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EIGENVALUE RATIOS FOR THE REGULAR STURM-LIOUVILLE SYSTEM

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Abstract. Following the method of Ashbaugh-Benguria in Comm. Math. Phys. **124** (1989), 403–415; J. Differential Equations **103** (1993), 205–219, we prove an upper estimate of the arbitrary eigenvalue ratio (μ_m/μ_n) for the regular Sturm-Liouville system. This upper estimate is sharp for Neumann boundary conditions. We also discuss the sign of μ_1 and include an elementary proof of a useful trigonometric inequality first given in the aforementioned articles.

1. Introduction

A regular Sturm-Liouville system on $(0,1)$ [4, p.300] is the regular Sturm-Liouville equation:

$$-(p(x)y')' + q(x)y = \mu w(x)y \ , \tag{1}$$

where

$$p \in C^1[0,1], \qquad q, w \ \in C[0,1], \qquad p, w > 0; \tag{2}$$

with separated endpoint boundary conditions

$$\begin{cases} y(0)\cos\gamma - p(0)y'(0)\sin\gamma = 0, \\ y(1)\cos\delta - p(1)y'(1)\sin\delta = 0, \end{cases} \tag{3}$$

where $0 \le \gamma < \pi$, $0 < \delta \le \pi$. γ and δ are in fact the initial and final phases respectively in the classical Prüfer substitution.

It is well known that the above system has an increasing sequence of simple real eigenvalues

$$\mu_1 \ < \ \mu_2 \ < \ \cdots \ < \ \mu_n < \ \mu_{n+1} \ < \ \cdots \ ,$$

and the eigenfunction corresponding to μ_n has exactly n zeros in $(0,1)$. Sometimes, an additional eigenvalue $\mu_0 < \mu_1$ may also occur, and the corresponding eigenfunction has no zero in $(0,1)$.

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Recently, Ashbaugh and Benguria [3] proved an optimal upper estimate of the eigenvalue ratio λ_m/λ_n $(m > n \geq 1)$ for the Dirichlet problem when $\gamma = 0$ and $\delta = \pi$, with the assumptions that

$$q \geq 0 \quad \text{and} \quad 0 < k \leq pw \leq K \ .$$

Based on a modified Prüfer substitution and trigonometric inequality, they proved that

$$\frac{\lambda_m}{\lambda_n} \leq \left(\lceil \frac{m}{n} \rceil\right)^2 \frac{K}{k} \ ,$$

with equality when and only when $q \equiv 0$ and $k = K$. This is a generalization of their previous work on one-dimensional Schrödinger operators [1, 2]. (In this paper, we adopt the following notation. Let $\lceil s \rceil$ denote the value of the ceiling function at s, i.e. $\lceil s \rceil = \min\{n \in \mathbf{N},\ n \geq s\}$, and $\lfloor s \rfloor$ denote the value of the floor function at s, $\lfloor s \rfloor = \max\{n \in \mathbf{N},\ n \leq s\}$.)

We extend their result to the general regular Sturm-Liouville system. The main idea is to allow the modified phase to take up a negative initial value and in the induction process we divide the interval $[0, 1]$ into two parts by the point where the phase first attains $\pi/2$. Our main theorem is

Theorem 1. *For the regular Sturm-Liouville system* (1)–(3), *if* $q(x) \geq 0$ *and* $0 < k \leq pw \leq K$, *then for any integers* $m > n \geq 1$ *discussed below, the eigenvalues* μ_m *and* μ_n *are positive, and*

$$\frac{\mu_m}{\mu_n} \leq l^2 \, \frac{K}{k}$$

where l *is determined by the boundary conditions as follows:*

	$\pi \geq \pi - \gamma \geq \pi/2$	$\pi/2 > \pi - \gamma > 0$
$\pi \geq \delta \geq \pi/2$	$2(\lfloor \frac{1}{2} \lceil \frac{m}{n} \rceil \rfloor) + 1$	$2 \lceil \frac{m}{n} \rceil - 1$
$\pi/2 > \delta > 0$	$2 \lceil \frac{m}{n} \rceil - 1$	$2 \lceil \frac{m}{n} \rceil - 1$ (*when* $n > 1$)

The above theorem has an interesting corollary on the Neumann boundary conditions. The inequality is sharp there!

