

## NON-SIMPLE LINKS WITH TUNNEL NUMBER ONE

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**ABSTRACT.** We determine all non-simple links which admit an unknotting tunnel, i.e. links which contain an essential annulus or torus in its exterior and have tunnel number one.

A link  $l$  is *non-simple* if there is an essential annulus or torus properly embedded in the exterior of  $l$ ,  $E(l)$ , that is, there is an annulus or torus which is incompressible,  $\partial$ -incompressible, and non  $\partial$ -parallel. A link  $l$  admits an unknotting tunnel if there is a properly embedded arc  $\tau$  in  $E(l)$  such that  $E(l \cup \tau)$  is a handlebody; in that case we say that  $l$  has tunnel number one. If  $l$  is not a knot and admits an unknotting tunnel, then necessarily  $l$  has two components, say  $l = l_1 \cup l_2$ , and any unknotting tunnel for  $l$  connects  $l_1$  to  $l_2$ .

In this paper we give a complete list of all non-simple links which admit an unknotting tunnel. Knots admitting unknotting tunnels are classified in [MS], [E]. First we show that if  $l$  is a non-simple link which has an unknotting tunnel  $\tau$ , then there is an essential annulus  $A$  in  $E(l)$  disjoint from  $\tau$ . This divides the non-simple links into two types, depending if  $\partial A = a_1 \cup a_2$  lie on different, or on the same component of  $\partial E(l)$ . In Theorem 1 (2) we give a list of the non-simple links having an annulus as in the first (second) case.

**Lemma.** *Let  $l$  be a non-simple link with an unknotting tunnel  $\tau$ . Then there exist an essential annulus  $A$  in  $E(l)$  such that  $A \cap \tau = \emptyset$ .*

*Proof.* Let  $M = E(l \cup \tau)$ .  $M$  is a genus 2-handlebody.  $E(l)$  is obtained by adding a 2-handle to  $M$  along a curve  $\alpha$  on  $\partial M$  which is a cocore of the tunnel  $\tau$ . Clearly  $\partial M$  is compressible,  $\partial M \setminus \alpha$  is incompressible, and  $\alpha$  is a separating curve on  $\partial M$ . If there is an essential annulus or torus in  $E(l)$ , then by Theorems 1 and 2 of [E], there is an essential (incompressible, not parallel to a surface in  $\partial M$ ) annulus  $A$  in  $M$  disjoint from  $\alpha$ , that is, an essential annulus in  $E(l)$  disjoint from  $\tau$ .  $\square$

**Theorem 1.** *Let  $l$  be a non-simple link with unknotting tunnel  $\tau$ , and  $A$  an essential annulus disjoint from  $\tau$ . Suppose that  $\partial A = a_1 \cup a_2$  lie on different components of  $\partial E(l)$ . Then one of the following cases occurs:*

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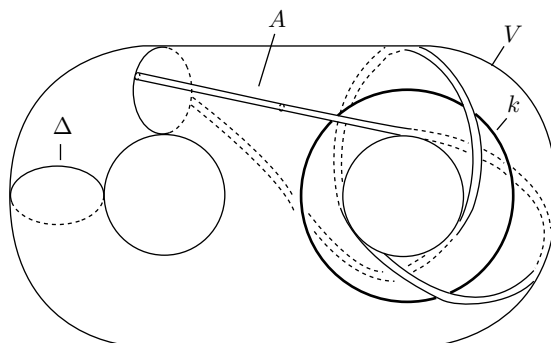


