PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 5, May 1996

UNCOUNTABLY MANY C^0 CONFORMALLY DISTINCT LORENTZ SURFACES AND A FINITENESS THEOREM

ROBERT W. SMYTH

(Communicated by Christopher Croke)

ABSTRACT. This paper describes an uncountable family of Lorentz surfaces realized as rectangular regions in the Minkowski 2-plane E_1^2. A simple C^0 conformal invariant is defined which assigns a different real value to each Lorentz surface in the family. While these surfaces provide uncountably many C^0 conformally distinct, bounded, convex subsets of E_1^2 which are each symmetric about a properly embedded timelike curve and about a properly embedded spacelike curve, it is shown that there are only 21 C^0 conformally distinct, bounded, convex subsets of E_1^2 which are symmetric about some null line.

1. PRELIMINARIES

A *Lorentz surface* $\mathcal{L} = (S, [h])$ is by definition an oriented, connected C^∞ 2-manifold S together with the collection of C^∞ metrics on S conformally equivalent to a given indefinite (*i.e.* Lorentzian) metric h. The metric $\tilde{h}$ is conformally equivalent to h on S if and only if there exists a C^∞ function $\lambda > 0$ on S such that $h = \lambda\tilde{h}$. Thus a Lorentz surface is the indefinite metric analog of a Riemann surface.

An equivalence class of metrics $[h]$ on S determines a naturally ordered pair of distinct C^∞ null direction fields X, Y on S. (See Lemma 1 on p.5 in [4].) An integral curve k of X (resp. Y) is called an X-line (resp. Y-line). Either an X-line or a Y-line may be referred to as a *null line*. A conformal homeomorphism from the Lorentz surface $\mathcal{L} = (S, [h])$ to the Lorentz surface $\hat{\mathcal{L}} = (\hat{S}, [\hat{h}])$ is an orientation preserving homeomorphism from S to $\hat{S}$ that takes X-lines to X-lines and Y-lines to Y-lines. The characterization of conformal diffeomorphisms in the indefinite metric case on p.21 in [4] yields the above definition as a natural generalization.

The fundamental example of a Lorentz surface is the Minkowski 2-plane E_1^2 which is defined to be the x, y-plane together with the class of metrics conformally equivalent to $dxdy$. The X-lines of E_1^2 are the horizontal lines, and the Y-lines are the vertical lines. Although not every Lorentz surface can be embedded in E_1^2 (see [1] or [2]), the open, connected subsets of E_1^2 provide a rich collection of examples of Lorentz surfaces. For example, an uncountable collection of subsets of E_1^2 is presented in [3], no two members of which are C^1 conformally diffeomorphic. In this paper we present uncountably many subsets of E_1^2 no two of which are C^0 conformally homeomorphic.

Received by the editors October 20, 1994.
1991 *Mathematics Subject Classification.* Primary 53C50, 53A30.
Key words and phrases. Indefinite metric, conformal geometry.

We specify a 'time orientation' on E_1^2 by choosing left to right as the preferred orientation on the horizontal lines. This choice together with the standard orientation on the plane determines bottom to top to be the preferred orientation on the vertical lines. All open, connected subsets of E_1^2 inherit this time orientation. A time orientation preserving conformal homeomorphism from one subset of E_1^2 to another respects the preferred orientation on the horizontal lines. Hence it also respects the preferred orientation on the vertical lines.