Theorem 2. *For the regular Sturm-Liouville system* (1) *and* (2) *with Neumann boundary conditions* $(\delta = \gamma = \pi/2)$, *if* $q \geq 0$ *and* $0 < k \leq pw \leq K$, *then for any integers* $m > n \geq 1$,

$$\frac{\mu_m}{\mu_n} \leq \left\{2(\lfloor \frac{1}{2} \lceil \frac{m}{n} \rceil \rfloor) + 1\right\}^2 \frac{K}{k} \ ,$$

with equality when and only when m *is an odd multiple of* n, $q \equiv 0$, *and* $k = K$.

We note that for any integer l, the function

$$2(\lfloor \frac{l}{2} \rfloor) + 1 \ = \ \begin{cases} l & \text{when } l \text{ is odd,} \\ l + 1 & \text{when } l \text{ is even,} \end{cases}$$

so that $\lceil m/n \rceil$ is odd when and only when

$$2(\lfloor \frac{1}{2} \lceil \frac{m}{n} \rceil \rfloor) + 1 \ = \ \lceil \frac{m}{n} \rceil \ .$$

Thus Theorem 2 is analogous to Ashbaugh-Benguria's result for Dirichlet boundary conditions above.

At the end of this section, we shall give an elementary proof of a trigonometric inequality in [2, 3]. This inequality is interesting itself and is critical in the method.

In section 2, we introduce the modified Prüfer substitution and the Comparison Theorem. The function $\tan^{-1}(s\tan\theta)$ is also analyzed there. The main theorems are proved in section 3. In section 4, we discuss the sign of μ_1 and obtain an upper estimate for μ_m/μ_1 for $0<\delta, \pi-\gamma<\pi/2$ when the potential function q is sufficiently large.

Theorem 3. *Let $c>1$. Then for $0<\theta<\lfloor c\rfloor\pi/c$,*

$$|\sin c\theta|<c\sin\theta\ .$$

Proof. When c is integral-valued, the inequality can easily be proved by mathematical induction, using the compound angle formula for the sine function. On the other hand, since $f(x)=\sin x$ is a strictly concave function in $[0,\pi]$, we have for any $\phi\in(0,\pi)$,

$$\sin(\frac{\phi}{c})>\frac{1}{c}\sin\phi\ .$$

Thus for arbitrary $c\in(1,2)$, let $\phi=c\theta$; the above inequality implies $c\sin\theta>\sin c\theta$. Inductively, suppose the above inequality holds for any $c\in[1,j]$ for $j\geq 2$. Let $c=p+1\in(j,j+1)$. It is easy to show that $|\sin c\theta|<c\sin\theta$ when $0<p\theta<\lfloor p\rfloor\pi$. And when $\frac{\pi}{2}<\frac{\lfloor p\rfloor\pi}{p}\leq\theta<\frac{\lfloor p\rfloor+1}{p+1}\pi$,

$$0<(\lfloor p\rfloor+1)\pi-(p+1)\theta\leq\pi-\theta<\frac{\pi}{2}.$$

Therefore

$$|\sin(p+1)\theta|\leq\sin\theta<(p+1)\sin\theta\ .$$

The proof is complete. □

Corollary 4. *If $c\geq 1$ and $|\theta|\leq\lfloor c\rfloor\pi/c$, then $\sin^2 c\theta\leq c^2\sin^2\theta$.*

2. Preliminaries

We shall apply the modified Prüfer substitution as introduced in [3] to the regular Sturm-Liouville system. Let

$$\begin{aligned} y &= r(x)\sin a\sqrt{\mu}\theta\ , \\ py' &= a\sqrt{\mu}r(x)\cos a\sqrt{\mu}\theta\ . \end{aligned}$$

Then

$$r^2=y^2+\frac{(py')^2}{a^2\mu}\quad \textit{and}\quad \tan a\sqrt{\mu}\theta=\frac{a\sqrt{\mu}y}{py'}\ .$$

