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FAILURE OF THE DENJOY THEOREM FOR QUASIREGULAR MAPS IN DIMENSION $n \geq 3$

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Abstract. In 1929 L. V. Ahlfors proved the Denjoy conjecture which states that the order of an entire holomorphic function of the plane must be at least k if the map has at least $2k$ finite asymptotic values. In this paper, we prove that the Denjoy theorem has no counterpart in the classical form for quasiregular maps in dimensions $n \geq 3$. We construct a quasiregular map of $\mathbb{R}^n$, $n \geq 3$, with a bounded order but with infinitely many asymptotic limits. Our method also gives a new construction for a counterexample of Lindelöf's theorem for quasiregular maps of B^n, $n \geq 3$.

1. Introduction

A continuous map $f : G \to \mathbb{R}^n$ of a domain G in $\mathbb{R}^n$, $n \geq 2$, is called *quasiregular* (qr) if

$$f \in W^1_{n,\mathrm{loc}}(G)\,, \tag{1.1}$$

and there exists K, $1 \leq K < \infty$, such that

$$|f'(x)|^n \leq K J_f(x) \quad \text{a.e.} \tag{1.2}$$

The condition (1.1) means that for all $D \Subset G$ the coordinate functions of f belong to the Sobolev space $W^1_n(D)$ of functions in $L^n(D)$ whose distributional first order partial derivatives are also L^n integrable in D. In the above definition $f'(x)$ is the formal derivative of f at x defined by means of the partial derivatives, $|f'(x)|$ is the operator norm of $f'(x)$, and $J_f(x)$ is the Jacobian determinant of f at x. In this article we call a qr map K–*quasiregular* if (1.2) is satisfied. The definition extends immediately to the case $f : M \to N$ where M and N are connected oriented Riemannian n–manifolds. A quasiregular homeomorphism is by definition *quasiconformal*. For properties of qr maps we refer to books [4] by Yu. G. Reshetnyak, [8] by M. Vuorinen, and [6] by the second author.

Many geometric properties of analytic functions of one complex variable have their counterparts in the theory of quasiregular maps in the Euclidean space. For

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example, Picard type theorems on omitted values are known as well as value distribution in the spirit of Ahlfors' theory of covering surfaces; see [6].

In 1907 A. Denjoy conjectured that if an entire complex analytic function has at least $2k$ finite asymptotic values, then the order must be at least k. In 1921 T. Carleman proved a weaker form where $2k$ is replaced by $5k$. Finally, in 1929 L. V. Ahlfors [1] settled the sharp result. The sharp result for qr maps in the plane was proved by J. Jenkins [3].

It has been an open problem for some time to determine the situation of the relationship between asymptotic values and order for quasiregular maps in space. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonconstant quasiregular map, we define the *order* μ_f and *lower order* λ_f of f by

$$\mu_f = \limsup_{r\to\infty}(n-1)\frac{\log\log M(r)}{\log r}\ ,$$
$$\lambda_f = \liminf_{r\to\infty}(n-1)\frac{\log\log M(r)}{\log r}\ ,$$

where

$$M(r) = \sup_{|x|=r} |f(x)|.$$

It was proved in [7] that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonconstant K–quasiregular map with at least one asymptotic value in $\mathbb{R}^n$, then the lower order satisfies

$$\lambda_f \geq c(n, K) > 0.$$

This result follows also from arguments in [2].

The purpose of this paper is to show that for dimensions $n > 2$ there is no lower bound for the order that tends to ∞ as the number of asymptotic values grows to ∞. More precisely, we will prove the following theorem.

1.3. **Theorem.** *For each $n > 2$ there is a quasiregular map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mu_f \leq 1$ and f has infinitely many asymptotic values.*

The method also gives a new proof of the counterexample to Lindelöf's theorem presented originally in [5]. This is described in Section 3.

2. Proof of Theorem 1.3

We shall give the proof for dimension $n = 3$. The method extends to higher dimensions in a straightforward manner. The construction of the map f in Theorem 1.3 is based on a modification of the Zorich map. We refer to [6, p. 15] for the usual Zorich map. To start the construction of f, we fix a triangulation of $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ by translating the closed triangles $A_i,\ i = 1, \dots, 4$, in Figure 1 by $x \mapsto x + (2p, 2q),\ p, q \in \mathbb{Z}$. We call the set of 2-simplexes M^2. Let $a_i,\ i = 1, 2, \dots$, be distinct points of $B^3(1/2)\cap\mathbb{R}^2$. We choose a sequence $D_1, D_2, \dots$ of sectors in $\mathbb{R}^2$ such that their mutual distances satisfy $d(D_i, D_j) \geq 8$. Each a_i will be an asymptotic value of f along the axis of D_i.

