ON CLOSE TO LINEAR COCYCLES

H. B. KEYNES, N. G. MARKLEY, AND M. SEARS

(Communicated by Linda Keen)

Abstract. If we have a flow \((X, \mathbb{Z}^m)\) and a cocycle \(h\) on this flow, \(h : X \times \mathbb{Z}^m \to \mathbb{R}^m\), then \(h\) is called close to linear if \(h\) can be written as the direct sum of a linear (constant) cocycle and a cocycle in the closure of the coboundaries. Many of the desirable consequences of linearity hold for such cocycles and, in fact, a close to linear cocycle is cohomologous to a cocycle which is norm close to linear. Furthermore in the uniquely ergodic case all cocycles are close to linear. We also establish that a close to linear cocycle which is covering is cohomologous to one with the special property that it can be extended by piecewise linearity to an invertible cocycle from \(X \times \mathbb{R}^m\) to itself. This implies that a suspension obtained from a close to linear cocycle is isomorphic to a time change of the suspension obtained from the identity cocycle.

1. The Space of Cocycles

This paper is one of a sequence ([2], [1], [3], [4]) designed to understand the structure of the space of continuous cocycles and the suspension flows they can be used to produce. Here we identify and study a particularly well-behaved class which we call “close to linear”.

Let \(X\) be a compact metric space and let \(\mathbb{Z}^m\) denote the integer lattice in \(\mathbb{R}^m\), \(m\)-dimensional Euclidean space. We will assume that \(\mathbb{Z}^m\) acts as a group of commuting homeomorphisms on \(X\), that is, we have a flow \((X, \mathbb{Z}^m)\). A cocycle for such a flow is a continuous map \(h : X \times \mathbb{Z}^m \to \mathbb{R}^m\) such that for all \(x \in X, a, b \in \mathbb{Z}^m\) we have \(h(x, a + b) = h(x, a) + h(ax, b)\) where \(ax\) denotes the action of \(a\) on \(x\). This relationship is called the cocycle equation. Observe that the range of \(h\) could be \(\mathbb{R}^n\) for any \(n \geq 1\) and, indeed, by looking at the coordinate functions which are also cocycles we could do analysis by taking \(n = 1\). (This viewpoint is exploited in [4].) However, as we will see below, the case \(n = m\) is the appropriate environment in which to investigate the construction of \(\mathbb{R}^m\) flows using cocycles and we will restrict ourselves to that situation in this paper.

Let \(C\) denote the set of cocycles on \((X, \mathbb{Z}^m)\). Clearly \(C\) is a vector space over \(\mathbb{R}\). Using the norm \(|t| = \sum_{i=1}^{m} |t_i|\) for \(t = (t_1, \ldots, t_m) \in \mathbb{R}^m\),

\[
\|h\| = \sup \left\{ \frac{|h(x, a)|}{|a|} : x \in X \text{ and } a \in \mathbb{Z}^m \right\}
\]

defines a norm on \(C\). Using the cocycle equation it is not hard to show that

\[
\|h\| = \sup \{|h(x, e_j)| : x \in X \text{ and } 1 \leq j \leq m\}
\]

Received by the editors February 25, 1994 and, in revised form, November 11, 1994.
1991 Mathematics Subject Classification. Primary 58F25; Secondary 28D10, 54H20.

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1923
where $e_1, \ldots, e_m$ is the standard basis for $\mathbb{R}^m$. With this norm $\mathcal{C}$ turns out to be a separable Banach space.

There are several easy ways of obtaining cocycles. If $T \in \mathcal{L}$, the linear operators from $\mathbb{R}^m$ to itself, then $h(x, a) = T(a)$ defines a cocycle in $\mathcal{C}$. Conversely if $h \in \mathcal{C}$ and $h(x, a) = h(y, a)$ for all $x, y \in \mathcal{X}$ and $a \in \mathbb{Z}^m$, then the map $a \to h(x, a)$ is linear. Consequently $\mathcal{L}$ is a closed subspace of $\mathcal{C}$ which is called the space of constant cocycles.

The second easy way to produce cocycles is as follows. Suppose $f$ is a continuous function from $\mathcal{X}$ into $\mathbb{R}^m$. Define $h \in \mathcal{C}$ by setting $h(x, a) = f(ax) - f(x)$. Such a cocyle is called a coboundary and the coboundaries, $\mathcal{B}$, form another subspace of $\mathcal{C}$. If two cocycles differ by a coboundary we will say they are cohomologous. These ideas are important because cohomologous cocycles have essentially the same properties.

