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PRODUCTS OF ω^* IMAGES

M. BELL, L. SHAPIRO, AND P. SIMON

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ABSTRACT. Let ω^* be the Čech-Stone remainder $\beta\omega \setminus \omega$. We show that there exists a large class $\mathcal{O}$ of images of ω^* such that whenever $\mathcal{S}$ is a subset of $\mathcal{O}$ of cardinality at most the continuum, then $\omega^* \times \prod \mathcal{S}$ is again an image of ω^*. The class $\mathcal{O}$ contains all separable compact spaces, all compact spaces of weight at most ω_1 and all perfectly normal compact spaces.

1. INTRODUCTION

For a Tychonoff space X, βX represents the Čech-Stone compactification of X and for $A \subset X$, A^* represents the subspace $cl_{\beta X}(A) \setminus X$. If $f : X \to Y$, then $\beta f : \beta X \to \beta Y$ is the Čech-Stone extension of f and $f^* = \beta f \restriction X^*$. The first infinite ordinal ω is given the discrete topology.

W. Just [Ju89] has proven that $\omega^* \times \omega^*$ is consistently not a continuous image of ω^*. So it is consistent that there are two ω^* images whose product is not an ω^* image. This creates a limit to naive product results and a desire to find a general product result that includes many of the ω^* image results appearing in the literature. This is the theme of this paper. We will define a subclass $\mathcal{O}$ of the class of all ω^* images and show (The Product Theorem) that if $\mathcal{S}$ is a subset of $\mathcal{O}$ of cardinality at most the continuum, then $\omega^* \times \prod \mathcal{S}$ is an ω^* image. In the succeeding sections, we will show that $\mathcal{O}$ contains all separable compact spaces, all compact spaces of weight at most ω_1, all perfectly normal compact spaces, all zero-dimensional, orderable ω^* images, and all ω^* images of weight $< \mathfrak{b}$.

In this paper all spaces are assumed to be Hausdorff and all mappings between spaces are assumed to be continuous surjections. If X is homeomorphic to Y, then this is denoted by $X \approx Y$. If F is a family of functions with a common domain X and ranges Y_f for $f \in F$, then $\triangle F$ represents the diagonal function with domain X and range $\prod_{f \in F} Y_f$ defined by $\triangle F(x) = (f(x))_{f \in F}$. A Boolean space is a zero-dimensional, compact space, and $CO(X)$ denotes the algebra of all clopen subsets of a space X. Thus, $CO(\omega^*)$ is isomorphic to the quotient algebra $\mathcal{P}(\omega)$ modulo the ideal of finite sets. The cardinality of the continuum is denoted by $\mathfrak{c}$. For an

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introduction to the theory of ω^* images we refer the reader to the article by van Mill [vM84].

2. Orthogonality and products

Definition 2.1. Two functions $f : X \to Y$ and $g : X \to Z$ are called **orthogonal** **($f \perp g$)** iff $f \triangle g : X \to Y \times Z$ is a surjection. If the sets are Boolean spaces and the functions are mappings, then the dual notion is as follows: If A and B are two subalgebras of a boolean algebra C, then A and B are called **independent** if for every non-0 $a \in A$ and for every non-0 $b \in B$, $a \wedge b \neq 0$.

Definition 2.2. X is an **orthogonal ω^* image** if there exists a finite-to-one surjection $q : \omega \to \omega$ and a mapping $f : \omega^* \to X$ with $f \perp q^*$. If X is a Boolean space, then the dual notion is as follows: A subalgebra B of $CO(\omega^*)$ is called an **orthogonal subalgebra** if there exists a partition $\mathcal{R} = \{R_k : k < \omega\}$ of ω into finite sets such that for every non-empty $b \in B$, if $b = A^*$ where $A \subset \omega$, then $\{k < \omega : A \cap R_k \neq \emptyset\}$ is a co-finite subset of ω.