Also from (1), we derive

$$\begin{aligned} \frac{d\theta}{dx} &= \frac{1}{p}-(\frac{1}{p}-\frac{w}{a^2})\sin^2 a\sqrt{\mu}\theta-\frac{q}{a^2\mu}\sin^2 a\sqrt{\mu}\theta \\ &= \Phi_{(a,\mu)}(x,\theta)\ . \end{aligned} \tag{4}$$

Eq. (4) represents different first-order equations for different values of a and μ. Moreover, suppose y_m is the eigenfunction associated with the eigenvalue μ_m; the corresponding Eq. (4) has boundary values

$$\begin{cases} a\sqrt{\mu_m}\,\theta(0) = \tan^{-1}(a\sqrt{\mu_m}\tan\gamma), & 0\leq\gamma<\pi\ , \\ a\sqrt{\mu_m}\,\theta(1) = \tan^{-1}(a\sqrt{\mu_m}\tan\delta)+m\pi, & 0<\delta\leq\pi\ . \end{cases} \tag{5}$$

The inverse function $\tan^{-1}$ takes value in $[0, \pi]$ instead of the usual $(-\frac{\pi}{2}, \frac{\pi}{2})$. In particular, let $\tan^{-1}(a\sqrt{\mu}\tan\phi) = \phi$ where $\phi = 0, \frac{\pi}{2}, \pi$.

Lemma 5. *Fix any* $\gamma \in [0, \pi)$; *the function* f *defined as*

$$f_\gamma(s) = \frac{1}{s}[\tan^{-1}(s\tan\gamma) - \pi]$$

is strictly increasing in $s \in \mathbf{R}^+$.

Proof. First $f_0(s) = -\frac{\pi}{s}$ and $f_{\frac{\pi}{2}}(s) = -\frac{\pi}{2s}$. Both functions are strictly increasing. For the other f_γ's,

$$f'_\gamma(s) = \frac{s\tan\gamma - [\tan^{-1}(s\tan\gamma) - \pi](1 + s^2\tan^2\gamma)}{s^2(1 + s^2\tan^2\gamma)}.$$

Let $\varphi = \tan^{-1}(s\tan\gamma) \in [0, \pi)$. Then the numerator of f'_γ is given by

$$u(\varphi) = \tan\varphi + (\pi - \varphi)(1 + \tan^2\varphi)\ .$$

Since u is positive for $0 < \gamma < \frac{\pi}{2}$, the corresponding f_γ is strictly increasing. When $\frac{\pi}{2} < \gamma < \pi$, then $\frac{\pi}{2} < \varphi < \pi$ and $\tan\varphi < 0$, and

$$u'(\varphi) = 2(\pi - \varphi)\tan\varphi\sec^2\varphi < 0.$$

Since $u(\pi) = 0$, it follows that $u(\varphi) > 0$ for $\frac{\pi}{2} < \varphi < \pi$. Therefore $f'_\gamma > 0$ and f_γ is strictly increasing. □

Corollary 6. *For any* $\varphi \in [0, \pi]$ *and* $j \in \mathbf{N}$, *if* $1 \le s \le j$, *then*

$$\tan^{-1}(s\tan\varphi) + (j - 1)\pi \ge j\varphi.$$

Proof. It is evident that the inequality holds when $\varphi = \pi$. When $\varphi \in [0, \pi)$, we have from Lemma 5 that $\tan^{-1}(s\tan\varphi) - \pi \ge s(\varphi - \pi)$. Hence

$$\tan^{-1}(s\tan\varphi) - \varphi \ge (s - 1)(\varphi - \pi).$$

And since $1 \le s \le j$,

$$\tan^{-1}(s\tan\varphi) - \varphi \ge (j - 1)(\varphi - \pi).$$

□

Lemma 7. *For any* $j \in \mathbf{N}$, $c \in \mathbf{R}$, *define the function*

$$g(\varphi) = \frac{1}{\varphi}\{j\pi + \tan^{-1}(c\tan\varphi)\}$$

for $\varphi \in (0, \pi)$. *If* $1 < c \le g(\varphi)$, *then* $g'(\varphi) < 0$. *Hence if* $1 < c \le g(\varphi_0)$ *for some* $\varphi_0 \in (0, \pi)$, *then for any* $0 < \varphi < \varphi_0$, $c \le g(\varphi_0) < g(\varphi)$.