Form the closed subspace $\mathcal{D} = \mathcal{L} + \mathbb{R}$. It is the subspace $\mathcal{D}$ which is the subject of study of this paper. We refer to elements of $\mathcal{D}$ as “close to linear”. In essence $\mathcal{D}$ consists of cocycles cohomologous to linear cocycles (which are essentially trivial) and limits of sequences of such cocycles. We want to understand the set $\mathcal{D}$; it may be rather rich (see §2), but we will establish that the members of $\mathcal{D}$ are all “well-behaved” cocycles (see §4). Of course in terms of the cocycle norm, a linear cocycle plus a coboundary may be very far away from a linear one. However, we will show that many consequences of linearity persist in $\mathcal{D}$. Furthermore it will turn out that every element of $\mathcal{D}$ is cohomologous to a cocycle which is close to a linear one in the norm sense. These ideas will justify the terminology.

Cocycles are important tools in the construction of certain special $\mathbb{R}^m$ flows, namely the $\mathbb{R}^m$ suspensions. So assume we have a flow $(\mathcal{X}, \mathbb{Z}^m)$ and a cocycle $h$. Form the space $\mathcal{X} \times \mathbb{R}^m$ and note that there is a trivial $\mathbb{R}^m$ action on this space given by $(x, t)s = (x, t + s)$. Also for each $a \in \mathbb{Z}^m$, define a homeomorphism $T_a : \mathcal{X} \times \mathbb{R}^m \to \mathcal{X} \times \mathbb{R}^m$ by $T_a(x, t) = (ax, t + h(x, a))$. It is obvious that each $T_a$ commutes with the $\mathbb{R}^m$ action and, in fact, the group of these maps gives a $\mathbb{Z}^m$ action on $\mathcal{X} \times \mathbb{R}^m$ because $h$ satisfies the cocycle equation. We now form the quotient space

$$X_h = \mathcal{X} \times \mathbb{R}^m / \{T_a : a \in \mathbb{Z}^m\}$$

and let $\pi$ be the canonical projection from $\mathcal{X} \times \mathbb{R}^m$ to $X_h$. We thus obtain an $\mathbb{R}^m$ flow $(X_h, \mathbb{R}^m)$ which we call the $\mathbb{R}^m$ suspension of $\mathcal{X}$ given by $h$. In the case $m = 1$ this construction is just the usual flow under a function since in one dimension every cocycle is given by $h(x, n) = \sum_{i=0}^{n-1} f(ix)$ for $n > 0$ and a similar formula for $n < 0$, where $f$ is a continuous function. It is easy to check that $X_h$ is a compact Hausdorff space in the case where $f > 0$, but even in the general one-dimensional case, it is not clear that $X_h$ is well behaved in a topological sense or that $X$ is embedded as a global section in the flow. The relationship between properties of $h$ and the corresponding suspension flow $(X_h, \mathbb{R}^m)$ was investigated in detail in [1]. The main results we will need from that paper are restated here for completeness.

**Definition 1.** A cocycle $h \in \mathcal{C}$ is called covering if

(a) $X_h$ is a Hausdorff space,

(b) $\pi$ is a local homeomorphism.

If in addition $\pi$ is one-to-one on $\mathcal{X} \times \{0\}$, then $h$ is called embedding.
Note that when $h$ is an embedding cocycle, $X$ is naturally embedded in $X_h$ as a global section.

**Theorem 1.** Suppose $(X,\mathbb{Z}^m)$ has a free dense orbit. A cocycle $h$ is covering if and only if $|h(x,a)| \to \infty$ uniformly in $x$ as $|a| \to \infty$.

This theorem is used extensively in what follows. We thus will impose the Standing Assumption that $(X,\mathbb{Z}^m)$ has a free dense orbit.

**Theorem 2.** If $h : X \times \mathbb{Z}^m \to \mathbb{R}^m$ is covering, then $X_h$ is a compact metric space. Moreover, there are constants $M_1$ and $M_2$ such that $M_1|a| \leq |h(x,a)| \leq M_2|a|$ for all $a \in \mathbb{Z}^m$ with $|a|$ sufficiently large.

