PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 5, May 1996

THE SECOND ITERATE OF A MAP WITH DENSE ORBIT

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(Communicated by James E. West)

Abstract. Suppose that X is a Hausdorff topological space having no isolated points and that $f : X \to X$ is continuous. We show that if the orbit of a point $x \in X$ under f is dense in X while the orbit of x under $f \circ f$ is not, then the space X decomposes into three sets relative to which the dynamics of f are easy to describe. This decomposition has the following consequence: suppose that x has dense orbit under f and that the closure of the set of points of X having odd period under f has nonempty interior; then x has dense orbit under $f \circ f$.

1. Introduction

Suppose that T is a linear function on a complex Banach space B and that the orbit of $b \in B$ under T is dense in B; then for each positive integer n, the orbit of b under T^n is also dense. This remarkable result was recently obtained by S. I. Ansari ([1, Theorem 1]). Her proof makes use of the fact that once T has a single vector b with dense orbit, it has a dense *connected* set of vectors having dense orbit.

Inspired by Ansari's work, we move in this paper to a topological-space setting in which a continuous map f acts on a Hausdorff space X having no isolated points (no linearity assumptions on space or map). We examine what must happen if a point x has dense orbit under f while failing to have dense orbit under $f \circ f$. Our main result, the Separation Theorem, yields Ansari's theorem in the $n = 2$ case; more important, it leads to a decomposition of X into three sets relative to which the dynamics of f are easy to describe. This decomposition, in turn, yields the following: suppose that f acts chaotically on X and that the closure of the set of points of X having odd period under f has nonempty interior; then $f \circ f$ acts chaotically on X.

2. The Separation Theorem

In this section and the one that follows, X denotes a Hausdorff topological space having no isolated points, and f denotes a continuous map from X to X. We use $f^{[n]}$ to represent f composed with itself n times:

$$f^{[n]} = f \circ f \circ f \circ \cdots \circ f, \ n \text{ times}$$

(take $f^{[0]}$ to be the identity map). For a subset S of X,

$$f^{-[n]}(S) := \{w \in X : f^{[n]}(w) \in S\}.$$

Received by the editors June 1, 1994.
1991 *Mathematics Subject Classification.* Primary 54H20; Secondary 47A15, 58F13.

For $x \in X$, the *orbit* of x under f, denoted $\mathrm{Orb}(f, x)$, is given by

$$\mathrm{Orb}(f, x) = \{f^{[n]}(x) : n = 0, 1, 2, \dots\}.$$

Let $\mathcal{D}$ be the set of elements of X that have dense orbit under f. We show that if $\mathrm{Orb}(f, x)$ is dense in X while $\mathrm{Orb}(f^{[2]}, x)$ is not, then there exists a separation of $\mathcal{D}$. Before proving this separation theorem, we present two simple lemmas.

Lemma 2.1. *Suppose that $\mathcal{D}$ is not empty. Then $\mathcal{D}$ is a dense subset of X fully invariant under f:*

$$f(\mathcal{D}) \subset \mathcal{D} \quad \textit{and} \quad f^{-[1]}(\mathcal{D}) \subset \mathcal{D}.$$

Proof. Because X is a Hausdorff space having no isolated points, removing a finite set from a dense set leaves a dense set. Let x be in $\mathcal{D}$. Then $\mathrm{Orb}(f, f(x))$ is dense in X since it is the dense set $\mathrm{Orb}(f, x)$ minus the singleton set $\{x\}$. Hence $f(x) \in \mathcal{D}$, and $\mathcal{D}$ is invariant under f. Observe that as a consequence of this invariance, given x is in $\mathcal{D}$, $\mathrm{Orb}(f, x)$ must be contained in $\mathcal{D}$; however, $\mathrm{Orb}(f, x)$ is dense in X and thus $\mathcal{D}$ is dense in X as well.