Proof. For $\varphi \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$,

$$\begin{aligned} g'(\varphi) &= \frac{-1}{\varphi^2}[j\pi + \tan^{-1}(c\tan\varphi)] + \frac{1}{\varphi}[\frac{c\sec^2\varphi}{1 + c^2\tan^2\varphi}] \\ &< \frac{1}{\varphi}[c - \frac{1}{\varphi}(j\pi + \tan^{-1}(c\tan\varphi))] \\ &\le 0\ . \end{aligned}$$

Also, by L'Hôpital's rule,

$$\begin{aligned} g'(\frac{\pi}{2}) &= \lim_{\phi\to\pi/2} \frac{j\pi + \tan^{-1}(c\tan\phi) - (2j+1)\phi}{\phi(\phi-\pi/2)} \\ &= -\frac{2(2j+1)}{\pi} + \frac{2}{\pi}\lim_{\phi\to\pi/2} \frac{\tan^{-1}(c\tan\phi)-\pi/2}{\phi-\pi/2} \\ &= \frac{2}{c\pi}\,[1-(2j+1)c] \\ &< 0\ . \end{aligned}$$

□

The modified Prüfer substitution turns the problem into a first order differential equation. Hence we can use a comparison theorem to compare the growth of solutions for two different first order initial value problems. We state the following modified version of [4, p.29] for future use.

Comparison Theorem. *Consider two differential equations on* $[0,1]$,

$$\begin{aligned} \theta_1'(x) &= F(x,\theta_1(x))\ , \\ \theta_2'(x) &= G(x,\theta_2(x))\ . \end{aligned}$$

Suppose F *or* G *is Lipschitz in* θ, *and* $F(x,\theta) \le G(x,\theta)$ *on* $[0,1]\times I$ *for some interval* I. *If* $\theta_1(0)\le\theta_2(0)$ *and* $\theta_2(x)$ *lies in the interior of* I *for every* x *in* $(0,1)$, *then* $\theta_1\le\theta_2$ *on* $[0,1]$. *In fact, take any* $x_0\in[0,1]$, *either* $\theta_1(x_0)<\theta_2(x_0)$ *or* $\theta_1=\theta_2$ *on* $[0,x_0]$.

3. Proof of main theorems

Theorem 8. *For the regular Sturm-Liouville system* (1)–(3), *if* $q\ge 0$, $0<k\le pw\le K$ *and* $\mu_1>0$, *then for any integer* $m\ge 1$,
(a)

$$\frac{\mu_m}{\mu_1} \le (\frac{\lfloor m/2\rfloor\pi + \phi_{k\mu_m}}{\phi_{K\mu_1}})^2\frac{K}{k}\ ,$$

where

$$\phi = \min\{\delta,\pi-\gamma\}\ , \qquad \phi_s = \tan^{-1}(\sqrt{s}\tan\phi)\ .$$

(b)

$$\frac{\mu_m}{\mu_1} < (\frac{(m-1)\pi + \varphi_{k\mu_m}}{\varphi_{K\mu_1}})^2\frac{K}{k}\ , \tag{6}$$

where

$$\varphi = \max\{\delta,\pi-\gamma\}\ , \quad \textit{and} \quad \varphi_s = \tan^{-1}(\sqrt{s}\tan\varphi)\ .$$

Proof. For (a), we actually need to show

$$\frac{\mu_m}{\mu_1} \le \left(\max\{\frac{\lfloor m/2\rfloor\pi+\delta_{k\mu_m}}{\delta_{K\mu_1}},\ \frac{(\lfloor m/2\rfloor+1)\pi-\gamma_{k\mu_m}}{\pi-\gamma_{K\mu_1}}\}\right)^2\frac{K}{k} \tag{7}$$