Let $z : \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$ be a Zorich type map such that each cylinder $(\operatorname{int} A) \times \mathbb{R}^1,\ A \in M^2$, is mapped onto either $\mathbb{H}_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$ or $\mathbb{H}_- = \{x \in \mathbb{R}^3 :$

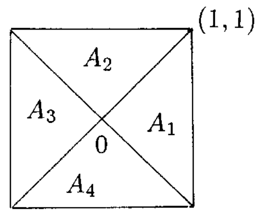

FIGURE 1

$x_3 < 0\}$ and that $|z(x)| = e^{x_3}$. More precisely, each (closed) triangle $A^c = \{x \in \mathbb{R}^3 : (x_1, x_2) \in A,\ x_3 = c\},\ A \in M^2$, is mapped onto either the hemisphere $S^2_+(e^c)$ or $S^2_-(e^c)$. If $zA^c = S^2_+(e^c)$, a triangle next to A^c in the plane $x_3 = c$ is mapped onto $S^2_-(e^c)$.

Next we define a function $h : \mathbb{R}^2 \to \mathbb{R}^1$ by

$$h(x) = \begin{cases} 0\,, & \text{if } x \in \mathbb{R}^2 \setminus \bigcup_i D_i\,; \\ -\frac{1}{2} d(x, \partial D_i)\,, & \text{if } x \in D_i\,, \end{cases}$$

where $d(x, \partial D_i)$ is the distance of x to the boundary of D_i. Let $G \subset \mathbb{R}^3$ be the domain $\{x \in \mathbb{R}^3 : x_3 > h(x_1, x_2)\}$. We define a map $g : \bar{G} \to \mathbb{R}^3$ as follows. In $\bar{\mathbb{H}}_+$, we set $g = z$. We want that $g(x) \to a_i$ as $x_3 \to -\infty$ in the component of $G \setminus \mathbb{H}_+$ corresponding to D_i. Call this component G_i. Let $T_i : \mathbb{R}^3 \to \mathbb{R}^3$ be the bilipschitz map with the properties that $T_i = \mathrm{id}$ in $\mathbb{R}^3 \setminus B^3$, $T_i(0) = a_i$, and T_i maps each line segment $[0, x]$, where $x \in S^2$, linearly onto the line segment $[a_i, x]$. We set $g = T_i \circ z$ in G_i. In fact, $g = T_i \circ z$ in $\bar{G}$ for each i since $zG_i = B^3 \setminus \{0\}$ and the G_i are disjoint. Furthermore, g maps the boundary $\partial A \times \mathbb{R}^1$ of each cylinder into the plane since z does so and $T_i \mathbb{R}^2 = \mathbb{R}^2$.

Let $F_1 : \mathbb{R}^3 \to \mathbb{R}^3$ be a map $F_1(x) = \big(x_1, x_2, x_3 - h(x_1, x_2)\big)$. Clearly F_1 is bilipschitz and maps G onto $\mathbb{H}_+$. Let F_2 be a map of the cylinder $C_0 = (A_1 \cup \cdots \cup A_4) \times [0, \infty[$ obtained by lifting the ray $\{0\} \times [0, \infty[$ by $x \mapsto x + (0, 0, 1)$ and extending to each $A_i \times [0, \infty[$ linearly such that $F_2 = \mathrm{id}$ on $\partial(A_1 \cup \cdots \cup A_4) \times \mathbb{R}^1$. Thus $F_2(x) = (x_1, x_2, x_3 + 1 - \max\{|x_1|, |x_2|\})$. We extend F_2 to $\bar{\mathbb{H}}_+$ by conjugating with $x \mapsto x + (2p, 2q, 0)$, $p, q \in \mathbb{Z}$, and call the extended map F_2, too.

We set $C_{p,q} = C_0 + (2p, 2q, 0)$. Then $f_1 = g \circ F_1^{-1} \circ F_2^{-1} \mid F_2 \bar{\mathbb{H}}_+$ maps each boundary part $(\partial F_2 \mathbb{H}_+) \cap F_2 C_{p,q}$ either (a) 2 to 1 onto the unit sphere S^2 if $C_{p,q} \cap (\bigcup D_i) = \emptyset$, or (b) onto a union of 4 topological half spheres if $C_{p,q} \cap (\bigcup D_i) \neq \emptyset$. In the case (b), the diameter of the image is approximately $\exp\big(-d(C_{p,q}, \partial(\bigcup D_i))\big)$.