In fact if $h$ is a covering cocycle into a space $\mathbb{R}^n$ and $X_h$ turns out to be compact, then $n$ must be equal to $m$. This justifies our restriction to $n = m$.

It is thus clear that the covering cocycles from $X \times \mathbb{Z}^m$ to $\mathbb{R}^m$ play a key role in constructing well-behaved $\mathbb{R}^m$ suspension actions. Furthermore the suspension flows are important in understanding general minimal $\mathbb{R}^m$ actions. (See [3].)

As an easy consequence of Theorem 1, we have:

**Remark 1.** If $L \in \mathcal{L}$, then $L$ is covering if and only if $L$ is invertible as a linear map.

**Remark 2.** If two covering cocycles are cohomologous, then the corresponding suspensions are isomorphic as flows. Any cocycle cohomologous to a covering cocycle is covering.

Note that by Remark 2, $\mathcal{L} + \mathcal{B}$ generates the same suspension flows as $\mathcal{L}$ itself. We want to understand what new suspension flows come from the closure points of $\mathcal{L} + \mathcal{B}$ in $\mathcal{C}$.

Covering cocycles may exhibit a stronger property which is closely related to the structure of the phase space and orbits of the corresponding suspension. This involves extending $h$ from $X \times \mathbb{Z}^m$ to $X \times \mathbb{R}^m$.

Let $I^m = [0,1]^m$ be the unit cube in $\mathbb{R}^m$. By the standard triangulation of $I^m$ we mean the complex $K$ with $|K| = I^m$ consisting of the $m$-simplices

\[ \mathcal{S}_\sigma = \{0,e_{\sigma 1},e_{\sigma 1} + e_{\sigma 2},\ldots,e_{\sigma 1} + \ldots + e_{\sigma m}\} \]

and all their faces generated by all permutations $\sigma$ of $(1,\ldots,m)$. It is easy to see that

\[ |\mathcal{S}_\sigma| = \{t \in I^m : 1 \geq t_{\sigma 1} \geq t_{\sigma 2} \geq \ldots \geq t_{\sigma m} \geq 0\}. \]

Now for each $x \in X$, $h(x,\cdot)$ can be extended to a map $H(x,\cdot) : \mathbb{R}^m \to \mathbb{R}^m$. To do this give $\mathbb{R}^m$ a simplicial structure with $K + \mathbb{Z}^m$ and view $h(x,\cdot)$ as a vertex map into $\mathbb{R}^m$. Then let $H(x,\cdot)$ be the piecewise-linear extension of $h(x,\cdot)$ on $|K + \mathbb{Z}^m| = \mathbb{R}^m$ to $\mathbb{R}^m$. It is clear that $H$ is a jointly continuous map and $H(x,t + a) = h(x,a) + H(ax,t)$ where $t \in \mathbb{R}^m$ and $a \in \mathbb{Z}^m$.

If $H(x,\cdot)$ is one-to-one and onto for all $x \in X$ the cocycle $h$ is invertible in the sense of [2] where they were called “cocycles for suspensions”. If $h$ is invertible, then there is an orbit-preserving homeomorphism from $X_h$ onto $X_{id}$ (the suspension corresponding to the identity map on $\mathbb{Z}^m$). (Flows with this property are called conjugate.) So the space $X_h$ is topologically independent of $h$ and the dynamics differ only in a time change—the orbits are the same as those of $X_{id}$. In fact $h$ is invertible in the sense that $H(x,\theta(x,t)) = t$ where $\theta(x,t)$ is the required
(continuous) time change. Note that in view of Remark 2 all this holds if \( h \) is merely cohomologous to an invertible cocycle.

We first establish that \( H \) being onto holds for all covering cocycles.

**Proposition 1.** If \( h \) is covering, then \( H(x, \cdot) \) is onto \( \mathbb{R}^m \) for all \( x \in X \).

**Proof.** Fix \( x \) and denote \( H(x, \cdot) \) by \( H \). If \( h \) is covering, \( H \) can be extended continuously to an \( m \)-dimensional sphere by defining \( H(x, \infty) = \infty \). Now if \( H \) is not onto, then \( H \) can be thought of as a map from \( S^m \) to \( \mathbb{R}^m \) (denoted by \( H \) again) where \( S^m \) is the \( m \)-sphere of radius \( r \) and \( r \) is chosen so large that \(|H(x, t) - H(x, s)| \geq 1\) for \(|t-s| \geq r\). (That this can be done is a consequence of the fact that \(|h(x, a)| \geq M_1|a|\) for \( a \) sufficiently large.) By the Borsuk-Ulam theorem, we now have a pair of antipodal points \( s \) and \(-s \) say, with \( H(x, s) = H(x, -s) \). But \(|s - (s)| = 2r\) which is a contradiction.