($\gamma_s = \tan^{-1}(\sqrt{s}\tan\gamma)$ and $\delta_s=\tan^{-1}(\sqrt{s}\tan\delta)$). The two choices arise because the eigenvalues μ_m for (1) are invariant under the transformation $x\longmapsto 1-x$, while the phases at the boundary become $\pi-\delta$ and $\pi-\gamma$. Then Lemma 7 asserts that the choice of the maximum is equivalent to the minimum of the alternative phases (i.e. $\min\{\delta_{K\mu_1}, \pi-\gamma_{K\mu_1}\}$). Hence we may simply assume that

$$\delta_{K\mu_1} \le \pi - \gamma_{K\mu_1} \quad \text{or} \quad \delta \le \pi - \gamma \tag{8}$$

and show that

$$\frac{\mu_m}{\mu_1} \le (\frac{\lfloor m/2 \rfloor \pi + \delta_{k\mu_m}}{\delta_{K\mu_1}})^2 \frac{K}{k} \ .$$

We compare two different equations of form (4),

$$\frac{d\theta_j}{dx} = F_j(x,\theta_j) \qquad\qquad (\ j=1,m\ )\ ,$$

where

$$\begin{cases} F_1(x,\theta) = \Phi_{\sqrt{K},\mu_1}(x,\theta), \\ F_m(x,\theta) = \Phi_{\sqrt{k},\mu_m}(x,\theta). \end{cases}$$

We let θ_j $(j=1,m)$ denote the modified phase function of the jth eigenfunction, with initial values

$$\begin{aligned} \sqrt{K\mu_1}\,\theta_1(0) &= \tan^{-1}(\sqrt{K\mu_1}\tan\gamma) - \pi \\ &= \gamma_{K\mu_1} - \pi, \end{aligned}$$

$$\begin{aligned} \sqrt{k\mu_m}\,\theta_m(0) &= \tan^{-1}(\sqrt{k\mu_m}\tan\gamma) - \lfloor\frac{m+1}{2}\rfloor\pi \\ &= \gamma_{k\mu_m} - \lfloor\frac{m+1}{2}\rfloor\pi \ . \end{aligned}$$

Hence

$$\theta_1(1) = \frac{\delta_{K\mu_1}}{\sqrt{K\mu_1}}\ , \qquad \theta_m(1) = \frac{\lfloor m/2\rfloor\pi + \delta_{k\mu_m}}{\sqrt{k\mu_m}}\ .$$

Assume

$$\frac{\mu_m}{\mu_1} > (\frac{\lfloor m/2\rfloor\pi + \delta_{k\mu_m}}{\delta_{K\mu_1}})^2\frac{K}{k} \qquad \text{for some } m>1. \tag{9}$$

By (7), (8), and (9),

$$c \equiv \sqrt{\frac{k\mu_m}{K\mu_1}} > \frac{\lfloor\frac{m+1}{2}\rfloor\pi - \gamma_{k\mu_m}}{\pi - \gamma_{K\mu_1}}\ ,$$

and thus $\theta_1(0) < \theta_m(0)$. Then from $k\mu_m > K\mu_1$ and Corollary 4,

$$\frac{\sin^2\sqrt{K\mu_1}\theta}{K\mu_1} \le \frac{\sin^2\sqrt{k\mu_m}\theta}{k\mu_m}$$

for $\theta \in I = [-A, A]$, $A = \frac{\lfloor c\rfloor\pi}{\sqrt{k\mu_m}}$, Hence $F_1(x,\theta) \le F_m(x,\theta)$ for fixed $x\in[0,1]$ and $\theta\in I$. Therefore, if $\lfloor c\rfloor \ge \lfloor m/2\rfloor + 1$, then $\theta_m(x)$ lies in $(-A,A)$ for all $x\in(0,1)$. It follows from the Comparison Theorem that $\theta_1(1) < \theta_m(1)$ and a contradiction with (9) is achieved. Indeed, if $\lfloor c\rfloor < \lfloor m/2\rfloor + 1$, then by (9) and Corollary 6,

$$\lfloor c\rfloor \ge \lfloor\frac{\lfloor m/2\rfloor\pi + \delta_{k\mu_m}}{\delta_{K\mu_1}}\rfloor = \lfloor\frac{\lfloor m/2\rfloor\pi + \tan^{-1}(c\tan\delta_{K\mu_1})}{\delta_{K\mu_1}}\rfloor \ge \lfloor m/2\rfloor + 1\ ,$$

a contradiction. The proof of (a) is complete.