Let $\pi_1 : E_1^2 \to \mathbb{R}$ and $\pi_2 : E_1^2 \to \mathbb{R}$ denote projection of E_1^2 onto its first and second coordinates respectively. Suppose that S is an open, convex subset of E_1^2, and that $S' = f(S)$ is the image in E_1^2 of S under a conformal homeomorphism $f : S \to S'$. Then f is given by $f(x, y) = (f_1(x), f_2(y))$ with $f_1 : \pi_1(S) \to \pi_1(S')$ and $f_2 : \pi_2(S) \to \pi_2(S')$ both continuous, and both strictly increasing or both strictly decreasing. If S and S' are bounded, the intervals $\pi_1(S) = (a, b)$, $\pi_1(S') = (c, d)$, $\pi_2(S) = (\alpha, \beta)$ and $\pi_2(S') = (\gamma, \delta)$ are bounded as well. Furthermore, f_1 and f_2 have unique continuous extensions $\bar{f}_1 : [a, b] \to [c, d]$ and $\bar{f}_2 : [\alpha, \beta] \to [\gamma, \delta]$. Thus f has a unique continuous one-one extension from $[a, b] \times [\alpha, \beta]$ onto $[c, d] \times [\gamma, \delta]$ which restricts to a homeomorphism $\bar{f} : \bar{S} \overset{onto}{\to} \bar{S}'$ given by $\bar{f} = (\bar{f}_1, \bar{f}_2)$.

2. Examples

For each $\alpha > 0$ let R_α be the rectangle in the x, y-plane with vertices at $(0, 0), (-1, 1)$ and (α, α). We show that distinct values of α yield C^0 conformally distinct Lorentz surfaces $(R_\alpha, [dxdy])$. Given a point P on the boundary of R_α other than the top or bottom (resp. right or left) corner, denote by $l_\alpha(P)$ (resp. $m_\alpha(P)$) the X-line (resp. Y-line) of $(R_\alpha, [dxdy])$ which has P as one of its endpoints. Given a null line k of $(R_\alpha, [dxdy])$ define k^+ (resp. k^-) to be the right most (resp. left most) point of k if k is an X-line, and the highest (resp. lowest) point of k if k is a Y-line. Let $P_\alpha^0 = (0, 0)$, the lowest corner of R_α. Let $k_\alpha^1 = m_\alpha(P_\alpha^0)$ and $P_\alpha^1 = (k_\alpha^1)^+$. Suppose $P_\alpha^1, ..., P_\alpha^n$ have been defined. If P_α^n is a corner of R_α, let $k_\alpha^{n+1} = k_\alpha^n$. Otherwise let

$$k_\alpha^{n+1} = \begin{cases} l_\alpha(P_\alpha^n) & \text{if } k_\alpha^n = m_\alpha(P_\alpha^n), \\ m_\alpha(P_\alpha^n) & \text{if } k_\alpha^n = l_\alpha(P_\alpha^n). \end{cases}$$

Define P_α^{n+1} by

$$P_\alpha^{n+1} = \begin{cases} (k_\alpha^{n+1})^+ & \text{if } P_\alpha^n = (k_\alpha^{n+1})^-, \\ (k_\alpha^{n+1})^- & \text{if } P_\alpha^n = (k_\alpha^{n+1})^+. \end{cases}$$

Thus $\{P_\alpha^i\}_{i=0}^\infty$ is an infinite sequence of boundary points such that any two consecutive points are the endpoints of a common null line. (See Figure 1.) We define the functions ν_α^+ and ν_α^- on ∂R_α so that ν_α^+ is 1 along both sides of R_α with slope 1 and zero otherwise, while ν_α^- is 1 along both sides of R_α with slope -1 and zero otherwise. Notice that ν_α^+ and ν_α^- both equal 1 on the corners of R_α.

Consider the group Γ_α generated by the reflections of E_1^2 in the four lines $y = \pm x$, $y = x + 2$ and $y = -x + 2\alpha$ along the sides of R_α. The set R_α together with its translates $R_\alpha + r(-1, 1) + s(\alpha, \alpha)$, for any $r, s \in \mathbb{Z}$, are the fundamental regions under the action of Γ_α on the plane. The grid G_α of lines $y = x + 2r$ and $y = -x + 2\alpha s$, for $r, s \in \mathbb{Z}$, is identified with ∂R_α under the action of Γ_α. The lattice points $L_\alpha = \{r(-1, 1) + s(\alpha, \alpha) : r, s \in \mathbb{Z}\}$ are identified with the corners of R_α. We use the action of Γ_α to extend the definitions of ν_α^+ and ν_α^- to G_α. The points where the grid G_α intersects the non-negative y-axis form a sequence $\{Q_\alpha^i\}_{i=0}^\infty$ whose entries