Thus if \( h \) is covering and \( H \) is one-to-one at each \( x \in X \), then \( h \) is invertible. Furthermore the extension of \( h \) is piecewise-linear in the sense of the discussion above. If \( h \) is covering and \( H \) is one-to-one for all \( x \in X \), then we call \( h \) a piecewise-linear invertible cocycle (P-L invertible).

The following sections will bring these concepts together in a natural way for \( \mathcal{D} = \mathcal{L} + \overline{\mathcal{B}} \). In §2 we will note that \( \mathcal{D} \) is normally a rich subspace of \( \mathcal{L} \) worthy of study in its own right. In §3 we will establish that an element of \( \mathcal{D} \) has a close relationship to a linear cocycle motivating the term “close to linear” for its elements. We also establish that if \((X, Z^m)\) is uniquely ergodic (i.e. there is only one probability measure on \( X \) which is invariant under \( Z^m \)), then in fact \( \mathcal{D} \) is all of \( \mathcal{L} \) so that in this case all the results apply to all cocycles. Finally in §4 we will show that each covering cocycle in \( \mathcal{D} \) is cohomologous to a P-L invertible cocycle so that the suspensions obtained from members of \( \mathcal{D} \) are all conjugate to the identity suspension.

2. The subspace \( \mathcal{D} = \mathcal{L} + \overline{\mathcal{B}} \)

We first establish that \( \mathcal{D} \) is actually a direct sum.

**Proposition 2.** \( \mathcal{D} = \mathcal{L} + \overline{\mathcal{B}} \).

**Proof.** Suppose \( L \in \mathcal{L} \cap \overline{\mathcal{B}} \) and \( L \neq 0 \). Choose a unit vector \( v \) such that \( Lv \neq 0 \) and find a sequence \( \{a_k\} \) in \( Z^m \) with \( |a_k - kv| \leq m \). In this case \( |La_k| \geq Bk \) for some \( B > 0 \) and \( k \) large enough. Now choose \( f \in \mathcal{B} \) such that \( \|L - f\| < B/2 \). Then \( (B/2)|a_k| \geq |La_k - f(x, a_k)| \geq |La_k| - |f(x, a_k)| \geq Bk - C \) where \( C \) is some bound for \( f \). Thus \( C \geq Bk - (B/2)|a_k| = (B/2)k + (B/2)(|v| - |a_k|) \geq (B/2)k - (B/2)m \). Letting \( k \to \infty \) gives a contradiction.

In view of Proposition 2, if \( h \in \mathcal{D} \) we can associate with \( h \) a fixed \( L_h \in \mathcal{L} \) and \( f_h \in \overline{\mathcal{B}} \) so that \( h \) has the (unique) decomposition \( h = L_h + f_h \). We immediately obtain \( \overline{\mathcal{B}} = \{h \in \mathcal{D} : L_h = 0\} \).

The following lemma is helpful in dealing with elements of \( \overline{\mathcal{B}} \).

**Lemma 1.** If \( f \in \overline{\mathcal{B}} \), then \( |f(x, a)|/|a| \to 0 \) uniformly in \( x \) as \( |a| \to \infty \).

**Proof.** Given \( \varepsilon > 0 \), let \( g \in \mathcal{B} \) with \( \|g - f\| < \varepsilon/2 \). As \( g \) is bounded, choose \( A > 0 \) so that \( |g(x, a)|/|a| < \varepsilon/2 \) when \( |a| \geq A \). Now \( |f(x, a)| \leq |f(x, a) - g(x, a)| + |g(x, a)| \) and so \( |f(x, a)|/|a| < \varepsilon \) whenever \( |a| \geq A \).
Corollary 1. If $h \in \mathcal{D}$, then $|h(x,a) - L_h(a)|/|a| \to 0$ uniformly in $x$ as $|a| \to \infty$.