FIGURE 1

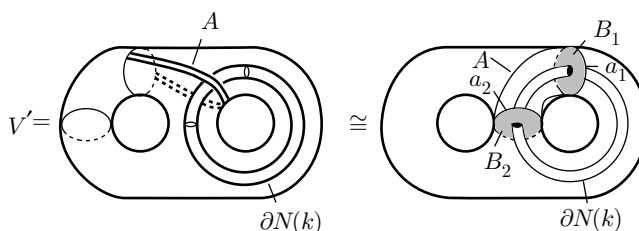


FIGURE 2

- (i) If  $a_1$  is a meridian of  $N(l_1)$  and  $a_2$  is (necessarily) isotopic to  $l_2$  in  $N(l_2)$ , then  $l$  is a connected sum of a two-bridge knot  $k$  and the Hopf link.
- (ii) If  $a_1$  is isotopic to  $l_1^n$  ( $n \geq 2$ ) in  $N(l_1)$  and  $a_2$  is isotopic to  $l_2$  in  $N(l_2)$ , then  $l_2$  is a torus knot and  $l_1$  is its core.
- (iii) If  $a_i$  is isotopic to  $l_i$  in  $N(l_i)$  ( $i = 1, 2$ ), then  $l$  is a torus link with type  $(2, 2p)$ ,  $p \in \mathbb{N}$ .
- (iv) If  $a_i$  is isotopic to  $l_i^{n_i}$  ( $n_i \geq 2$ ) in  $N(l_i)$  ( $i = 1, 2$ ), then  $l$  is the Hopf link.

*Proof.* By the definition of a tunnel,  $E(l \cup \tau)$  is a genus two handlebody  $V$ . Since each component of  $\partial A$  lies on a different component of  $\partial E(l)$  and  $A$  is essential in  $E(l)$ ,  $A$  is a non-separating essential annulus in  $V$ . According to [K, Lemma 3.2 (ii)], we can assume that the pair  $(V, A)$  is like in Figure 1.

Note that there exists a meridian disk  $\Delta$  of  $V$  such that  $\Delta \cap A = \emptyset$ . Now, we consider the knot  $k$  in  $V$  indicated in Figure 1. Then  $V' = V \setminus N^\circ(k)$  is as shown in Figure 2.

Let  $\alpha$  be a simple closed curve on  $\partial V$  which is a cocore of the tunnel  $\tau$ . Since each component of  $\partial A$  lie on a different component of  $E(l)$ ,  $\alpha$  separates  $a_1$  from  $a_2$  on  $\partial V$ . Let  $B_1, B_2$  be the annuli indicated in Figure 2. Then  $\{B_1, B_2\}$  cuts  $V'$  into a genus two handlebody  $V''$ . Let  $a'_i, a''_i$  be copies of  $a_i$  ( $i = 1, 2$ ) on  $\partial V''$ . (See Figure 3.)

We can consider  $(V'', \partial N(k))$  as a trivial tangle. Since the curve  $\alpha$  separates  $\{a'_1, a''_1\}$  from  $\{a'_2, a''_2\}$ , it follows that  $\alpha$  is as shown in Figure 4(i). Then, we can assume that  $V'$  is as shown in Figure 4(ii). By attaching a 2-handle to  $V'$  along  $\alpha$ , we have  $\Sigma = E(l) \setminus N^\circ(k)$ . Now, after moving  $\partial N(l_2)$  through  $k$  along the annulus