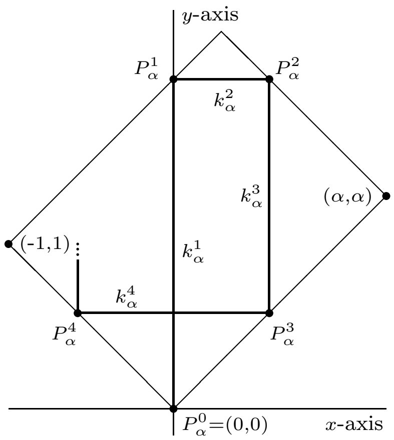

FIGURE 1

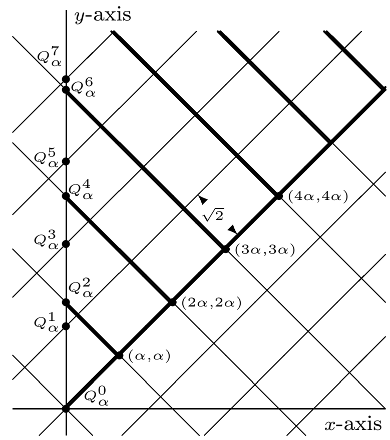

FIGURE 2

are indexed in the order of their y values. The action of Γ_α identifies the non-negative y-axis with the set $(\{P^0_\alpha\} \cup k^1_\alpha) \cup (\{P^1_\alpha\} \cup k^2_\alpha) \cup \cdots$, and the point Q^i_α with P^i_α for $i = 0, ..., n$.

Let L^s_α be the right isosceles triangle with vertices at $Q^0_\alpha = (0,0), (0, 2\alpha s)$, and $(\alpha s, \alpha s)$. The point $(0, 2\alpha s)$ is in the sequence $\{Q^i_\alpha\}$, so we may define n_s to be the integer such that $Q^{n_s}_\alpha = (0, 2\alpha s)$. The length of each leg of L^s_α is $\sqrt{2}\alpha s = \sqrt{2}\alpha \sum_{i=1}^{n_s} \nu^-_\alpha(Q^i_\alpha)$. We also have the following estimate on the length of each leg of L^s_α:

$$\sqrt{2}\sum_{i=1}^{n_s} \nu^+_\alpha(Q^i_\alpha) \leq \sqrt{2}\alpha \sum_{i=1}^{n_s} \nu^-_\alpha(Q^i_\alpha) \leq \sqrt{2}\left(\sum_{i=1}^{n_s} \nu^+_\alpha(Q^i_\alpha) + 1\right).$$

The term $\sum_{i=1}^{n_s} \nu^+_\alpha(Q^i_\alpha)$ counts the lines of slope 1 in the grid G_α that cross the upper leg of L^s_α, excluding the line on which the lower leg of L^s_α sits. (See Figure 2.) Since

$$\lim_{s\to\infty} \frac{\sqrt{2}}{\sqrt{2}\sum_{i=1}^{n_s} \nu^-_\alpha(Q^i_\alpha)} = 0$$

we have

$$\alpha = \lim_{s\to\infty} \frac{\sum_{i=1}^{n_s} \nu^+_\alpha(Q^i_\alpha)}{\sum_{i=1}^{n_s} \nu^-_\alpha(Q^i_\alpha)} = \lim_{s\to\infty} \frac{\sum_{i=1}^{n_s} \nu^+_\alpha(P^i_\alpha)}{\sum_{i=1}^{n_s} \nu^-_\alpha(P^i_\alpha)}. \tag{2.1}$$