Next we extend f_1 to $\bar{\mathbb{H}}_+$. Let $A_i^{p,q} = A_i + (2p, 2q)$, where A_i is as in Figure 1, and let $U_i^{p,q}$ be the part of $\mathbb{H}_+ \setminus F_2 \mathbb{H}_+$ whose vertical projection on $\mathbb{R}^2$ is $A_i^{p,q}$. Thus $U_i^{p,q}$ is a tedrahedron whose base is $A_i^{p,q}$ (Figure 2).

Consider first the case (a). Then f_1 maps the 2-simplex abc onto a half-sphere of S^2, say S^2_+, such that the boundary of abc is mapped onto the unit circle $S^1 \subset \mathbb{R}^2$. We extend f_1 to $\bar{U}_i^{p,q}$ by mapping $\bar{U}_i^{p,q}$ onto $\bar{B}^3_+$ such that the other faces of $U_i^{p,q}$ are mapped into $\mathbb{R}^2$ and that $d \mapsto 0$. A tedrahedron next to $U_i^{p,q}$ is mapped onto $\bar{B}^3_-$ in a similar manner.

In the case (b) we do a natural modification of this. Now f_1 maps the 2-simplex abc onto a topological half sphere, call this S, and f_1 is extended to $\bar{U}_i^{p,q}$ as above

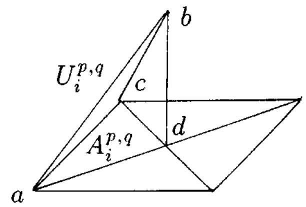

FIGURE 2

by mapping $\bar{U}_i^{p,q}$ onto the closed topological ball bounded by S and a part of $\mathbb{R}^2$ such that $d \mapsto a_i$.

We obtain an extended map $f_1 : \bar{\mathbb{H}}_+ \to \mathbb{R}^3$ with the properties that $f_1\mathbb{R}^2 = \mathbb{R}^2$ and that $f_1(x) \to a_i$ as $x \to \infty$ along the axis of the sector D_i.

Finally, we define $f : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$f(x) = \begin{cases} f_1(x)\,, & \text{if } x \in \bar{\mathbb{H}}_+\,; \\ \bar{f}_1(\bar{x})\,, & \text{if } x \in \mathbb{H}_-\,, \end{cases}$$

where $\bar{x} = (x_1, x_2, -x_3)$ is the reflection of $x = (x_1, x_2, x_3)$ in $\mathbb{R}^2$. Then f has the desired properties of Theorem 1.3.

3. A counterexample to Lindelöf's theorem

In this section we shall show that the method from Section 2 also gives a new counterexample to Lindelöf's theorem for quasiregular maps of B^n for $n \geq 3$. Such an example was originally given in [5] (see also [6, p. 193]) and the statement is formulated as follows.

3.1. **Theorem.** *For each $n \geq 3$ there exist a bounded qr mapping f of B^n and a point $b \in \partial B^n$ such that f has infinitely many asymptotic values at b and no angular limit at b.*

Proof. Also now the proof is given for $n = 3$ and it extends to other dimensions easily. We again identify $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ and set

$$\begin{aligned} X &= [0,1]\times]0,1] \subset \mathbb{R}^2, \\ A^* &= \{x \in \mathbb{R}^3 : (x_1, x_2) \in A, -x_2 \leq x_3 \leq x_2\} \text{ if } A \subset X. \end{aligned}$$

We shall construct a quasiregular map of $V = \operatorname{int} X^*$ with infinitely many asymptotic values at $0 \in \partial X^*$. Since V is bilipschitz equivalent to B^n, the first statement of Theorem 3.1 follows. We start out with a dyadic Whitney type subdivision of X into squares shown in Figure 3. For $k = 1, 2, ...$ set

$$\begin{aligned} Z_0 &= \{x \in X : \left(\frac{x_1}{4}\right)^2 \leq x_2\}\,, \\ X_k &= \{x \in X : \frac{1}{2}\left(\frac{x_1}{4}\right)^{4k-2} \leq x_2 \leq \left(\frac{x_1}{4}\right)^{4k-2}\}\,, \\ Y_k &= \{x \in X : \left(\frac{x_1}{4}\right)^{4k} \leq x_2 \leq \frac{1}{2}\left(\frac{x_1}{4}\right)^{4k-2}\}\,, \\ Z_k &= \{x \in X : \left(\frac{x_1}{4}\right)^{4k+2} \leq x_2 \leq \left(\frac{x_1}{4}\right)^{4k}\}\,. \end{aligned}$$