Let us observe that a continuous image of an orthogonal ω^* image is again an orthogonal ω^* image.

In this paper, we will find it convenient to view ω in one of its alternate forms. Let $W = \{(k, j) \in \omega \times \omega : j \leq k\}$ be given the discrete topology (so $W^* \approx \omega^*$), let $p : W \to \omega$ be the projection $p(k, j) = k$, and for each $k < \omega$, let $W_k = p^{-1}(k)$. As p is a finite-to-one surjection, we have that $p^* : W^* \to \omega^*$ is an open surjection. Thus we have

($\bigstar$) $\forall$ open $V \subset W^*$ with $p^*(V) = \omega^*$ $\exists$ clopen $V' \subset V$ with $p^*(V') = \omega^*$.

Henceforth W, W_k, W^*, p, and p^* will always have the above meanings. They will be frequently used in this paper. The following demonstrates the canonical nature of W.

Proposition 2.3. *If X is an orthogonal ω^* image, then there exists a mapping $f : W^* \to X$ such that $f \perp p^*$.*

Proof. Let q be a finite-to-one surjection $q : \omega \to \omega$ and let g be a mapping $g : \omega^* \to X$ such that $g \perp q^*$. Since $g \perp q^*$, for every $x \in X$ we have that $q^*(g^{-1}(x)) = \omega^*$. This means that for every $x \in X$ and for every $A \subset \omega$ such that $g^{-1}(x) \subset A^*$, $\{n < \omega : A \cap q^{-1}(n) \neq \emptyset\}$ is a co-finite subset of ω. We will produce a finite-to-one surjection $\varphi : W \to \omega$ such that whenever $A \subset \omega$ satisfies that $\{n < \omega : A \cap q^{-1}(n) \neq \emptyset\}$ is co-finite in ω, then $\{n < \omega : \varphi^{-1}(A) \cap W_n \neq \emptyset\}$ is co-finite in ω. Putting $f = g \circ \varphi^*$, we see that $f \perp p^*$. To construct φ, first get an increasing function $\psi : \omega \to \omega$ such that for every $n < \omega$, $\psi(n) \geq |q^{-1}(n)|$. For every $i < \psi(0)$ let $\varphi_i : W_i \to \omega$ be arbitrary. For every $n \geq 0$ and for every i with $\psi(n) \leq i < \psi(n+1)$ let φ_i be a surjection from W_i onto $q^{-1}(n)$. Define $\varphi : W \to \omega$ so that for every $i < \omega$, $\varphi \restriction W_i = \varphi_i$. □

Proposition 2.4. *The product of an ω^* image and an orthogonal ω^* image is an ω^* image.*

Proof. This is an immediate consequence of the definition of orthogonal ω^* image. □

It follows from Just's result that ω^* need not be an orthogonal ω^* image.

Theorem 2.5 (The Product Theorem)**.** *The product of at most $\mathfrak{c}$ many orthogonal ω^* images is an orthogonal ω^* image.*