Suppose f is a time orientation preserving conformal homeomorphism from $(R_\alpha, [dxdy])$ to $(R_\beta, [dxdy])$. Then $\bar{f}(k^+) = (f(k))^+$ and $\bar{f}(k^-) = (f(k))^-$ for any null line k of $(R_\alpha, [dxdy])$, so that $\bar{f}$ takes each edge and vertex of ∂R_α to the corresponding edge or vertex of ∂R_β. Hence

$$\nu_\alpha^+ = \nu_\beta^+ \circ \bar{f}, \tag{2.2}$$

$$\nu_\alpha^- = \nu_\beta^- \circ \bar{f}, \tag{2.3}$$

and from $\bar{f}(P_\alpha^0) = P_\beta^0$ we conclude that $f(k_\alpha^1) = k_\beta^1$ and $\bar{f}(P_\alpha^1) = \bar{f}((k_\alpha^1)^+) = (f(k_\alpha^1))^+ = (k_\beta^1)^+ = P_\beta^1$. We wish to show that $\bar{f}(P_\alpha^i) = P_\beta^i$ for $i = 0, 1, \dots$. Suppose inductively that $\bar{f}(P_\alpha^i) = P_\beta^i$ and $f(k_\alpha^i) = k_\beta^i$ for $i = 1, \dots, n$. This inductive hypothesis together with the supposition that f is time orientation preserving implies that

(i) $P_\alpha^i = (k_\alpha^i)^+$ if and only if $P_\beta^i = (k_\beta^i)^+$,

(ii) $P_\alpha^i = (k_\alpha^i)^-$ if and only if $P_\beta^i = (k_\beta^i)^-$,

(iii) $k_\alpha^i = l_\alpha(P_\alpha^i)$ if and only if $k_\beta^i = f(k_\alpha^i) = f(l_\alpha(P_\alpha^i)) = l_\beta(\bar{f}(P_\alpha^i)) = l_\beta(P_\beta^i)$, and

(iv) $k_\alpha^i = m_\alpha(P_\alpha^i)$ if and only if $k_\beta^i = f(k_\alpha^i) = f(m_\alpha(P_\alpha^i)) = m_\beta(\bar{f}(P_\alpha^i)) = m_\beta(P_\beta^i)$, for $i = 1, \dots, n$.

Suppose P_α^n is a corner of R_α. Then P_β^n is a corner of R_β, and we have $f(k_\alpha^{n+1}) = f(k_\alpha^n) = k_\beta^n = k_\beta^{n+1}$. Otherwise we have

$$f(k_\alpha^{n+1}) = \begin{cases} f(l_\alpha(P_\alpha^n)) & \text{if } k_\alpha^n = m_\alpha(P_\alpha^n) \\ f(m_\alpha(P_\alpha^n)) & \text{if } k_\alpha^n = l_\alpha(P_\alpha^n) \end{cases}$$

$$= \begin{cases} l_\beta(f(P_\alpha^n)) & \text{if } k_\beta^n = m_\beta(P_\beta^n) \\ m_\beta(f(P_\alpha^n)) & \text{if } k_\beta^n = l_\beta(P_\beta^n) \end{cases} = k_\beta^{n+1}.$$

In either case we have

$$\bar{f}(P_\alpha^{n+1}) = \begin{cases} (\bar{f}(k_\alpha^{n+1}))^+ & \text{if } P_\alpha^n = (k_\alpha^{n+1})^- \\ (\bar{f}(k_\alpha^{n+1}))^- & \text{if } P_\alpha^n = (k_\alpha^{n+1})^+ \end{cases}$$

$$= \begin{cases} (k_\beta^{n+1})^+ & \text{if } P_\beta^n = (k_\beta^{n+1})^- \\ (k_\beta^{n+1})^- & \text{if } P_\beta^n = (k_\beta^{n+1})^+ \end{cases} = P_\beta^{n+1}$$