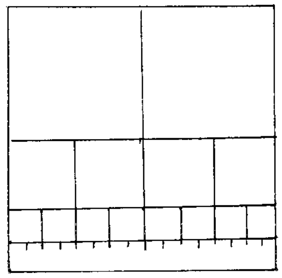

FIGURE 3

We modify the subdivision of Figure 3 as follows. For $k \geq 1$ let $\psi_k : X \to X$ be the map of the form $\psi_k(x_1, x_2) = (x_1, a_k(x_1)x_2 + b_k(x_1))$, where a_k and b_k are functions such that ψ_k keeps $\partial X_k \cap \partial Z_{k-1}$ fixed and moves $\partial Y_k \cap \partial Z_k$ onto $\partial X_k \cap \partial Y_k$. For $k \geq 0$ let Z_k' be the union of the (closed) squares of Figure 3 that are contained in Z_k. In our modification the sets Z_k' are kept unchanged. Let $W_k \subset X$ be the closed part between Z_{k-1} and Z_k'. Write $\psi_k(W_k) = \tilde{W}_k$. This means that the squares touching $X_k \cup Y_k$ are squeezed upwards. Then we fill the part $W_k \setminus \tilde{W}_k$ as follows. Each square Q in Z_k' touching W_k defines a column $\hat{Q} = \{x \in W_k \setminus \tilde{W}_k : x_1 \in \pi_1 Q\}$, where $\pi_1(x_1, x_2) = x_1$. Let the side length of Q be a. We fill the column $\hat{Q}$ by squares congruent to Q plus a leftover part nearest to $\tilde{W}_k$ whose smaller vertical side length s is required to satisfy $s \in [a, 2a[$.

Next we triangulate this new subdivision of X by joining vertices by line segments and without adding any new vertices. Call the simplicial 2–complex P' with space X. Finally we perform the geometric barycentric subdivision of P' and obtain a 2–complex P. The complex P has the property that each vertex in $\operatorname{int} X$ belongs to an even number of 2–simplices; we let P^2 be the 2–simplices of P.

To construct the required map f of V we apply the method from Section 2 as follows. Each group of four triangles A_i, $i = 1, \ldots, 4$ (Figure 1), with their translations by $x \mapsto x + (2p, 2q)$, $p, q \in \mathbb{Z}$, corresponds now to one 2–simplex in P'. The center in Figure 1 and its translations correspond to the barycenters of the triangulation P'. We start with a Zorich–type map $\tilde{z}$ restricted to $X^* \cap \bar{\mathbb{H}}_+$. The cylinders are $A^* \cap \bar{\mathbb{H}}_+$, $A \in P^2$, and we fix $\tilde{z}$ so that $\tilde{z}$ maps the boundary part $\{x \in X^* : x_3 = x_2\}$ onto $S^2(2)$ and each A onto a topological half sphere whose diameter is approximately $\exp\bigl(-d(A^* \cap \mathbb{H}_+)/d(A)\bigr)$, where $d(E)$ is the diameter of a set E. In addition we require that $|\tilde{z}| \leq 1$ in X and $|\tilde{z}| = 1$ in Z_k, $k \geq 0$. Our map $\tilde{z}$ corresponds to $z \circ F_1^{-1}$ in Section 2. Then we perform shifting $\tilde{F}_2$ in the x_3 direction similar to F_2 by using the barycenters for points of maximal shifts. Define $\tilde{g}$ by $\tilde{g} = T_i \circ \tilde{z}$ in $(X_i^* \cup Y_i^*) \cap \bar{\mathbb{H}}_+$, where T_i is as in Section 2. We extend $\tilde{f}_1 = \tilde{g} \circ \tilde{F}_2^{-1} \mid \tilde{F}_2(X^* \cap \mathbb{H}_+)$ similarly as before. Let A' be a 2–simplex in P'. We divide the treatment again into two cases: (a) if A' does not meet any $X_i \cup Y_i$, $i \geq 1$, we map the barycenter of A' to 0; (b) if $A' \cap (X_i \cup Y_i) \neq \emptyset$, the barycenter is mapped to a_i. We get a map $\tilde{f}_1$ of $X^* \cap \mathbb{H}_+$, which we extend by reflection to $\tilde{f} : X^* \to \bar{B}^3(2)$. The restriction $\tilde{f} \mid V$ is quasiregular and has the asymptotic limit a_i along the path $t \mapsto (t, \frac{1}{2}(\frac{t}{4})^{4i-2}, 0) \in V$ as $t \to 0$. With a bilipschitz map $F : \bar{V} \to \bar{B}^n$ we also see that $f = \tilde{f} \circ F^{-1} \mid B^n$ has no angular limit at $F(0)$. Theorem 3.1 is proved.

Addendum

After this article was completed we received a manuscript "On a method of Holopainen and Rickman" by D. Drasin. He constructs an entire quasiregular map on $\mathbb{R}^n$, $n \geq 3$, of order $n-1$ with every $a \in \mathbb{R}^n$ asymptotic.

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