Proof. Invoking Proposition 2.3, for every $s \in S \subset 2^\omega$ let $f_s : W^* \to X_s$ satisfy $f_s \perp p^*$. For every $k < \omega$ let $U_k = \{t : t$ is a function with domain 2^k and range $W_k\}$. Let $U = \bigcup_{k<\omega} U_k$ have the discrete topology (so $U^* \approx \omega^*$). If $t \in U_k$, then define $r(t) = k$. As $r : U \to \omega$ is a finite-to-one surjection, $r^* : U^* \to \omega^*$ is a surjection. For every $s \in S$ define $p_s : U \to W$ by $p_s(t) = t(s \restriction k)$ where $t \in U_k$, i.e., $p_s \restriction U_k$ is the projection from U_k onto the factor indexed by $s \restriction k$. As p_s is a finite-to-one surjection, $p_s^* : U^* \to \omega^*$ is a surjection. We claim that $r^* \perp \triangle\{f_s \circ p_s^* : s \in S\}$. By compactness, it suffices to show that for any finite set $T \subset S$, $r^* \perp \triangle\{f_s \circ p_s^* : s \in T\}$; for which, in turn, it suffices to show that for any collection of non-empty sets $\{V, V_s : s \in T\}$ where V is clopen in ω^* and V_s is open in X_s for $s \in T$, that $r^{*-1}(V) \cap \bigcap_{s\in T} (f_s \circ p_s^*)^{-1}(V_s) \neq \emptyset$. Let us check this. Since $f_s \perp p^*$, we have $p^*(f_s^{-1}(V_s)) = \omega^*$. So by ($\bigstar$), we can choose clopen $V_s' \subset W^*$ with $V_s' \subset f_s^{-1}(V_s)$ and $p^*(V_s') = \omega^*$. We will prove that $r^{*-1}(V) \cap \bigcap_{s\in T} p_s^{*-1}(V_s') \neq \emptyset$. Let $A \subset \omega$, $A_s \subset W$ for $s \in T$ be subsets with $V = A^*$, $V_s' = A_s^*$. A is infinite and $p(A_s)$ is co-finite for every $s \in T$ since $p^*(A_s^*) = \omega^*$. So $A \cap \bigcap_{s\in T} p(A_s)$ is infinite. Let $k \in A \cap \bigcap_{s\in T} p(A_s)$ be big enough to distinguish all members of T, i.e., if $s \neq s'$, s,$s' \in T$, then $s \restriction k \neq s' \restriction k$. For every such k, there is a mapping $t \in U_k$ such that for every $s \in T$, $t(s \restriction k) \in A_s$. Since any t like this is in $r^{-1}(A) \cap \bigcap_{s\in T} p_s^{-1}(A_s)$, the set $r^{-1}(A) \cap \bigcap_{s\in T} p_s^{-1}(A_s)$ is infinite. So $r^{*-1}(V) \cap \bigcap_{s\in T} p_s^{*-1}(V_s') \neq \emptyset$. □

We now strengthen a classical result.

Proposition 2.6. *Every separable, compact space is an orthogonal ω^* image.*

Proof. It suffices to show that $\beta\omega$ is an orthogonal ω^* image. Let $q : W \to \omega$ be the second projection $q(k, j) = j$. As q is an infinite-to-one surjection, $q^* : W^* \to \beta\omega$ is a surjection. Since for every $j < \omega$, $\{k < \omega : q^{-1}(j) \cap W_k \neq \emptyset\}$ is a co-finite subset of ω, we see that $q^* \perp p^*$. □

3. Weight ω_1

Definition 3.1. A **thick clopen** subset $V \subset W^*$ is a clopen $V \subset W^*$ with $V = A^*$ where $A \subset W$ satisfies $\lim_{k\to\infty} |A \cap W_k| = \infty$. A **thick zeroset** $Z \subset W^*$ is a zeroset $Z \subset W^*$ with $Z = \bigcap_{k<\omega} V_k$ where $\{V_k\}$ is a decreasing sequence of thick clopen subsets. If G is a clopen subset of W^*, then a **thick mapping** $f : G \to X$ is a mapping $f : G \to X$ with $f^{-1}(x)$ a thick zeroset for every $x \in X$.

We now list some basic properties of thick sets.

T1. If $V \subset V'$ are clopen sets and V is thick, then V' is thick.
If $Z \subset Z'$ are zerosets and Z is thick, then Z' is thick.

T2. If $\{Z_k\}$ is a decreasing sequence of thick zerosets, then $\bigcap_{k<\omega} Z_k$ is a thick zeroset.

T3. A thick zeroset contains a thick clopen set.
A thick clopen set contains two disjoint thick clopen sets.
T4. A thick mapping $f : W^* \to X$ satisfies $f \perp p^*$.