We now turn to the situation for covering cocycles.

Proposition 3. If $h$ is covering, then $h + \overline{B}$ consists of covering cocycles.

Proof. As $h$ is covering, $|h(x,a)| \geq M_1|a|$ for some $M_1 > 0$ and $a$ with $|a|$ sufficiently large. If $f \in \overline{B}$, then $|h(x,a) + f(x,a)|/|a| \geq |h(x,a)|/|a| - |f(x,a)|/|a| \geq M_1/2$ for $|a|$ sufficiently large by Lemma 1.

Corollary 2. If $h \in \mathcal{D}$, then $h$ is covering if and only if $L_h$ is invertible.

Proof. If $h$ is covering, then $|h(x,a)|/|a| \geq M_1$ for $|a|$ large enough and $L_h$ must be invertible by Corollary 1. The converse follows from the proposition and Remark 1.

Theorem 3. Suppose $\mathcal{B}$ is not closed. Then $\mathcal{C}$ contains non-trivial covering cocycles (i.e. covering cocycles not cohomologous to constant ones) and these are dense in $\mathcal{D}$.

Proof. Suppose $f \in \overline{B} - \mathcal{B}$ and $L \in \mathcal{L}$ is invertible. Then $L + f$ is covering by Corollary 2. If $L + f$ is cohomologous to a constant cocycle $M$ say, then $L + f = M + g$ where $g$ is a coboundary; so by Proposition 2, $f = g \in \mathcal{B}$, which is a contradiction. Now if $h \in \mathcal{D}$, then $h = L_h + f_h$. Then $L' + f_h$ can be made as close to $h$ as we like with $L'$ invertible since the invertible linear maps are dense in $\mathcal{L}$. Now if $f_h \in \mathcal{B}$ and we choose $f \in \overline{B} - \mathcal{B}$ with $\|f\|$ sufficiently small, then $f_h + f \notin \mathcal{B}$ and $L' + f_h + f$ is arbitrarily close to $h$.

We show in [4] that under our standing assumption of a free dense orbit, $\mathcal{B} \neq \mathcal{B}$ for real-valued cocycles. Of course this applies to the situation here also by considering the coordinate maps. Thus, in fact, the supposition that $\mathcal{B}$ is not closed is unnecessary in the statement of the theorem provided $(X,\mathbb{Z}^m)$ has a free dense orbit.

Proposition 4. The covering cocycles are open in $\mathcal{C}$.

Proof. This is an easy consequence of the definition of the norm and the fact that a cocycle is covering if and only if $|h(x,a)| \geq M_1|a|$ for some $M_1 > 0$ and $|a|$ sufficiently large.

Corollary 3. The covering cocycles form an open dense set in $\mathcal{D}$.

3. LINEARISATION IN $\mathcal{D}$

The next result indicates a precise sense in which $h \in \mathcal{D}$ is “close to linear”.

Theorem 4. Let $h \in \mathcal{C}$. The following are equivalent:

(a) $h \in \mathcal{D}$.
(b) There is a linear cocycle $L \in \mathcal{L}$ such that we can find a sequence of cocycles
\{ $h_n$ \} all cohomologous to $h$ with $h_n \to L$.
(c) The distance between the sets $\mathcal{L}$ and $h + \mathcal{B}$ is 0.

Proof. (a) $\Rightarrow$ (b). If $h \in \mathcal{D}$, let $f_n \in \mathcal{B}$ be a sequence with $f_n \to f_h$. Now $h - f_n \to L_h$ and $h - f_n$ is cohomologous to $h$.

(b) $\Rightarrow$ (c). Obvious.
(c)⇒(a). If \( d(h + B, L) = 0 \), then there exist \( f_n \in B \) and \( L_n \in L \) such that \( \|h + f_n - L_n\| \leq 1/n \). Thus \( f_n - L_n \) is a sequence in \( D \) converging to \( -h \). So \( h \in D = D \).