That $f^{-[1]}(\mathcal{D}) \subset \mathcal{D}$ is clear because the orbit of a given $y \in f^{-[1]}(\mathcal{D})$ contains $\mathrm{Orb}(f, f(y))$, which is dense in X. ☐

Lemma 2.2. *Suppose that $x \in X$, that $h : X \to X$ is continuous, and that G is the complement of the closure of* $\mathrm{Orb}(h, x)$. *Then for every nonnegative integer k, $h^{-[k]}(G) \subset G$.*

Proof. Let k be a nonnegative integer and suppose that $h^{-[k]}(G)$ intersects the closure of $\mathrm{Orb}(h, x)$. Then because $h^{-[k]}(G)$ is open, it must intersect $\mathrm{Orb}(h, x)$; that is, there is a nonnegative integer m such that $h^{[m]}(x)$ belongs to $h^{-[k]}(G)$. We conclude that $h^{[k+m]}(x)$ is in G, a contradiction. Thus $h^{-[k]}(G)$ does not intersect the closure of $\mathrm{Orb}(h, x)$, or $h^{-[k]}(G) \subset G$. ☐

Theorem 2.3 (Separation Theorem)**.** *Suppose that $x \in X$ is such that* $\mathrm{Orb}(f, x)$ *is dense in X. The following are equivalent:*

(a) $\mathrm{Orb}(f^{[2]}, x)$ *is not dense in X;*

(b) *There exists a separation $\mathcal{D}_1$, $\mathcal{D}_2$ of $\mathcal{D}$ such that each of the sets $\mathcal{D}_1$ and $\mathcal{D}_2$ is invariant under $f^{[2]}$.*

Proof. That (b) implies (a) is easy. Because x is in $\mathcal{D}$, it is either in $\mathcal{D}_1$ or in $\mathcal{D}_2$. By the invariance of these sets under $f^{[2]}$, $\mathrm{Orb}(f^{[2]}, x)$ is contained entirely in $\mathcal{D}_1$ or entirely in $\mathcal{D}_2$ and thus can't be dense in X.

Now suppose that $\mathrm{Orb}(f^{[2]}, x)$ is not dense in X. Let G be the complement of the closure of $\mathrm{Orb}(f^{[2]}, x)$ so that G is nonempty and open. By Lemma 2.2 (with $h = f^{[2]}$), for each nonnegative integer k, $f^{-[2k]}(G)$ is contained in G.

We claim that $f^{-[1]}(G)$ must be contained in the complement of G. To see this, suppose that $f^{-[1]}(G)$ intersects G. Because $G \cap f^{-[1]}(G)$ is open and $\mathrm{Orb}(f, x)$ is dense, there is a nonnegative integer j such that $f^{[j]}(x)$ belongs to $G \cap f^{-[1]}(G)$. Observe that j cannot be even ($f^{[j]}(x) \in G$) and that j cannot be odd ($f^{[j]}(x) \in f^{-[1]}(G)$ implies $f^{[j+1]}(x) \in G$ and $j + 1$ is even), a contradiction. Thus $f^{-[1]}(G)$ must be contained in the complement of G.

Let $S_1 = G$ and $S_2 = f^{-[1]}(G)$. Note S_1 and S_2 are open (f is continuous) and disjoint. Let w be in $\mathcal{D}$. Because G is open and w has dense orbit under f, there is a nonnegative integer m such that $f^{[m]}(w) \in G$. Thus w belongs to

$f^{-[m]}(G)$ and is either in S_1 (if m is even) or S_2 (if $m = 2k+1$ is odd, because then $f(w) \in f^{-[2k]}(G) \subset G$); we have $\mathcal{D} \subset S_1 \cup S_2$. Because $\mathcal{D}$ is dense, both $S_1 \cap \mathcal{D}$ and $S_2 \cap \mathcal{D}$ are nonempty. Thus the pair $\mathcal{D}_1 := S_1 \cap \mathcal{D}$ and $\mathcal{D}_2 := S_2 \cap \mathcal{D}$ is a separation of $\mathcal{D}$.