Let us prove the first part of T3. Let $Z \subset W^*$ with $Z = \bigcap_{j<\omega} V_j$ where $\{V_j\}$ is a decreasing sequence of thick clopen sets. For $j < \omega$ let $A_j \subset W$ such that $V_j = A_j^*$. Thus we have (a) $j < k$ implies $A_k \setminus A_j$ is finite and (b) for every j, $\lim_{k\to\infty} |A_j \cap W_k| = \infty$. By induction, using (a) and (b), define an increasing sequence $\{r_n\} \subset \omega$ such that for every $k \geq r_n$, $A_n \cap W_k \subset \bigcap_{i<n} A_i \cap W_k$ and $|A_n \cap W_k| \geq n$. For each $n > 0$ and for each i with $r_n \leq i < r_{n+1}$ let F_i be a subset of $A_n \cap W_i$ of cardinality n. If $F = \bigcup_{i<\omega} F_i$, then F^* is a thick clopen set with $F^* \subset Z$.

We now give a topological proof of a strengthening of a result of Parovičenko [Pa63] using some ideas in Błaszczyk and Szymański [BS80].

Proposition 3.2. *Let X be a compact metric space and let $f : W^* \to X$ be a thick mapping. Let E and F be closed in X with $X = E \cup F$. Then, there exists a clopen $G \subset W^*$ such that $f \upharpoonright G$ is a thick mapping onto E and $f \upharpoonright (W^* \setminus G)$ is a thick mapping onto F.*

Proof. Let D be a countable, dense subset of $E \cap F$. For every $d \in D$ (by T3) choose disjoint, thick clopen subsets U_d and V_d of $f^{-1}(d)$. Thus $G_0 = f^{-1}(E \setminus F) \cup \bigcup_{d\in D} U_d$ and $G_1 = f^{-1}(F \setminus E) \cup \bigcup_{d\in D} V_d$ are disjoint cozerosets of W^*. Let G be clopen in W^* with $G_0 \subset G$ and $G_1 \subset (W^* \setminus G)$. Then $f \upharpoonright G$ is a mapping onto E. Let $x \in E$ and let Z be a zeroset neighbourhood of x in E. Then $Z \cap ((E \setminus F) \cup D) \neq \emptyset$. If $z \in Z \cap (E \setminus F)$, then, as $f^{-1}(z)$ is a thick zeroset and $f^{-1}(z) \subset (f \upharpoonright G)^{-1}(Z)$, $(f \upharpoonright G)^{-1}(Z)$ is a thick zeroset (by T1). If $z \in Z \cap D$, then, as U_z is a thick clopen set and $U_z \subset (f \upharpoonright G)^{-1}(Z)$, $(f \upharpoonright G)^{-1}(Z)$ is a thick zeroset. Since x is the intersection of a countable, decreasing sequence of its zeroset neighbourhoods, we have that $(f \upharpoonright G)^{-1}(x)$ is a thick zeroset (by T2). Similarly, we have that $f \upharpoonright (W^* \setminus G)$ is a thick mapping onto F. □

Theorem 3.3. *Every compact space X of weight at most ω_1 is an orthogonal ω^* image.*

Proof. Since every ω^* image is the image of a zero-dimensional ω^* image of the same weight, it suffices to assume that X is Boolean. Via an embedding into the Cantor cube 2^{ω_1} we may express X as an inverse limit space, $X = \underline{\lim}\{X_\alpha, p_\alpha^\beta, \alpha < \beta < \omega_1\}$ such that:

(i) β limit implies $X_\beta = \underline{\lim}\{X_\alpha, p_\alpha^\gamma, \alpha < \gamma < \beta\}$ (the spectrum is continuous).
(ii) X_α is a zero-dimensional, compact metric space.
(iii) $X_{\alpha+1} = X_{\alpha+1}^0 \cup X_{\alpha+1}^1$ where $X_{\alpha+1}^0$ and $X_{\alpha+1}^1$ are disjoint, clopen subsets of $X_{\alpha+1}$ and for $i = 0, 1$, $p_\alpha^{\alpha+1} \upharpoonright X_{\alpha+1}^i$ is one-to-one (short projections are "simple").
(iv) $|X_0| = 1$.