Given \( h \in C \), there is a natural way of constructing cohomologous cocycles which average the values of \( h \) over a central portion of each orbit. Choose \( \ell \) to be a positive integer and define \( P_\ell(x) = (1/\ell^m) \sum h(x, a) \) where the sum is taken over all \( a \in \mathbb{Z}^m \) such that \( 1 \leq a_j \leq \ell \) for all \( 1 \leq j \leq m \). Since \( P_\ell \) is continuous, we can use it to define a coboundary and set \( h_\ell(x, a) = P_\ell(ax) - P_\ell(x) + h(x, a) \). Thus \( h_\ell \) is cohomologous to \( h \). Now

\[
\begin{align*}
    h_\ell(x, e_i) &= h(x, e_i) + \frac{1}{\ell^m} \sum \{ h(e_i, a) - h(x, a) : 1 \leq a_j \leq \ell, 1 \leq j \leq m \} \\
    &= h(x, e_i) + \frac{1}{\ell^m} \sum \{ h(ax, e_i) - h(x, e_i) : 1 \leq a_j \leq \ell, 1 \leq j \leq m \} \\
    &= \frac{1}{\ell^m} \sum \{ h(ax, e_i) : 1 \leq a_j \leq \ell, 1 \leq j \leq m \}.
\end{align*}
\]

This argument to construct the cohomologous cocycles was shown to us by H Furstenberg.

In the one-dimensional case it can be shown that the sign of \( \int h(x, 1) d\mu \) is the same for every invariant measure \( \mu \), say positive. (See Theorem 1.12 and its proof in [1].) If \( h_\ell(x, 1) \) is not eventually strictly positive, the above expansion for \( h_\ell \) can be used to produce an invariant measure with non-positive integral. Thus in the one-dimensional case a covering cocycle generated by an unrestricted continuous function is cohomologous to one generated by a strictly positive or negative function. Thus for real suspensions one can restrict oneself to that simpler situation without loss of generality.

We can now characterise \( D \) in terms of the integrals of \( h \). Define an operator \( A^*_\ell : C(X) \to \mathbb{R} \) by \( A^*_\ell(f) = \frac{1}{\ell^m} \sum f(ax) \) where the sum is over all \( a \) with \( 1 \leq a_i \leq \ell \).

**Lemma 2.** \( \int_X f(x) d\mu = 0 \) for all invariant probability measures \( \mu \) if and only if \( A^*_\ell(f) \to 0 \) uniformly in \( x \) as \( \ell \to \infty \).

**Proof.** Suppose that \( A^*_\ell(f) \) does not converge to \( 0 \) uniformly in \( x \) as \( \ell \to \infty \). We then have \( \varepsilon > 0 \), \( \ell_k \to \infty \), \( \{x_k\} \) with \( |A^*_\ell(f)| \geq \varepsilon \) for all \( k \). Since these linear operators are all in the unit ball, we can suppose \( A^*_\ell(f) \) converges weakly to a linear operator \( \theta \), say. It is easy to check that \( \theta \) is invariant in the sense that \( \theta(ag) = \theta(g) \) \((a \in \mathbb{Z}^m)\) where \( ag(x) = g(ax) \). So \( \theta \) is represented by an invariant probability measure \( \mu \) and \( \int_X f(x) d\mu \neq 0 \).

The converse is immediate because \( A^*_\ell(f) \) is an ergodic average of \( f \).

**Theorem 5.** Let \( h \in C \) and \( \mu \) be an invariant probability measure on \((X, \mathbb{Z}^m)\), and define the linear map \( L_\mu(a) = \int_X h(x, a) d\mu \). Then \( h \in D \) if and only if the maps \( L_\mu \) are all the same, i.e. the integral of \( h \) does not depend on the choice of \( \mu \).

**Proof.** Let \( h \in D \) and fix \( a \in \mathbb{Z}^m \). By Corollary 1

\[
\left| \frac{1}{k} h(x, ka) - L_\mu(a) \right| \to 0
\]
uniformly in $x$ as $k \to \infty$. But by the ergodic theorem applied to the integer action induced by $a$

$$
\frac{1}{k} h(x, ka) = \frac{1}{k} \sum_{j=0}^{k-1} h(jax, a)
$$

must converge a.e. $\mu$ to an integrable function whose integral is $\int_X h(x, a) d\mu$. Thus

$$
L_h(a) = \int_X L_h(a) d\mu = \int_X h(x, a) d\mu
$$

and so $L_\mu = L_h$ for any $\mu$.