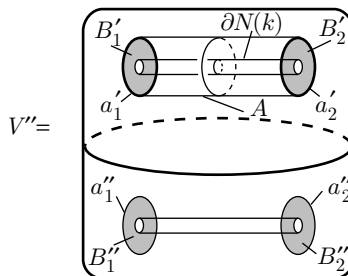


FIGURE 3

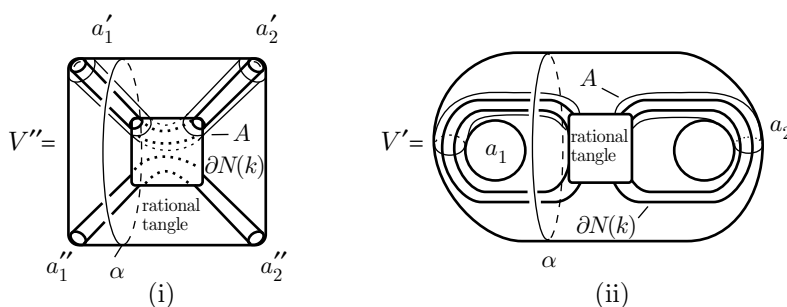


FIGURE 4

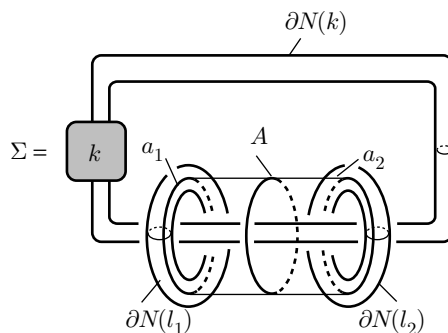


FIGURE 5

$A, \Sigma$  looks like in Figure 5.  $\partial N(k)$  as in Figure 5 determines the exterior of a knot  $K$  in  $S^3$ . Note that  $K$  is a two-bridge knot.

Case (i). Since  $a_2$  is isotopic to  $l_2$  in  $N(l_2)$ ,  $\Sigma' = \Sigma \cup N(l_2)$  is indicated in Figure 6(i).

We can exchange  $\partial N(l_1)$  for  $\partial N(k)$  by a homeomorphism of  $\Sigma$ . (See Figure 6(ii). If we cut  $\Sigma'$  along the annulus  $C$  shown in Figure 6, it is easy to find the homeomorphism. It can also be seen applying the bulb trick, where  $\partial N(l_1)$  becomes knotted while  $\partial N(k)$  is being unknotted.) But because  $a_1$  is a meridian of  $N(l_1)$ , it follows that  $(S^3, l)$  is as shown in Figure 6(iii), by a surgery description. The manifold is  $S^3$ , so  $s/r = \pm 1/n$ ,  $n \in \mathbb{N}$ .

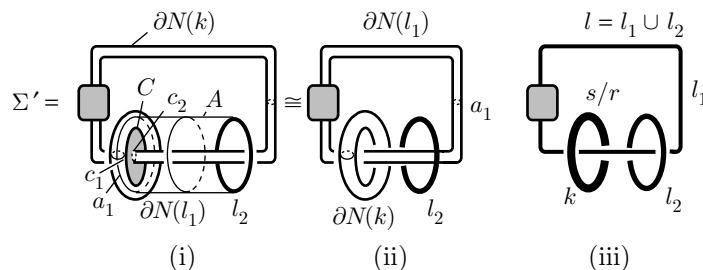


FIGURE 6

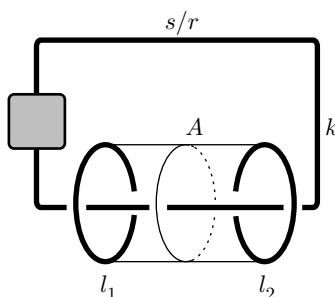


FIGURE 7

This completes the proof of Case (i).

Case (ii). Since  $a_2$  is isotopic to  $l_2$  in  $N(l_2)$ ,  $\Sigma' = \Sigma \cup N(l_2)$  is indicated in Figure 6(i). Let  $X = N(l_1) \cup N(C) \cup N(k)$ ,  $c_1 = N(l_1) \cap C$ ,  $c_2 = N(k) \cap C$ . Then  $S^3 = X \cup E(X)$ ,  $E(X)$  is homeomorphic to  $E(K)$ , and  $a_1$  is a meridian of  $E(K)$ . Now, we consider the following two cases:

- (a)  $K$  is a non-trivial two-bridge knot, or
- (b)  $K$  is a trivial knot.

Case (ii-a).  $X$  must be a solid torus, for otherwise  $X \cup E(X)$  could not be  $S^3$ . Since  $a_1 \cap C = \emptyset$ ,  $a_1$  is a curve on  $\partial X$ . Then  $a_1$  is not a meridian of  $X$ , because  $a_1$  is isotopic to  $l_1^n (n \geq 2)$  in  $N(l_1)$ . Now  $S^3 = X \cup_{a_1} E(X)$ . This is a contradiction, because every non-trivial two-bridge knot has Property P by [T]. Hence Case (ii-a) does not occur.