We will construct a system of thick mappings $\{f_\alpha : W^* \to X_\alpha : \alpha < \omega_1\}$ such that $f_\alpha = p_\alpha^\beta \circ f_\beta$, $\alpha < \beta < \omega_1$. Since for every $\alpha < \omega_1$, $f_\alpha \perp p^*$, by compactness we will get that $f_{\omega_1} = \underline{\lim}\{f_\alpha : \alpha < \omega_1\}$ satisfies $f_{\omega_1} \perp p^*$.

Limit Step. If β is a limit, then let $f_\beta = \underline{\lim}\{f_\alpha : \alpha < \beta\}$. Property T2 implies that f_β is a thick mapping.

Successor Step. If $\beta = \alpha + 1$, then let $E = p_\alpha^{\alpha+1}(X^0_{\alpha+1})$ and $F = p_\alpha^{\alpha+1}(X^1_{\alpha+1})$. Apply Proposition 3.2 to get a clopen $G \subset W^*$ with $f_\alpha \restriction G$ a thick mapping onto E and $f_\alpha \restriction (W^* \setminus G)$ a thick mapping onto F. Define $f_{\alpha+1} : W^* \to X_{\alpha+1}$ by

$$f_{\alpha+1}(z) = \begin{cases} (p_\alpha^{\alpha+1} \restriction X^0_{\alpha+1})^{-1}(f_\alpha(z)) & \text{if } z \in G, \\ (p_\alpha^{\alpha+1} \restriction X^1_{\alpha+1})^{-1}(f_\alpha(z)) & \text{if } z \notin G. \end{cases}$$

$f_{\alpha+1}$ is a thick mapping because

$$f^{-1}_{\alpha+1}(x) = \begin{cases} (f_\alpha \restriction G)^{-1}(p_\alpha^{\alpha+1}(x)) & \text{if } x \in X^0_{\alpha+1}, \\ (f_\alpha \restriction (W^* \setminus G))^{-1}(p_\alpha^{\alpha+1}(x)) & \text{if } x \in X^1_{\alpha+1}, \end{cases}$$

and both $f_\alpha \restriction G$ and $f_\alpha \restriction (W^* \setminus G)$ are thick mappings. ☐

4. Perfectly normal

Proposition 4.1. *Let X, Y, $\overline{Y}$ and Z be compact spaces and let $f : X \to Y$, $g : X \to Z$, $h : \overline{Y} \to Y$, and $k : X \to \overline{Y}$ be mappings such that $f \perp g$, $f = h \circ k$, and h is irreducible. Then, $k \perp g$.*

Proof. Let $y \in \overline{Y}$ and let O_y be a neighbourhood of y. Irreducibility of h implies that there exists a non-empty, open $V \subset Y$ with $h^{-1}(V) \subset O_y$. Therefore, $k^{-1}(h^{-1}(V)) = f^{-1}(V) \subset k^{-1}(O_y)$. But $g(f^{-1}(V)) = Z$ since $f \perp g$. So $g(k^{-1}(O_y)) = Z$. Thus $g(k^{-1}(y)) = Z$. ☐

We now strengthen a result of Przymusiński [Pr82]. For the reader's convenience, we sketch his arguments and indicate how we apply Proposition 4.1. The following Lifting Lemma appears in his paper.

