Proposition 4.1. The following Lifting Lemma appears in his paper.

**Lemma 4.2.** Let $X$ be compact and perfectly normal, $Z$ a closed subspace of $X \times I$ and suppose that the restriction $\pi \restriction Z : Z \to X$ of the projection $\pi : X \times I \to X$ is irreducible. If $f : \omega^* \to X$ is a continuous mapping of $\omega^*$ onto $X$, then there exists a continuous mapping $g : \omega^* \to Z$ of $\omega^*$ onto $Z$ such that $f = \pi \circ g$.

**Theorem 4.3.** Every perfectly normal compact space $X$ is an orthogonal $\omega^*$ image.

**Proof.** We express $X$ as the limit space of an inverse spectrum

$$X = \lim \{X_\alpha, \pi^\alpha, \alpha < \beta < \kappa\}$$

with limit projections $\pi_\alpha : X \to X_\alpha$ such that:

(a) $\beta$ limit implies $X_\beta = \lim \{X_\alpha, \pi^\alpha, \alpha < \gamma < \beta\}$
(b) $w(X_0) \leq \omega_1$
(c) $\pi_0$ is irreducible (therefore all other projections are irreducible)
(d) $X_{\alpha+1}$ is a closed subset of $X_\alpha \times I$ and $\pi^{\alpha+1}_\alpha$ is the restriction to $X_{\alpha+1}$ of the projection of $X_\alpha \times I$ onto $X_\alpha$. 

Parts (b) and (c) are achieved by using the result of Šapirovski [Sa74] that every perfectly normal compact space admits a $\pi$-base of cardinality at most $\omega_1$. By Theorem 3.3, let $f_0 : W^* \to X_0$ satisfy $f_0 \perp p^*$. By induction on $\alpha \leq \kappa$, we construct $f_\alpha : W^* \to X_\alpha$ (where $X_\alpha = X$) such that $\beta < \alpha$ implies that $\pi_\beta \circ f_\alpha = f_\beta$. At successor stages use Lemma 4.2 to get $f_{\alpha+1}$ and at limit stages let $f_\kappa = \lim \{f_\beta : \beta < \alpha\}$. Thus we have $f_\kappa : W^* \to X$ with $f_0 = \pi_0 \circ f_\kappa$. Proposition 4.1 with $X = W^*$, $Y = X_0$, $\overline{Y} = X$, $Z = \omega^*$ and $f = f_0$, $g = p^*$, $h = \pi_0$, $k = f_\kappa$ implies that $f_\kappa \perp p^*$.

\section{Orderable}

\begin{theorem}
Every zero-dimensional, orderable $\omega^*$ image $L$ is an orthogonal $\omega^*$ image.
\end{theorem}

\begin{proof}
Let $\prec$ be a compatible order for $L$. Let $L' = \{x \in L : x$ is a left neighbour of a jump in $\prec\}$. Since $L$ is Boolean, $G = \{\{y \in L : y \preceq x\} : x \in L'\}$ is a generating set for $CO(L)$. It suffices to define $\phi : L' \to \mathcal{P}(W)$ such that for every $x, y \in L'$, $x < y$ implies that (a) $\phi(x) \setminus \phi(y)$ is finite and $\phi(y) \setminus \phi(x)$ is infinite and (b) $\{k < \omega : (\phi(y) \setminus \phi(x)) \cap W_k \neq \emptyset\}$ is a co-finite subset of $\omega$. Because then the function that sends $\{y \in L : y \preceq x\}$ to $\phi(x)^*$ will extend to an embedding of $CO(L)$ into $CO(W^*)$ (by (a)) onto an orthogonal subalgebra of $CO(W^*)$ (by (b)). Since $L$ is an $\omega^*$ image, let $\psi : L' \to \mathcal{P}(\omega)$ satisfy that for every $x, y \in L'$, $x < y$ iff $\psi(x) \setminus \psi(y)$ is finite. Finally, define $\phi : L' \to \mathcal{P}(W)$ by $\phi(x) = \{(k, j) : j \leq \psi(x) \cap k\}$.
\end{proof}