We complete the proof by showing $f(\mathcal{D}_2) \subset \mathcal{D}_1$ and $f(\mathcal{D}_1) \subset \mathcal{D}_2$. Suppose that $t \in \mathcal{D}_2$. Then because $t \in S_2$, $f(t)$ belongs to $G = S_1$. Because $f(t)$ also belongs to $\mathcal{D}$ (Lemma 2.1), $f(t) \in S_1 \cap \mathcal{D} = \mathcal{D}_1$. Now, let $v \in \mathcal{D}_1$ so that, in particular, v is in $G = S_1$. Suppose $f(v)$ also belongs to G; then $G \cap f^{-[1]}(G)$ is not empty (containing, e.g., v), but this contradicts $S_1 \cap S_2 = \emptyset$. Thus $f(v)$ is contained in the complement of $S_1 = G$. In particular, $f(v)$ cannot be in $\mathcal{D}_1$; however, $f(v)$ is in $\mathcal{D}$ so that $f(v)$ must belong to $\mathcal{D}_2$. □

In the following section, we present some applications of the Separation Theorem.

3. The Decomposition Theorem

The following theorem describes the behavior of orbits of f relative to the open sets S_1 and S_2 that yielded the separation obtained in Theorem 2.3. The referee has pointed out to the author that for inverval maps this theorem is known.

Theorem 3.1 (Decomposition Theorem)**.** *Suppose that* $x \in X$ *is such that* $\mathrm{Orb}(f, x)$ *is dense in* X*, while* $\mathrm{Orb}(f^{[2]}, x)$ *is not dense. Let* S_1 *be the complement of the closure of* $\mathrm{Orb}(f^{[2]}, x)$*, and let* $S_2 = f^{-[1]}(S_1)$*. Then*

(a) $S_1 \cap S_2 = \emptyset$ *and* $S_1 \cup S_2$ *is dense in* X*;*
(b) $f(S_2) \subset S_1$*;*
(c) $f(S_1) \subset X \backslash S_1$*;*
(d) $X \backslash (S_1 \cup S_2)$ *is invariant under* f*.*

Proof. Part (b) follows immediately from the definition of S_2. Parts (a) and (c) are corollaries of the proof of the Separation Theorem. The second paragraph of that proof shows that S_2 is contained in the complement of S_1, while the fourth shows $S_1 \cup S_2$ contains the dense set $\mathcal{D}$ and hence is dense itself. The last paragraph of the proof shows that (c) holds (the hypothesis that v belong to $\mathcal{D}$ wasn't used to show that $f(v)$ belongs the closure of $\mathrm{Orb}(f^{[2]}, x)$).

Now we establish part (d). Let $w \in X \backslash (S_1 \cup S_2)$. If $f(w)$ were in S_1, then w would belong to $f^{-[1]}(S_1) = S_2$. If $f(w)$ were in S_2, then $f^{[2]}(w)$ would belong to S_1, and thus by Lemma 2.2, w would belong to S_1. Hence, $f(w)$ must belong to $X \backslash (S_1 \cup S_2)$. □

Given that the hypotheses of the preceding theorem hold, we see that an orbit that begins in S_1 either steps to S_2 and returns, or steps into $X \backslash (S_1 \cup S_2)$ and remains there. An orbit that starts in S_2 steps into S_1 and either returns or steps into the invariant set $X \backslash (S_1 \cup S_2)$. Observe that neither S_1 nor S_2 may contain a point having odd period under f. Also observe that the closed set $X \backslash (S_1 \cup S_2)$ has no interior ($S_1 \cup S_2$ is dense).