To prove (b), we compare the differential equations $\theta_j'(x) = F_j(x, \theta_j)$ $(j = 1, m)$ on $(0,1)$ with another set of initial values

$$\theta_1(0) = \frac{\gamma_{K\mu_1} - \pi}{\sqrt{K\mu_1}} \quad \text{and} \quad \theta_m(0) = \frac{\gamma_{k\mu_m} - \pi}{\sqrt{k\mu_m}} \ .$$

It follows that

$$\theta_1(1) = \frac{\delta_{K\mu_1}}{\sqrt{K\mu_1}} \quad \text{and} \quad \theta_m(1) = \frac{(m-1)\pi + \delta_{k\mu_m}}{\sqrt{k\mu_m}} \ .$$

Assume

$$\frac{\mu_m}{\mu_1} \geq \left(\frac{(m-1)\pi + \delta_{k\mu_m}}{\delta_{K\mu_1}}\right)^2 \frac{K}{k} \qquad \text{for some } m > 1. \tag{10}$$

Owing to Lemma 5, $\theta_1(0) < \theta_m(0)$. Thus $\theta_1(1) < \theta_m(1)$ if $\lfloor c \rfloor \geq m$. Indeed, if $\lfloor c \rfloor < m$, by Corollary 6 and (10),

$$\lfloor c \rfloor \geq \left\lfloor \frac{(m-1)\pi + \delta_{k\mu_m}}{\delta_{K\mu_1}} \right\rfloor = \left\lfloor \frac{(m-1)\pi + \tan^{-1}(c \tan \delta_{K\mu_1})}{\delta_{K\mu_1}} \right\rfloor \geq m \ ,$$

a contradiction. Hence half of (6) is proved. The transformation $x \longmapsto 1 - x$ produces the rest. Therefore (b) is also valid. □

Theorem 9. *Suppose all the conditions in Theorem* 8 *are satisfied.*

(a) *If* $\pi/2 \leq \min\{\delta, \pi - \gamma\}$, *then for all* $m > n \geq 1$,

$$\frac{\mu_m}{\mu_n} \ \leq \ \left(2\left\lfloor \frac{1}{2} \left\lceil \frac{m}{n} \right\rceil \right\rfloor + 1\right)^2 \frac{K}{k} \ .$$

(b) *If* $\pi/2 > \max\{\delta, \pi - \gamma\}$, *then for all* $m > n > 1$,

$$\frac{\mu_m}{\mu_n} \ < \ \left(2\left\lceil \frac{m}{n} \right\rceil - 1\right)^2 \frac{K}{k} \ .$$

Proof. We follow the method in [2]. First assume that n divides m. Use mathematical induction in n. The case $n = 1$ follows from Theorem 8. Suppose part (a) holds from $m = hn$ $(h \in \mathbf{N})$. Fix $i \in N$; for each $j < i$, let $N_j(\mu_i)$ denote the point in $(0,1)$ such that the phase function of the ith eigenfunction first attains $\pi/2$ $(\text{mod } \pi)$ after j zeros in (0,1). Let $\omega_1 = N_1(\mu_{n+1})$ and $\omega_2 = N_h(\mu_{h(n+1)})$. If $\omega_1 \leq \omega_2$, we consider the same regular Sturm-Liouville system on $(0, \omega_1)$. Now for each $h \in \mathbf{N}$, suppose $\widetilde{\mu_h}$ is the hth eigenvalue. Then $\mu_{h(n+1)} \leq \widetilde{\mu_h}$. By Theorem 8 and Lemma 7,

$$\frac{\mu_{h(n+1)}}{\mu_{n+1}} \leq \frac{\widetilde{\mu_h}}{\widetilde{\mu_1}} \leq \left(2\left\lfloor \frac{h}{2} \right\rfloor + 1\right)^2 \frac{K}{k} \ . \tag{11}$$