which completes the induction. Thus we may conclude that $\bar{f}(P_\alpha^i) = P_\beta^i$ for $i = 0, 1, \dots$. Since f is time orientation preserving this implies $\nu_\beta^+(P_\beta^i) = \nu_\beta^+(\bar{f}(P_\alpha^i)) = \nu_\alpha^+(P_\alpha^i)$ and $\nu_\beta^-(P_\beta^i) = \nu_\beta^-(\bar{f}(P_\alpha^i)) = \nu_\alpha^-(P_\alpha^i)$ for $i = 0, 1, \dots$. (See equations (2.2) and (2.3).) Hence equation (2.1) yields

$$\alpha = \lim_{s \to \infty} \frac{\sum_{i=1}^{n_s} \nu_\alpha^+(P_\alpha^i)}{\sum_{i=1}^{n_s} \nu_\alpha^-(P_\alpha^i)} = \lim_{s \to \infty} \frac{\sum_{i=1}^{n_s} \nu_\beta^+(P_\beta^i)}{\sum_{i=1}^{n_s} \nu_\beta^-(P_\beta^i)} = \beta.$$

Now suppose that f is a conformal homeomorphism from $(R_\alpha, [dxdy])$ to $(R_\beta, [dxdy])$ that does not preserve time orientation. Let r_β be 180 degree rotation about the center of the rectangle R_β. The composition $r_\beta \circ f$ is a time orientation preserving conformal homeomorphism from $(R_\alpha, [dxdy])$ to $(R_\beta, [dxdy])$. Thus we again have $\alpha = \beta$ using the argument above. Although all rectangles in E_1^2 with sides parallel to the x or y coordinate axes are C^∞ conformally equivalent to E_1^2, we have established the following result.

Theorem 1. *The Lorentz surfaces* $(R_\alpha, [dxdy])$ *with* $\alpha > 0$ *are distinct under conformal homeomorphism.*

3. Remarks

In [1], Kulkarni introduces the conformal boundary $\partial\mathcal{L}$ of any simply connected, time oriented Lorentz surface $\mathcal{L} = (S, [dxdy])$. (A detailed description of $\partial\mathcal{L}$ is given in chapter 4 in [4].) For the Lorentz surfaces $\mathcal{L}_\alpha = (R_\alpha, [dxdy])$ in Theorem 1, the conformal boundary $\partial\mathcal{L}_\alpha$ can be identified with the boundary ∂R_α of R_α in E_1^2, and the topology on $\bar{\mathcal{L}}_\alpha = \mathcal{L}_\alpha \cup \partial\mathcal{L}_\alpha$ coincides with the topology induced on $\bar{R}_\alpha = R_\alpha \cup \partial R_\alpha$ by the topology of E_1^2. The *rank* of a conformal boundary point p is the number of null lines of R_α that begin or end at p. For each $\alpha > 0$, $(R_\alpha, [dxdy])$ has four isolated rank one conformal boundary points, and four open arcs of rank two conformal boundary points. Thus $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ for $\alpha \neq \beta$ cannot be distinguished by the ranks of their conformal boundary points. Any simply connected Lorentz surface can be thought of as a twice transversely foliated disc, with the X-lines as the horizontal foliation and the Y-lines as the vertical foliation. (See [2].) It is clear that the leaf spaces are preserved under conformal homeomorphism. For each of the Lorentz surfaces $\mathcal{L}_\alpha$ in Theorem 1 both of the leaf spaces, horizontal and vertical, are homeomorphic to $\mathbb{R}$. The family of Lorentz surfaces $\{\mathcal{L}_\alpha : \alpha > 0\}$ is a counterexample to Theorem 4.9 in [1].