Let $Q$ denote the lexicographic ordered space $2^{\omega_1}$. It is known (Bell [Be90]) that $Q$ is an $\omega^*$ image, so we get the following corollary.

\begin{corollary}
$Q$ is an orthogonal $\omega^*$ image.
\end{corollary}

\begin{example}
The Alexandroff one-point compactification of the discrete space $\varepsilon$ is an orthogonal $\omega^*$ image because it is an image of $Q$.
\end{example}

\begin{corollary}
Every first countable orderable compact space is an orthogonal $\omega^*$ image.
\end{corollary}

\begin{proof}
This follows because Maurice [Ma64] has proved that every first countable orderable compact space is an image of $Q$.
\end{proof}

\begin{question}
Can zero-dimensionality be removed from the hypotheses of Theorem 5.1, i.e., is every orderable $\omega^*$ image an orthogonal $\omega^*$ image?
\end{question}

\section{Weight $< b$}

Recall that if $f, g \in \omega^\omega$, then $f \prec g$ means that eventually $f(k) < g(k)$ and that $b$ is by definition the least cardinality of an $\prec$-unbounded subset of $\omega^\omega$.

\begin{theorem}
Every $\omega^*$ image $X$ of weight $< b$ is an orthogonal $\omega^*$ image.
\end{theorem}

\begin{proof}
It suffices to assume that $X$ is Boolean. Let $B$ be a subalgebra of $CO(\omega^*)$ of cardinality $< b$. For every $b \in B$, choose $M_b \subset \omega$ such that $b = M_b^\omega$. For every $b \in B$, define $f_b \in \omega^\omega$ by $f_b(k) = \min\{n \in M_b : n \geq k\}$. Let $g \in \omega^\omega$ such that for every $b \in B$, $f_b < g$ and for every $k < \omega$, $k < g(k)$. Let $R = \{(k, j) : k < j < g(k)\}$ endowed with the discrete topology (therefore $R^* \approx \omega^*$), let for every $k < \omega$, $R_k = \{(k, j) : k \leq j < g(k)\}$, and let $R = \{R_k : k < \omega\}$. For every $b \in B$, define $A_b = \{(k, j) \in R : j \in M_b\}$. Then, $C = \{A_b^* : b \in B\}$ is a subalgebra of $CO(R^*)$,
$B \cong C$ by the natural map $b \mapsto A^*_b$, and $C$ is an orthogonal subalgebra of $CO(R^*)$ because for every $b \neq \emptyset$, $\{k < \omega : A_k \cap R_k \neq \emptyset\}$ is co-finite in $\omega$. \hfill \Box

A family $\mathcal{A}$ of subsets of $\omega$ is said to be strongly centered if whenever $\mathcal{F}$ is a finite subset of $\mathcal{A}$, then $\bigcap \mathcal{F}$ is infinite. Recall that $p$ is the least cardinality of a strongly centered family $\mathcal{A}$ such that there does not exist an infinite $X \subseteq \omega$ with the property that for every $A \in \mathcal{A}$, $X \setminus A$ is finite. It is a basic fact that $\omega_1 \leq p \leq c$.

We now strengthen the result of van Douwen and Przymusiński [DP80] that every compact space of weight $< p$ is an $\omega^*$ image.

**Corollary 6.2.** Every compact space $X$ of weight $< p$ is an orthogonal $\omega^*$ image.

**Proof.** Since $X$ is an $\omega^*$ image and $p \leq \mathfrak{b}$ (cf. van Douwen [vD84]), we can apply Theorem 6.1. \hfill \Box

It follows from this corollary that Martin’s Axiom implies that all compact spaces of weight less than $\mathfrak{c}$ are orthogonal $\omega^*$ images.

In conclusion, we summarize our results relating to well-known classes of spaces.

**Summation 6.3.** Let $\mathcal{D}$ be the class of all compact spaces which have at least one of the following properties: separable, weight at most $\omega_1$, perfectly normal, first countable orderable, and weight $< p$. If $\mathcal{S}$ is a subset of $\mathcal{D}$ of cardinality at most $\mathfrak{c}$, then $\omega^* \times \prod \mathcal{S}$ is an image of $\omega^*$.

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