On the other hand, if $\omega_1 > \omega_2$, we perform the transformation $t = 1 - x$ and consider the regular Sturm-Liouville system on $(0, 1 - \omega_1)$ with a new separated boundary condition

$$\begin{aligned} y(0)\cos\delta - p(0)y'(0)\sin\delta &= 0 \ , \\ y'(1 - \omega_1) &= 0 \ . \end{aligned}$$

Therefore

$$\frac{\mu_{h(n+1)}}{\mu_{n+1}} \ \leq \ \frac{\widetilde{\mu_{hn}}}{\widetilde{\mu_n}} \leq \left(2\left(\left\lfloor \frac{h}{2} \right\rfloor\right) + 1\right)^2 \frac{K}{k} \tag{12}$$

by the induction hypothesis. In general let $h = \lceil m/n \rceil$. The proof for (b) is similar. □

Proof of Theorem 1. In view of Proposition 10 below, it suffices to consider the case when $0 < \delta, \pi - \gamma < \pi/2$. Here $\mu_n > 0$ for $n \geq 2$. We apply the method in Theorem 9 to this case. Since the phase of one of the boundary points attains $\pi/2$ (mod π), Theorem 8 asserts that Eqs. (11) and (12) are still valid. ☐

Proof of Theorem 2. The inequality follows from Theorems 8 and 9. When equality holds and $n = 1$, then $\theta_1(1) = \theta_m(1)$. By the Comparison Theorem, $\theta_1(0) = \theta_m(0)$ and $F_1(x,\theta) = F_m(x,\theta)$. So $q \equiv 0$ and $k = K$. Moreover, since $\theta_1(x) = \theta_m(x)$ at $x = 0, 1$,

$$\sqrt{\frac{k\mu_m}{K\mu_1}} = 2\lfloor\frac{m+1}{2}\rfloor - 1 = 2\lfloor\frac{m}{2}\rfloor + 1 \ ,$$

which implies that m has to be odd. The converse is trivial. Then we use induction on n. The proof is similar to [2, Proposition 3.2] and will be omitted. ☐

4. Sign of μ_1 and other cases

While it is well known that when $q \geq 0$, μ_2 is always positive (cf. [4, p. 318]), the sign of μ_1 seems to be undetermined. In this section we show that when $q \geq 0$, μ_1 is positive except possibly when δ, $\pi - \gamma < \pi/2$. In the latter case, positive μ_1 is guaranteed when q is sufficiently large. An upper estimate for μ_m/μ_1 immediately follows.

After the transformation (see [3])

$$\frac{dt}{dx} = \frac{1}{p(x)} \quad , \qquad z(t) = y(x) \ ,$$

the regular Sturm-Liouville system as given in (1)–(3) becomes another regular Sturm-Liouville system on $(0, t_0)$,

$$-z''(t) + p(x(t))q(x(t))z(t) = \mu p(x(t))w(x(t))z(t) \ , \tag{13}$$

satisfying

$$\begin{cases} z(0)\cos\gamma - z'(0)\sin\gamma = 0, \\ z(t_0)\cos\delta - z'(t_0)\sin\delta = 0, \end{cases}$$

where $t_0 = \int_0^1 p(x)^{-1}dx$. Applying classical Prüfer substitution on Eq. (13),

$$z(t) = r(t)\sin\phi(t) \ , \qquad z'(t) = r(t)\cos\phi(t) \ ,$$

we have (cf. [4]) the following phase equation:

$$\begin{cases} \phi'(t) &= (\mu pw - pq)\sin^2\phi(t) + \cos^2\phi(t), \\ \phi(0) &= \gamma. \end{cases} \tag{14}$$

If $\mu_1 \leq 0$, then we compare Eqs. (13) and (14) with the differential equation (assuming $q \geq 0$),

$$\hat{z}'' = 0 \ , \tag{15}$$

and its phase equation

$$\begin{cases} \hat{\phi}'(t) &= \cos^2\hat{\phi}(t), \\ \hat{\phi}(0) &= \gamma. \end{cases}$$