Since dilations and translations on E_1^2 are conformal equivalences, any rectangle in E_1^2 tilted 45 degrees from the horizontal is conformally equivalent to some Lorentz surface $\mathcal{L}_\alpha$ with $\alpha > 0$. Thus the conformal class of such a rectangle is determined by the ratio of the length of the sides of slope 1 to the length of the sides of slope -1. Though all rectangles in $(\mathbb{R}^2, [dx^2 + dy^2])$ are conformally equivalent by the Riemann mapping theorem, it is interesting to note that a conformal map between two rectangles in $(\mathbb{R}^2, [dx^2 + dy^2])$ that extends to the boundary taking corners to corners exists if and only if the rectangles are similar. Finally, for the purpose of comparison with Theorem 2 below, note that each Lorentz surface $\mathcal{L}_\alpha$ is symmetric with respect to two lines through its center.

4. A finiteness theorem

In stark contrast with Riemann surfaces, conformally distinct, simply connected Lorentz surfaces abound. As the examples above indicate, even if we restrict our attention to elementary subsets of E_1^2 it is not difficult to find uncountably many C^0 conformally distinct Lorentz surfaces. (See also [3].) Nonetheless, the next result gives simple properties satisfied (up to conformal homeomorphism) by just a finite number of subsets of E_1^2.

Theorem 2. *Up to conformal homeomorphism there are exactly 21 (open) bounded, convex subsets of* E_1^2 *that are symmetric about some null line.*

Proof. Suppose S is a bounded, open, convex set in E_1^2 which is symmetric about some vertical line. By translation we may suppose S is symmetric about the y-axis. Since translation and dilation in each component are conformal homeomorphisms which preserve convexity and symmetry about the y-axis, we may suppose that S is circumscribed by the square with sides on the lines $x = \pm 1$ and $y = \pm 1$, so that

$$\inf_{p \in S} \pi_1(p) = \inf_{p \in S} \pi_2(p) = -1$$

and

$$\sup_{p\in S}\pi_1(p)=\sup_{p\in S}\pi_2(p)=1.$$

Under the hypotheses on S, $Bot_S \overset{\text{def}}{=} \{y=-1\}\cup\bar{S}$ can be written as $\{(x,y): y = -1, -a \le x \le a\}$ for some a, with $0\le a\le 1$. Similarly, $Top_S \overset{\text{def}}{=} \{y=1\}\cup\bar{S}$ can be written as $\{(x,y) : y = 1, -b \le x \le b\}$ for some b, with $0 \le b \le 1$. If $a = b = 0$, say S has Type 1. If $a = 0$ or $b = 0$, but not both, say S has Type 2. If $a = b \neq 0$, say S has Type 3. Finally, if $0 \neq a \neq b \neq 0$, say S has Type 4. Under the hypotheses on S, $Lft_S \overset{\text{def}}{=} \{x=-1\}\cap\bar{S}$ can be written as $\{(x,y) : x = -1,\ \alpha\le\beta\}$ for some α and β, with $-1\le\alpha\le\beta\le 1$. By symmetry $Rgt_S \overset{\text{def}}{=} \{x=1\}\cap\bar{S} = \{(x,y): x=1, \alpha\le y\le\beta\}$. If $\alpha=\beta$, say S has Type A. If $\alpha\neq\beta$, say S has Type B. (If S has Type 1 and Type A, say S has Type 1A, etc.)

We wish to show that there is exactly one surface of Type 1A up to conformal homeomorphism. Suppose S has Type 1A. Then $-1<\alpha<1$, since otherwise, by convexity of $\bar{S}$, Top_S or Bot_S would be a line segment contradicting the supposition that S has Type 1. For $n=0,1,\ldots$, let $f_n : [-1,0]\to[\alpha,1]$ be the piecewise-linear function whose graph has vertices at the upper endpoints of the portions of the lines $x=-1+(i/2^n)$ in $\bar{S}$ for $i=0,\ldots,2^n$. Similarly, let $\tilde{f}_n : [-1,0]\to[-1,\alpha]$ be the piecewise-linear function whose graph has vertices at the lower endpoints of the portions of the lines $x=-1+(i/2^n)$ in $\bar{S}$ for $i=0,\ldots,2^n$. The increasing (resp. decreasing) sequence of increasing (resp. decreasing) functions f_n (resp. $\tilde{f}_n$) converges pointwise to a strictly increasing (resp. decreasing) concave down (resp. up) function f (resp. $\tilde{f}$). It is easily seen that ∂S equals the union of the graphs of $f, f(-\cdot), \tilde{f}$, and $\tilde{f}(-\cdot)$. The mapping $F: S\to E_1^2$ defined by