Case (ii-b). We can consider  $X$  as a union of two solid tori attached along an annulus. So  $X$  must be a torus knot exterior, and  $l_1$  is its core. Since  $K$  is trivial,  $E(X) (\cong E(K))$  is a solid torus and  $l_2$  is its core. Then, we have that  $l_2$  is a torus knot and  $l_1$  is its core. This completes the proof of Case (ii).

Case (iii). Since  $a_i$  is isotopic to  $l_i$  in  $(N_i) (i = 1, 2)$ ,  $(S^3, l_1 \cup l_2)$  is indicated in Figure 7 by a surgery description, where  $s/r$  is a surgery coefficient.

Since  $A$  is incompressible, this Dehn surgery on  $K$  must be non-trivial. Every non-trivial two-bridge knot has Property P by [T], so  $K$  must be trivial, and  $r/s$  must be  $\pm 1/n$ . Then  $l$  is a torus link, with type  $(2, 2n)$ ,  $n \in \mathbb{N}$ .

This completes the proof of Case (iii).

Case (iv). Let  $Y = N(l_1) \cup N(A) \cup N(l_2)$ . Then  $S^3 = Y \cup E(Y)$ . Since  $a_i$  is isotopic to  $l_1^n (n \geq 2)$ ,  $Y$  must be a non-trivial torus knot exterior and  $E(Y)$  must

be a solid torus. Since  $l_1$  (resp.  $l_2$ ) is the core of  $N(l_1)$  (resp.  $N(l_2)$ ),  $l$  is the Hopf link.

This completes the proof of Case (iv).  $\square$

Note that any link satisfying one of Cases (i)–(iv) of Theorem 1 is non-simple and has an unknotting tunnel.

We describe two classes of non-simple links which admit unknotting tunnels.

(1) Let  $R$  be a standard solid torus in  $S^3$ . Let  $\mathcal{C}$  be the collection of all annuli in  $\partial R$ , which winds at least twice algebraically in the complementary solid torus of  $R$  in  $S^3$ , so for any such annulus  $S$ , a component of  $\partial S$  is a curve on  $\partial R$  of slope  $(p, q)$ , for some relative prime integers  $p$  and  $q$  with  $|p| \geq 2$ . Let  $l_1$  be the core of  $R$ . Let  $k$  be a nontrivial knot in  $S^3 \setminus l_1$  which has a 1-bridge presentation with respect to some annulus  $S$  of the collection  $\mathcal{C}$  (cf. [E, 4.3]), that is: (a)  $k$  can be isotoped to intersect transversally  $\partial R$  in two points, both points lying in  $S$ . Then  $k$  is divided into two arcs  $k_1, k_2$ . (b)  $k_i$ ,  $i = 1, 2$ , can be projected to  $S$  to a single arc, that is, there is a disk  $D_i$  such that  $\partial D_i = k_i \cup \gamma_i$ , where  $\gamma_i$  is a single arc in  $S$ , and  $\text{int} D_i \cap \partial R = \emptyset$ ,  $\text{int} D_i \cap k = \emptyset$ . Let  $l = l_1 \cup l_2$  be a link such that  $l_1$  is as above and  $l_2 = k$ . We say that  $l$  belongs to the class  $\mathcal{A}$ .