Obviously Eq. (15) has a solution $\hat{z}(t) = c_1 t + c_2$ where c_1, c_2 are constants satisfying

$$c_2 \cos\gamma - c_1 \sin\gamma = 0 \ . \tag{16}$$

From the Comparison Theorem, $\phi(t) \le \hat{\phi}(t)$ for all $t \ge 0$. We note that any solution of (14) has a zero in $(0,1)$ iff $\phi(t_1) = \pi$ for some $t_1 \in (0, t_0)$. Thus $\hat{\phi}(t_1) \ge \pi$. But the linear function $\hat{z}(t)$ has a zero in $(0, t_0)$ iff $(c_1 t_0 + c_2)c_2 < 0$ or $\frac{c_1}{c_2} t_0 + 1 < 0$. Hence by (16), $-t_0 < \tan\gamma < 0$. The transformation $t \longmapsto t_0 - t$ also gives $0 < \tan\delta < t_0$. Thus we have proved the following proposition.

Proposition 10. *If* $q \ge 0$, *then the eigenvalues* μ_1 *of a regular Sturm-Liouville system is positive, except possibly when*

$$-\int_0^1 p(x)^{-1} dx < \tan\gamma < 0 \quad \textit{and} \quad 0 < \tan\delta < \int_0^1 p(x)^{-1} dx \ ,$$

which imply $0 < \delta, \pi - \gamma < \pi/2$.

Theorem 11. *Suppose the regular Sturm-Liouville system* (1)–(3) *satisfies* $0 < k \le pw \le K$. *If* $\varphi = \max\{\delta, \pi - \gamma\} < \pi/2$ *and* $b \equiv \min_{[0,1]} pq > \cot^2 \varphi$, *then* $\mu_1 > 0$ *and*

$$\frac{\mu_m}{\mu_1} \le (m - \frac{1}{2})^2 \frac{\pi^2 K}{\zeta^2 k},$$

where

$$\zeta = \tan^{-1}[b \tan\varphi - \cot\varphi] \ .$$

Proof. In view of Theorem 8(b), we only need to prove that $\varphi_{K\mu_1} \ge \zeta$, i.e. $K\mu_1 > b - \cot^2\varphi$. WLOG, let $\varphi = \pi - \gamma$. Hence it suffices to show that $K\mu_1 > b - \cot^2\gamma$. Let $\xi = K\mu_1 - b$. We compare Eq. (13) with

$$-\tilde{z}'' = \xi \tilde{z} \ .$$

Then the phase equation becomes

$$\begin{cases} \frac{d\tilde{\phi}}{dt} = \xi \sin^2\tilde{\phi}(t) + \cos^2\tilde{\phi}(t), \\ \tilde{\phi}(0) = \gamma > \pi/2. \end{cases}$$

So $\phi(t) \le \tilde{\phi}(t)$ for all $t \ge 0$. If $\xi < \cot^2\gamma$, then $\tilde{\phi}'(t) < 0$ whenever $\tilde{\phi}(t) = \gamma$. On the other hand, $\tilde{\phi}'(t) > 0$ whenever $\tilde{\phi}(t) = 0$. It follows that $\tilde{\phi}(t)$ lies in $(0, \gamma]$ and never attains π, which means that $\tilde{z}$ has no zero in $(0, \infty)$. This result contradicts the fact that z has a zero in $(0, t_0)$. □

Our method of translating the phase function to the negative real axis works only when the eigenfunction has at least a zero in $(0,1)$. The method fails when we consider the zeroth eigenvalue μ_0. However, if one boundary condition is a Dirichlet one (say, $\gamma = 0$), then a straightforward use of the Ashbaugh-Benguria method would give

$$\frac{\mu_m}{\mu_1} \le (\frac{(m-1)\pi + \delta_{k\mu_m}}{\delta_{K\mu_1}})^2 \frac{K}{k}$$

provided that $\mu_1 > 0$ and arrive at an explicit upper estimate when $\delta \ge \pi/2$.

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