$$F(x,y)=\begin{cases}(x, f^{-1}(y)+1) & \text{if } y\ge\alpha,\\ (x,-\tilde{f}^{-1}(y)-1) & \text{if } y<\alpha\end{cases}$$

is a conformal homeomorphism from S to the interior of the 'diamond' with sides on the lines $y=\pm(x+1)$ and $y=\pm(1-x)$. (See Figure 3.)

There are exactly two surfaces of Type 4B up to conformal homeomorphism. Suppose S has Type 4B. By reflection about the origin, if necessary, we may suppose $b>a$. Thus $\alpha>1$ since otherwise we would have $a=1$. First consider the case $b=1$. This immediately implies $\beta=1$. By a construction similar to the one above we can see that there exists a strictly decreasing continuous function $f:[-1,-a]\to[-1,\alpha]$ such that the boundary of S is equal to the union of the graphs of f and $f(-\cdot)$, with Bot_S, Top_S, Rgt_S and Lft_S. Let $r,s:[-1,1]\to[-1,1]$ be continuous, onto and strictly increasing, with $s(\alpha)=0$ and r an odd function with $r(a)=\frac{1}{2}$. The mapping $F: S\to E_1^2$ defined by

$$F(x,y)=\begin{cases}(r(x),s(y)) & \text{if } y\ge\alpha,\\ (r(x),-2r(f^{-1}(y))-2) & \text{if } y<\alpha\end{cases}$$

is a conformal homeomorphism from S to 'homeplate,' the region bounded by the lines $x=\pm1$, $y=\pm1$ and $y=\pm2x-2$ pictured in Figure 3. Now consider the case $b\neq1$. This implies $\beta<1$. Arguing as before we obtain a strictly decreasing, continuous, onto function $f:[-1,-a]\to[-1,\alpha]$ and a strictly increasing, continuous, onto function $g:[-1,-b]\to[\beta,1]$ such that ∂S is equal to the union of the graphs of f, $f(-\cdot)$, g and $g(-\cdot)$ with Bot_S, Top_S, Rgt_S and Lft_S. Define r and s