If  $l$  is in the class  $\mathcal{A}$ , then  $l_2$  has a 1-bridge presentation with respect to some annulus  $S \subset \partial R$ , and it follows that  $l_2$  can be isotoped to lie in  $N(S)$ . Let  $T = \partial N(S)$ . If  $l_2$  is not parallel to the core of  $S$ , then the torus  $T$  is essential in  $E(l)$ . Note that in any case there is an essential annulus  $A$  in  $E(l)$ , which has its boundary on  $\partial E(l_1)$ ; this is as follows: Let  $\partial S = s_1 \cup s_2$ . There are annuli  $A_1$  and  $A_2$  in  $R$ , joining the curves  $s_1$  and  $s_2$  to respective curves on  $\partial N(l_1)$ ; let  $A = (T \setminus R) \cup A_1 \cup A_2$ . The link  $l$  admits an unknotting tunnel, which is a straight arc joining  $l_1$  and  $l_2$ , which is disjoint from the annulus  $A$ .

Note that if  $l \in \mathcal{A}$  and  $q \geq 2$ , then  $l_2$  is a satellite knot with tunnel number one.

The following is an alternative description of the class of links  $\mathcal{A}$ ; it follows from [E, 4.8] that these descriptions are equivalent: Let  $k_0$  be a torus knot of type  $(p, q)$  in  $S^3$ , where  $|p| \geq 2$ , so  $k_0$  may be trivial. Let  $L = k_1 \cup k_2$  be a 2-bridge link of type  $(\alpha, \beta)$  in  $S^3$ . Since  $k_2$  is a trivial knot, there is an orientation-preserving homeomorphism

$$f : S^3 \setminus N^\circ(k_2) \rightarrow N(k_0)$$

which takes a meridian  $m_2 \subset \partial N(k_2)$  of  $k_2$  to a fiber

$$h \subset \partial N(k_0) = \partial(S^3 \setminus N^\circ(k_0))$$

of the Seifert fibration  $D(-r/p, s/q)$  of  $S^3$ . We consider the link  $l = l_1 \cup l_2$ , where  $l_1$  is the core of the torus knot  $k_0$ , and  $l_2$  is the knot  $f(k_1)$ . The link  $l$  is determined by the symbol  $(\alpha, \beta; p, q)$ . In Figure 8 a member of this family is shown.

(2) Let  $k$  be a non-trivial  $(p, q)$ -torus knot or a  $(p, q)$ -cable of a  $(p_0, q_0)$ -torus knot, where  $p = qp_0q_0 \pm 1$ ; i.e.  $k$  is a torus knot or a cable knot with tunnel number one (cf. [MS], [Mo], [E, 4.6]).

Let  $k$  be a torus knot; then any unknotting tunnel for  $k$  is isotopic to one of the tunnels  $\tau_u, \tau_v, \tau_0$  described in [BRZ] or [MS]. There is an essential annulus  $A$  in  $E(k)$ , which divides it into two solid tori  $Q_1, Q_2$ . Choose representatives of  $\tau_u, \tau_v$ , and  $\tau_0$ , which are arcs properly embedded in  $E(k)$ , are disjoint from  $A$ ,  $\tau_u$  ( $\tau_v$ ) is contained in  $Q_1$  ( $Q_2$ ), and  $\tau_0$  is contained in either of  $Q_1$  or  $Q_2$ , for it is isotopic to an arc lying on  $A$ .

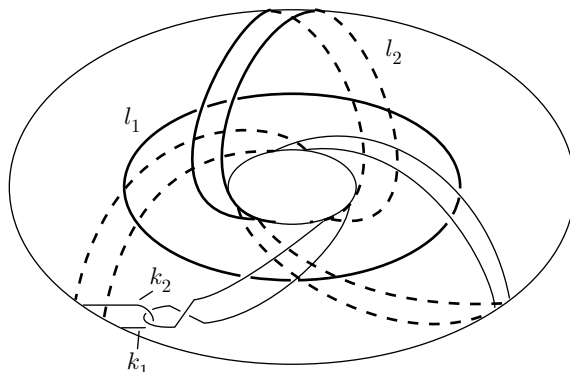


FIGURE 8

Let  $k$  be a cable of a torus knot as above. Any unknotting tunnel for  $k$  is isotopic to one of two unknotting tunnels described in [MS], which we denote by  $\tau_1, \tau_2$ . There is an essential annulus  $A$  in  $E(k)$ , which divide it into a solid torus  $Q_1$ , and the exterior  $E(k_0)$  of a torus knot  $k_0$ . Choose representatives of  $\tau_1$  and  $\tau_2$ , which are arcs properly embedded in  $E(k)$ , are disjoint from  $A$ , and are (necessarily) contained in  $E(k_0)$ .