Corollary 3.2. *Suppose that* $x \in X$ *is such that* $\mathrm{Orb}(f, x)$ *is dense in* X *and that the closure of the set of points of* X *having odd period under* f *has nonempty interior. Then* $\mathrm{Orb}(f^{[2]}, x)$ *is dense in* X*.*

Proof. If $\mathrm{Orb}(f^{[2]}, x)$ were not dense in X, then the set of points having odd period under f would be confined to the set $X \backslash (S_1 \cup S_2)$, a closed set with no interior. □

We say that f acts chaotically on X provided that f has an orbit dense in X and that the set of periodic points of f is dense in X (by [2] this definition is equivalent to that given by Devaney in [4]).

Corollary 3.3. *Suppose that f acts chaotically on X and that the closure of the set of points of X having odd period under f has nonempty interior. Then $f \circ f$ acts chaotically on X.*

Proof. Since any point periodic for f is periodic for $f \circ f$, the set of periodic points of $f \circ f$ is dense. Because f acts chaotically on X, there exists $x \in X$ such that $\mathrm{Orb}(f, x)$ is dense in X. By Corollary 3.2, $\mathrm{Orb}(f^{[2]}, x)$ is also dense in X so that $f \circ f$ has a dense orbit. Thus we see $f \circ f$ is chaotic, having the same sets of periodic points and points with dense orbit as does f. ☐

Our final result is the one that inspired this paper: Ansari's Theorem in the $n = 2$ case.

Theorem 3.4. *Suppose that B is a complex Banach space and $T : B \to B$ is bounded and linear. If for some $b \in B$, $\mathrm{Orb}(T, b)$ is dense in B, then $\mathrm{Orb}(T^2, b)$ is also dense in B.*

Proof. If $\mathrm{Orb}(T, b)$ is dense in B, then the set $E = \{p(T)b : p \text{ is a polynomial}\}\backslash\{0\}$ is a dense set of vectors in B, each element of which has dense orbit ([3]). Because E is *connected* and dense, the set $\mathcal{D}$ of vectors in B having dense orbit under T cannot be separated. Thus by the Separation Theorem, $\mathrm{Orb}(T^2, b)$ must be dense in B. ☐

That linear maps may have dense orbits was first observed by Rolewicz [6]. That linear maps may satisfy, say, Devaney's definition of chaos was first observed by Shapiro and Godefroy [5].

The following example illustrates the Decomposition Theorem.

Example. Let C_1 be the unit circle in the complex plane $\mathbf{C}$, and let L: $\mathbf{C} \to \mathbf{C}$ be given by $L(z) = 2z - 1$. Let $C_2 = L(C_1)$ so that C_2 is the circle of radius 2 with center -1 and that C_1 is internally tangent to C_2 at 1. Set $X = C_1 \cup C_2$ and topologize X using the euclidean metric. Define $g : X \to X$ by

$$g(z) = \begin{cases} L(z^2) & \text{if } z \in C_1, \\ \left(L^{-1}(z)\right)^2 & \text{if } z \in C_2. \end{cases}$$

Verifying that g is transitive is easy; thus, by the Baire Category Theorem, there is a point $x \in X$ having dense orbit under g. However, $g^{[2]}$ leaves both C_1 and C_2 invariant and thus has no dense orbits; in particular, $\mathrm{Orb}(g^{[2]}, x)$ is not dense in X. If x is, for instance, in C_1, then the reader may verify that $S_1 = C_2\backslash\{1\}$ and that $S_2 = C_1\backslash\{-1, 1\}$. Observe that $X\backslash(S_1 \cup S_2) = \{-1, 1\}$ is indeed invariant under g. We remark that the map g is chaotic (any $(4^n - 1)$-th root of unity is periodic for g and the image under L of any such root of unity is also periodic); hence, g is an example of a chaotic map whose second iterate is not chaotic.

We conclude this paper with the following natural question:

> *For $n > 2$, what kind of decomposition of X may be obtained if one assumes that f is (topologically) transitive on X while $f^{[n]}$ is not?*

Acknowledgment

I wish to thank Joel Shapiro for bringing Ansari's work ([1]) to my attention and for his valuable criticism of a preliminary version of this paper. I would also like to thank the referee for providing several helpful suggestions.

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