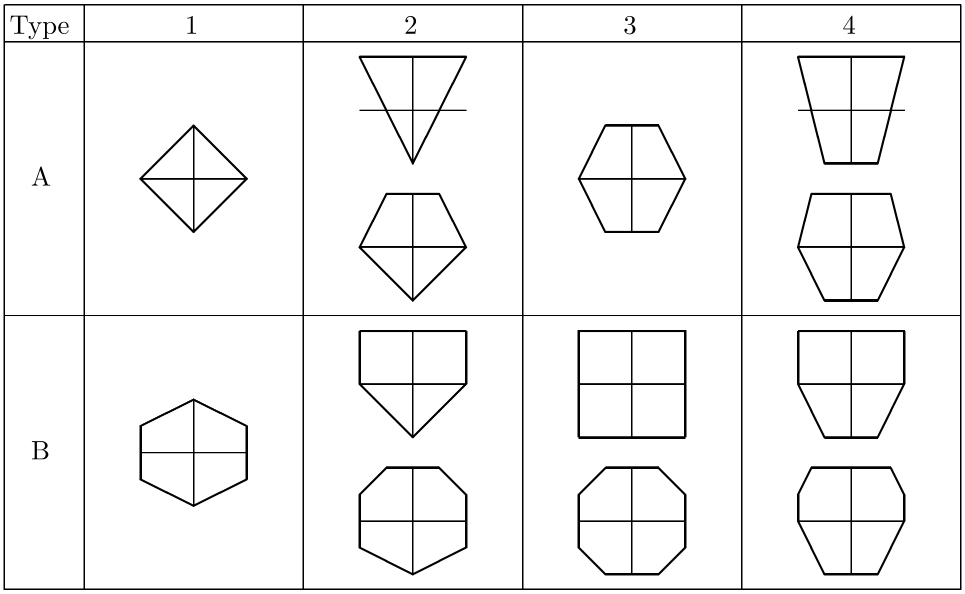

FIGURE 3

as above, but with the additional requirements that $s(\beta) = \frac{1}{2}$ and $r(b) = \frac{3}{4}$. The mapping $F : S \to E_1^2$ defined by

$$F(x, y) = \begin{cases} (r(x), 2r(g^{-1}(y)) + \frac{5}{2}) & \text{if } y \geq \beta, \\ (r(x), s(y)) & \text{if } \alpha \leq y < \beta, \\ (r(x), -2r(f^{-1}(y)) - 2) & \text{if } y < \alpha \end{cases}$$

is a conformal homeomorphism from S to the region bounded by the lines $x = \pm 1$, $y = \pm 1$, $y = \pm 2x + \frac{5}{2}$ and $y = \pm 2x - 2$ pictured in Figure 3.

Using the methods above, one can show that there is exactly one surface each of Type 1B and Type 3A, and exactly two surfaces each of Type 2A, Type 2B, Type 3B, and Type 4A. The rank of conformal boundary points C^0 conformally distinguishs any two surfaces of different Type. Up to conformal homeomorphism there are therefore exactly 13 bounded convex sets in E_1^2 that are symmetric about some vertical line. Similarly, there are exactly 13 bounded, convex sets in E_1^2 that are symmetric about some horizontal line. Since 5 surfaces (those of Types 1 and 3) belong to both lists, there are exactly 21 C^0 conformally distinct, bounded, convex sets in E_1^2 that are symmetric about some null line. $\square$

The Lorentz surfaces $\mathcal{L}_\alpha$ above show the necessity of the symmetry hypothesis in Theorem 2 above. A small alteration of the examples on p.21 in [4] suffices to show the necessity of the convexity hypothesis. In particular, one can construct infinitely many C^0 conformally distinct Lorentz surfaces in E_1^2 that are symmetric about both the x-axis and the y-axis, and have boundaries composed of horizontal and vertical segments only. Theorem 2 also fails dramatically if 'conformal homeomorphism' is replaced with 'conformal diffeomorphism'. (See the proof of Theorem 1 in [3].) The boundedness hypothesis is not necessary for a finiteness theorem of the type above. In fact it is not very difficult to show that any unbounded (open) convex subset of

E_1^2 that is symmetric about some null line must be conformally homeomorphic to the first pictured surface of Type 2A, 2B, 3B, 4A or 4B in Figure 3.

I thank Tilla Weinstein for extensive expository advice.

References

1. R. Kulkarni, *An analogue of the Riemann mapping theorem for Lorentz metrics*, Proc. R. Soc. Lond. A **401** (1985), 117-130. MR **87e:**53108
2. F. Luo and R. Stong, *An analogue of the Riemann mapping theorem for Lorentz metrics: Topological Embedding of a Twice Foliated Disc into the Plane*, preprint.
3. R. Smyth and T. Weinstein, *Conformally Homeomorphic Lorentz Surfaces Need Not Be Conformally Diffeomorphic*, Proc. Amer. Math. Soc. **123** (1995), 3499–3506. CMP 95:16
4. T. Weinstein, *An Introduction to Lorentz Surfaces*, DeGruyter Expositions in Math. (to appear).

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

E-mail address: `rsmyth@math.rutgers.edu`

Current address: Department of Mathematics, Georgian Court College, Lakewood, New Jersey 08701

E-mail address: `rsmyth@georgian.edu`