Conversely suppose $\int_X h(x, a) d\mu = L(a)$ where $L$ is independent of $\mu$. Then by Lemma 2 and the above definition of $h_\ell$

$$(h_\ell(x, e_i) - Le_i)_j = A^\ell_i((h(\cdot, e_i) - Le_i)_j) \to 0
$$

uniformly in $x$ for any $i, j = 1, \ldots, m$. Thus $h_\ell \to L$, and so $h \in D$ by Theorem 4.

\[\square\]

**Corollary 4.** If $(X, \mathbb{Z}^m)$ is uniquely ergodic, then $C = D$.

The question of when $C = D$ in general turns out to be a deeper issue. Note that if $m = 1$, unique ergodicity is both necessary and sufficient by Theorem 5 since every continuous function generates a cocycle. The general case is addressed in [4].

4. P-L invertible cocycles and $D$

We will establish that the P-L invertible cocycles form an open set in $C$. The results of §3 then enable us to establish a close relationship between the elements of $D$ and P-L invertible cocycles.

**Lemma 3.** Let $N > 0$ be an integer and $V = \mathbb{Z}^m \cap [-N, N]^m$. Suppose $f : V \to \mathbb{R}^m$ and let $F : [-N, N]^m \to \mathbb{R}^m$ be the piecewise-linear extension of $f$. Suppose $F$ is one-to-one. There is $\varepsilon > 0$ such that if $g : V \to \mathbb{R}^m$ and $\sup_{x \in V} |f(x) - g(x)| < \varepsilon$, then $G$ is one-to-one where $G$ is the piecewise-linear extension of $g$ to $[-N, N]^m$.

**Proof.** Let $\Xi$ be the collection of simplices $\{v + S : S$ is in the standard triangulation of $[0, 1]^m$ and $v \in \mathbb{Z}^m \cap [-N, N-1]^m\}$. If $S \in \Xi$, then $F$ is an invertible linear map on $S$ and so $G$ will be invertible on $S$ also for $g$ close enough to $f$. If $S_1, S_2 \in \Xi$ and $S_1$ and $S_2$ are disjoint, we can find a linear function $\gamma : \mathbb{R}^m \to \mathbb{R}$ and $c \in \mathbb{R}$ such that $\gamma(F(w)) > c$ for $w \in S_1$ and $\gamma(F(w)) < c$ for $w \in S_2$. If $g$ is close enough to $f$ we can ensure $\gamma(G(w)) > c$ for $w \in S_1$ and $\gamma(G(w)) < c$ for $w \in S_2$. Thus $G(S_1) \cap G(S_2) = \emptyset$.

Now suppose $S_1, S_2 \in \Xi$ with a common face $S'$ and assume $S'$ is a largest common face in terms of dimension. Let $v_1, \ldots, v_p$ be the vertices of $S'$ and then $f(v_1), \ldots, f(v_p)$ are the vertices of $F(S')$ which is a largest common face of $F(S_1)$ and $F(S_2)$. (Note that $F(S_1), F(S_2)$ and $F(S')$ are all simplices of the same dimension as $S_1$, $S_2$ and $S'$ respectively as $F$ is one-to-one.) Now let $H$ be a hyperplane separating $F(S_1)$ and $F(S_2)$ and containing $F(S')$.

We can choose $b_{p+1}, \ldots, b_m \in H$ such that if

$$
\gamma(w) = \det (w, f(v_2) - f(v_1), \ldots, b_m - f(v_1))
$$

then

$$
H = \{w : \gamma(w) = c\}$$
where \( c = \gamma(f(v_1)) \). Now for vertices of \( S_1 \) not in \( S' \), \( \gamma \) will be less than \( c \) (say), and then for vertices of \( S_2 \) not in \( S' \), it will be greater than \( c \). Now for \( g \) close enough to \( f \), the function \( \det(w, g(v_2) - g(v_1), \ldots, b_m - g(v_1)) \) will have the same property because the determinant depends continuously on its arguments. Thus \( G(S_1) \cap G(S_2) = G(S') \).

Since \( \Xi \) is finite, this completes the proof. \( \square \)

**Theorem 6.** The P-L invertible cocycles are open in \( \mathcal{C} \).

**Proof.** Let \( h \) be P-L invertible and \( M_1 > 0 \) such that \( |h(x,a)| \geq M_1|a| \) for \( |a| \) sufficiently large. \( (M_1 \text{ exists as } h \text{ is covering.}) \) Let \( g \in \mathcal{C} \) and \( \|h - g\| < M_1/2 \).