Let  $k$  and  $A$  be as above.  $\partial A$  divides  $\partial E(k)$  into two annuli; let  $B$  be one of these annuli (either one if  $k$  is a torus knot, and the one contained in  $E(k_0)$  if  $k$  is a cable of a torus knot  $k_0$ ). Embed  $B \times I$  into  $E(k)$ , where  $I = [0, 1]$  (and such that  $A \cap (B \times I) = \partial B \times \{0\}$ ). Clearly

$$\text{cl}(E(k) \setminus (B \times I)) = E(k).$$

If  $\tau$  is one of the chosen tunnels of  $k$ , assume that its endpoints lie on  $B \times \{1\}$ , say these are  $(x_0, 1)$  and  $(x_1, 1)$ , where  $x_0$  and  $x_1$  are distinct points in the interior of  $B$ . Let  $k'$  be a knot in  $S^3 - k$  divided into 2 simple arcs,  $k_1$  and  $k_2$ , such that  $k_1$  is an arc lying on  $B \times \{1\}$  with endpoints in  $x_0$  and  $x_1$ , and  $k_2$  is one of the standard tunnels for  $k$  described above. Define a link of two components  $l = l_1 \cup l_2$ , where  $l_1 = k$  and  $l_2 = k' = k_1 \cup k_2$ .  $l$  is a non-simple link, for  $A$  is an essential annulus in  $E(l)$ ;  $l$  has an unknotting tunnel, which is the arc  $\{x_0\} \times I$ . We say that  $l$  belongs to the class  $\mathcal{B}$ .

Note that arcs on an annulus with ends on two fixed points can be parameterized by the rational fractions  $\alpha/\beta$ , where  $\alpha$  is odd and  $\beta$  is even. (Pair of arcs in a sphere with endpoints in four fixed points are parameterized by  $Q \cup \{1/0\}$ ; such arcs are divided in three classes, depending of the points they join, and each class is parameterized by a class of fractions  $\alpha/\beta$ , where this class is determined by the parity of  $\alpha$  and  $\beta$ . Arcs in an annulus with endpoints in two fixed points are in one-to-one correspondence with any of the classes of arcs of the sphere. We can choose then the class parameterized by fractions  $\alpha/\beta$  where  $\alpha$  is odd and  $\beta$  is even.) A link  $l = l_1 \cup l_2$  in the class  $\mathcal{B}$  can be described by a symbol  $(p, q; \alpha, \beta; \tau_i)$  or a symbol  $(p, q; p_0, q_0; \alpha, \beta; \tau_i)$ , depending if  $l_1$  is a  $(p, q)$ -torus or a  $(p, q)$ -cable of a  $(p_0, q_0)$ -torus knot.  $(\alpha, \beta)$  and  $\tau_i$  denote the types of arcs in which  $l_2$  is divided, where  $\tau_i$  is one of the standard tunnels for  $l_1$ . In Figure 9 a member of  $\mathcal{B}$  is shown.