Lemma 4.2. *Let X be compact and perfectly normal, Z a closed subspace of $X \times I$ and suppose that the restriction $\pi \restriction Z : Z \to X$ of the projection $\pi : X \times I \to X$ is irreducible. If $f : \omega^* \to X$ is a continuous mapping of ω^* onto X, then there exists a continuous mapping $g : \omega^* \to Z$ of ω^* onto Z such that $f = \pi \circ g$.*

Theorem 4.3. *Every perfectly normal compact space X is an orthogonal ω^* image.*

Proof. We express X as the limit space of an inverse spectrum

$$X = \underline{\lim}\{X_\alpha, \pi^\beta_\alpha, \alpha < \beta < \kappa\}$$

with limit projections $\pi_\alpha : X \to X_\alpha$ such that:

(a) β limit implies $X_\beta = \underline{\lim}\{X_\alpha, \pi^\gamma_\alpha, \alpha < \gamma < \beta\}$
(b) $w(X_0) \leq \omega_1$
(c) π_0 is irreducible (therefore all other projections are irreducible)
(d) $X_{\alpha+1}$ is a closed subset of $X_\alpha \times I$ and $\pi^{\alpha+1}_\alpha$ is the restriction to $X_{\alpha+1}$ of the projection of $X_\alpha \times I$ onto X_α.

Parts (b) and (c) are achieved by using the result of Šapirovski [Sa74] that every perfectly normal compact space admits a π-base of cardinality at most ω_1. By Theorem 3.3, let $f_0 : W^* \to X_0$ satisfy $f_0 \perp p^*$. By induction on $\alpha \leq \kappa$, we construct $f_\alpha : W^* \to X_\alpha$ (where $X_\kappa = X$) such that $\beta < \alpha$ implies that $\pi^\alpha_\beta \circ f_\alpha = f_\beta$. At successor stages use Lemma 4.2 to get $f_{\alpha+1}$ and at limit stages let $f_\alpha = \underline{\lim}\{f_\beta : \beta < \alpha\}$. Thus we have $f_\kappa : W^* \to X$ with $f_0 = \pi_0 \circ f_\kappa$. Proposition 4.1 with $X = W^*$, $Y = X_0$, $\overline{Y} = X$, $Z = \omega^*$ and $f = f_0$, $g = p^*$, $h = \pi_0$, $k = f_\kappa$ implies that $f_\kappa \perp p^*$. □

5. Orderable

Theorem 5.1. *Every zero-dimensional, orderable ω^* image L is an orthogonal ω^* image.*

Proof. Let $\prec$ be a compatible order for L. Let $L' = \{x \in L : x \text{ is a left neighbour of a jump in } \prec\}$. Since L is Boolean, $G = \{\{y \in L : y \preceq x\} : x \in L'\}$ is a generating set for $CO(L)$. It suffices to define $\phi : L' \to \mathcal{P}(W)$ such that for every $x,y \in L'$, $x \prec y$ implies that (a) $\phi(x) \setminus \phi(y)$ is finite and $\phi(y) \setminus \phi(x)$ is infinite and (b) $\{k < \omega : (\phi(y) \setminus \phi(x)) \cap W_k \neq \emptyset\}$ is a co-finite subset of ω. Because then the function that sends $\{y \in L : y \preceq x\}$ to $\phi(x)^*$ will extend to an embedding of $CO(L)$ into $CO(W^*)$ (by (a)) onto an orthogonal subalgebra of $CO(W^*)$ (by (b)). Since L is an ω^* image, let $\psi : L' \to \mathcal{P}(\omega)$ satisfy that for every $x,y \in L'$, $x \prec y$ iff $\psi(x) \setminus \psi(y)$ is finite. Finally, define $\phi : L' \to \mathcal{P}(W)$ by $\phi(x) = \{(k,j) : j \leq |\psi(x) \cap k\,|\}$. □

Let Q denote the lexicographic ordered space 2^{ω_1}. It is known (Bell [Be90]) that Q is an ω^* image, so we get the following corollary.

Corollary 5.2. *Q is an orthogonal ω^* image.*

Example 5.3. The Alexandroff one-point compactification of the discrete space $\mathfrak{c}$ is an orthogonal ω^* image because it is an image of Q.

Corollary 5.4. *Every first countable orderable compact space is an orthogonal ω^* image.*

Proof. This follows because Maurice [Ma64] has proved that every first countable orderable compact space is an image of Q. □

Question 5.5. Can zero-dimensionality be removed from the hypotheses of Theorem 5.1, i.e., is every orderable ω^* image an orthogonal ω^* image?

6. Weight $< \mathfrak{b}$

Recall that if $f,g \in \omega^\omega$, then $f <_* g$ means that eventually $f(k) < g(k)$ and that $\mathfrak{b}$ is by definition the least cardinality of an $<_*$-unbounded subset of ω^ω.

Theorem 6.1. *Every ω^* image X of weight $< \mathfrak{b}$ is an orthogonal ω^* image.*

Proof. It suffices to assume that X is Boolean. Let B be a subalgebra of $CO(\omega^*)$ of cardinality $< \mathfrak{b}$. For every $b \in B$, choose $M_b \subset \omega$ such that $b = M_b^*$. For every $b \in B$, define $f_b \in \omega^\omega$ by $f_b(k) = \min\{n \in M_b : n \geq k\}$. Let $g \in \omega^\omega$ such that for every $b \in B$, $f_b <_* g$ and for every $k < \omega$, $k < g(k)$. Let $R = \{(k,j) : k \leq j < g(k)\}$ endowed with the discrete topology (therefore $R^* \approx \omega^*$), let for every $k < \omega$, $R_k = \{(k,j) : k \leq j < g(k)\}$, and let $\mathcal{R} = \{R_k : k < \omega\}$. For every $b \in B$, define $A_b = \{(k,j) \in R : j \in M_b\}$. Then, $C = \{A_b^* : b \in B\}$ is a subalgebra of $CO(R^*)$,

$B \cong C$ by the natural map $b \mapsto A_b^*$, and C is an orthogonal subalgebra of $CO(R^*)$ because for every $b \neq \emptyset$, $\{k < \omega : A_b \cap R_k \neq \emptyset\}$ is co-finite in ω. □

A family $\mathcal{A}$ of subsets of ω is said to be strongly centered if whenever $\mathcal{F}$ is a finite subset of $\mathcal{A}$, then $\bigcap \mathcal{F}$ is infinite. Recall that $\mathfrak{p}$ is the least cardinality of a strongly centered family $\mathcal{A}$ such that there does not exist an infinite $X \subset \omega$ with the property that for every $A \in \mathcal{A}$, $X \setminus A$ is finite. It is a basic fact that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$.

We now strengthen the result of van Douwen and Pryzmusiński [DP80] that every compact space of weight $< \mathfrak{p}$ is an ω^* image.

Corollary 6.2. *Every compact space X of weight $< \mathfrak{p}$ is an orthogonal ω^* image.*

Proof. Since X is an ω^* image and $\mathfrak{p} \leq \mathfrak{b}$ (cf. van Douwen [vD84]), we can apply Theorem 6.1. □

It follows from this corollary that Martin's Axiom implies that all compact spaces of weight less than $\mathfrak{c}$ are orthogonal ω^* images.

In conclusion, we summarize our results relating to well-known classes of spaces.

Summation 6.3. Let $\mathcal{D}$ be the class of all compact spaces which have at least one of the following properties: separable, weight at most ω_1, perfectly normal, first countable orderable, and weight $< \mathfrak{p}$. If $\mathcal{S}$ is a subset of $\mathcal{D}$ of cardinality at most $\mathfrak{c}$, then $\omega^* \times \prod \mathcal{S}$ is an image of ω^*.

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Department of Mathematics, University of Manitoba, Fort Garry Campus, Winnipeg, Canada R3T 2N2

E-mail address: mbell@cc.umanitoba.ca

Department of Mathematics, Academy of Labor and Social Relations, Lobachevskogo 90, Moscow, Russia 117454

E-mail address: lshapiro@glas.apc.org

Matematický Ústav, University Karlovy, Sokolovská 83, 18600 Praha 8, Czech Republic

E-mail address: psimon@ms.mff.cuni.cz