**Theorem 2.** *Let  $l$  be a non-simple link with unknotting tunnel  $\tau$ , and  $A$  an essential annulus disjoint from  $\tau$ . Suppose that  $\partial A = a_1 \cup a_2$  lie on the same component*

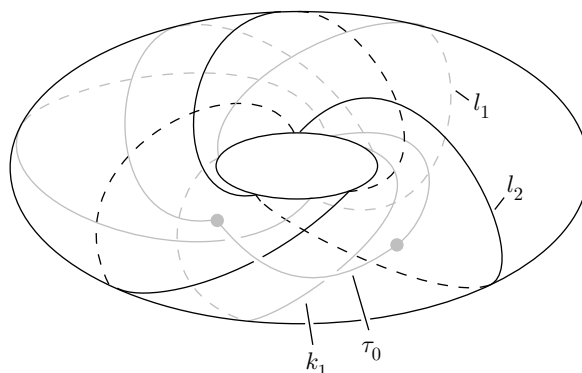


FIGURE 9

of  $\partial E(l)$ , say it lies on  $\partial N(l_1)$ . Then one of the following cases occurs:

- (i) If  $A$  is inessential in  $E(l_1)$ , then  $l$  is in  $\mathcal{A}$ .
- (ii) If  $A$  is essential in  $E(l_1)$ , then  $l$  is in  $\mathcal{B}$ .

*Proof.* Suppose that  $\partial A \subset \partial N(l_1)$ .  $A$  is incompressible in  $M = E(l \cup \tau)$ , and it is separating.  $M \setminus N^\circ(A)$  consists of two handlebodies.  $a_1$  cannot be a meridian of  $l_1$ , for in this case  $l$  would be a non-trivial connected sum  $l = k_1 \# k_2$ , where say,  $k_1$  is a knot. But  $E(k_1)$  is homeomorphic to a component of  $M \setminus N^\circ(A)$ , i.e., it is a handlebody; then  $k_1$  is the trivial knot, which is a contradiction. Therefore  $a_1$  is a curve of type  $(p, q)$  on  $\partial N(l_1)$ ,  $(p, q) \neq (1, 0)$ .

There are two possibilities:

*Case (i).*  $A$  is inessential in  $E(l_1)$ . Then  $A$  is parallel in  $E(l_1)$  to an annulus  $A'$  of type  $(p, q)$  in  $\partial N(l_1)$ .  $E(l_1) \setminus N^\circ(A)$  has two components; one is a solid torus in which  $l_2$  and  $\tau$  lie, and the other is homeomorphic to  $E(l_1)$ , but this has to be a handlebody, so  $E(l_1)$  is a solid torus and  $l_1$  is a trivial knot. Therefore  $|p| \geq 2$ .

Note that  $l_1 \cup \tau$  is an unknotting tunnel for  $l_2$ , so by [E, 4.4],  $l_2$  has a 1-bridge presentation with respect to  $\partial N(l_1)$ , and by the argument of [E, 4.7],  $l_2$  has a 1-bridge presentation w.r.t.  $A'$ . This shows that  $l \in \mathcal{A}$ .

*Case (ii).*  $A$  is essential in  $E(l_1)$ . Therefore  $l_1$  is a cable knot with tunnel number one, so  $l_1$  is a torus knot or a  $(p, q)$ -cable of a  $(p_0, q_0)$ -torus knot, where  $p = qp_0q_0 \pm 1$ .  $A$  is the standard annulus properly embedded in  $E(l_1)$ .  $A$  divides  $E(l_1)$  in two pieces; one is a solid torus, and the other is a manifold  $N$  homeomorphic to  $E(k)$ , where  $k$  is the companion of  $l_1$ , so  $k$  is the trivial knot or a certain torus knot. We can assume that  $l_2$  and  $\tau$  lie on  $N$ .  $A$  divides  $\partial E(l_1)$  on two annuli, let  $B$  be one of them.

Slide  $l_2$  over  $\tau$  to get an arc  $\tau'$  properly embedded in  $E(l_1)$ .  $\tau'$  is isotopic to one of the standard tunnels  $\gamma$  for  $E(l_1)$ .

*Claim.* There is an isotopy carrying  $\tau'$  to  $\gamma$  which keeps the annulus  $A$  fixed.

The isotopy which carries  $\tau'$  to  $\gamma$  transforms the annulus  $A$  into an annulus  $A'$ , which may intersect  $A$ . By looking at the intersections between the annuli and doing an innermost disk–outermost arc argument the isotopy can be deformed to one which fixes  $A$ , except possibly if  $\gamma = \tau_0$  and  $\tau'$  and  $\gamma$  lie on different components of  $E(k) \setminus A$ . But this can be fixed by assuming that if  $\gamma = \tau_0$ , then  $\gamma$  and  $\tau'$  must

lie on the same component  $E(k) \setminus A$ —there is no loss of generality in doing that. If  $\gamma \neq \tau_0$ , then necessarily  $\gamma$  and  $\tau'$  lie on the same side of  $E(k) \setminus A$ .

Let  $x_0, x_1$  be the endpoints of  $\gamma$ . We can assume that the endpoint of  $\tau$  lying on  $l_1$  coincides with one of the endpoints of  $\gamma$ , say this is  $x_0$ , and that the isotopy between  $\tau'$  and  $\gamma$  fixes that point. Consider an embedding of  $B \times I$  into  $E(k)$  so that  $A \cap (B \times I) = \partial B \times \{0\}$ . The isotopy carrying  $\tau'$  to  $\gamma$  can be made in two steps. First,  $\tau'$  is slid over  $B$  so that the complement of  $B \times I$  is fixed, and keeping fixed the endpoint of  $\tau'$  lying on  $x_0$ . Second,  $\tau'$  is isotoped to  $\gamma$  leaving fixed  $B \times I'$ , where  $I' = [0, a]$  is a subinterval of  $I$ .  $\gamma \cap (B \times I)$  consists of two straight arcs. This isotopy can be made in reverse order, in two steps. First, the endpoint of  $\gamma$  on  $x_0$  is fixed, and the other is slid over  $B$ , keeping fixed the complement of  $B \times I$ . Second,  $\gamma$  is isotoped, keeping fixed  $B \times I$ , and finally  $\tau'$  is slid over itself to get  $l_2 \cup \tau$ . This shows that  $l_2$  can be decomposed into two arcs,  $k_1, k_2$ , such that  $k_1$  is an arc in  $B \times I$ , consisting of a monotonic arc going from  $(x_1, 1)$  to  $(x_0, 0)$  union  $\tau$ , and  $k_2$  is an arc in the complement of  $B \times I$ , which can be thought equal to  $\gamma$ . Note that the arc  $k_1$  can be isotoped to a simple arc in  $B \times \{1\}$ , with endpoints  $(x_0, 1)$  and  $(x_1, 1)$ . This shows that  $l$  belongs to the class  $\mathcal{B}$ .  $\square$

**Note.** The list of links given in Theorem 1 is not disjoint from the list of links in Theorem 2. First note that case (iv) of Theorem 1 is contained in case (iii) of the same theorem. Links satisfying case (iii) of Theorem 1 also satisfy case (i) of Theorem 2. Links satisfying case (ii) of Theorem 1 satisfy both cases of Theorem 2. If  $l_1$  is a  $(p, q)$ -cable of a  $(p_0, q_0)$ -torus knot, where  $p = qp_0q_0 \pm 1$ , and if  $l_2$  is the core of the  $(p_0, q_0)$ -torus knot, then  $l = l_1 \cup l_2$  satisfy both cases of Theorem 2. However these are all the intersections among the different cases.

*Remark.* Tunnel number one knots are prime [N], [S]; but it follows from Theorem 1(ii) that the same is not true for links. K. Morimoto [M] proved that the only composite links which admit unknotting tunnels are sums of 2-bridge knots with the Hopf link. In Theorem 1(i) we give a different proof of this fact. A third proof can be given using a stronger version of the lemma (which also follows from [E]) and an argument similar to the first part of the proof of Theorem 2. A. C. Jones [J] proved that if a composite link has a 2-generator fundamental group, then it has two summands, one is a knot and the other is the Hopf link. The fact that a rank 2 composite link has a Hopf link summand was previously known by F. González-Acuña and H. Short; see *Knot surgery and primeness*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 89–102, page 96, line 32.

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