Note that \( \|g\| < M_1/2 + \|h\| = \alpha/m \), say. Using the cocycle equation, if \( t \in \mathbb{R}^m \) and \( a \in \mathbb{Z}^m \) is the integer part of \( t \), then \( |G(x,t) - g(x,a)| \leq m\|g\| \) for all \( x \in X \).

Now there is an integer \( N > 0 \) (independent of \( g \)) such that \( |G(x,t)| \geq \alpha \) for any \( x \in X \) and \( |t| \geq N \). If not, we can find a sequence \( g_n \in \mathcal{C} \) satisfying \( \|g_n - h\| < M_1/2, t_n \in \mathbb{R}^m \) with \( |t_n| \to \infty \) and \( x_n \in X \) such that \( |G_n(x_n,t_n)| < \alpha \). Thus if \( a_n \) is the integer part of \( t_n \), then \( |g_n(x_n,a_n)| < m\|g_n\| + \alpha \leq 2\alpha. \) But now
\[
M_1|a_n|/2 \geq |h(x_n,a_n) - g_n(x_n,a_n)| \\
\geq |h(x_n,a_n)| - |g_n(x_n,a_n)| \geq M_1|a_n| - m\|g_n\| - \alpha.
\]

So \( 2\alpha > \alpha + m\|g_n\| \geq M_1|a_n|/2 \) which is a contradiction as clearly \( |a_n| \to \infty \).

Now suppose \( g \) is not P-L invertible. An application of the cocycle equation shows that we can find \( x \in X, s \in I^m \) and \( t \in \mathbb{R}^m \) with \( G(x,s) = G(x,t) \). As \( |G(x,s)| \leq m\|g\| < \alpha \), we know that \( t \in [-N,N]^m \).

Using this \( N \), the lemma determines \( \varepsilon_x \) for \( h(x,\cdot) \) restricted to \( V \) for each \( x \in X \). There is \( \delta_x > 0 \) such that \( d(x,y) < \delta_x \) implies \( |h(x,a) - h(y,a)| < \varepsilon_x/2 \) for \( a \in V \). Cover \( X \) with balls of radius \( \delta_x \) about each \( x \) and select a finite subcover. Thus we have \( x_1, x_2, \ldots, x_n, \) say, with
\[
X = \bigcup_{i=1}^n \{ y : d(y, x_i) < \delta_x \}.
\]

Let \( \varepsilon = \min \{ \varepsilon_{x_1}, \varepsilon_{x_2}, \ldots, \varepsilon_{x_n}, M_1 \} \) and suppose that \( \|g - h\| < \varepsilon/2mN \). Now fix \( x \in X \) and choose \( x_i \) with \( d(x, x_i) < \delta_{x_i} \). Since \( H(x_i,\cdot) \) is invertible, it is enough to show that \( |g(x,a) - h(x,a)| < \varepsilon_x \) for \( a \in V \). But \( |g(x,a) - h(x,a)| \leq |g(x,a) - h(x,a)| + |h(x,a) - h(x,a)| \leq \|g - h\| |a| + \varepsilon_x/2 < \varepsilon_x \), for \( a \in V \) since \( |a| \leq mN \). \( \square \)

**Corollary 5.** Suppose \( h \in \mathcal{D} \) and \( h \) is covering. Then \( h \) is cohomologous to a P-L invertible cocycle.

**Proof.** This is immediate from Theorem 4, Corollary 2 and the theorem above since \( L \in \mathcal{L} \) is P-L invertible precisely if \( L \) is invertible in the usual sense. \( \square \)

**Corollary 6.** Suppose \( h \in \mathcal{D} \) and \( h \) is covering. Then the suspension \( (X_h, \mathbb{R}^m) \) is conjugate to the identity suspension \( (X_{id}, \mathbb{R}^m) \), i.e. \( (X_h, \mathbb{R}^m) \) is isomorphic to a time change of \( (X_{id}, \mathbb{R}^m) \).

**Proof.** An invertible cocycle is what was termed a “cocycle for a suspension” in [2], so this follows from the above corollary and Theorem 2.2 of [2]. \( \square \)

**Corollary 7.** If \( (X, \mathbb{Z}^m) \) is uniquely ergodic, then every \( \mathbb{R}^m \) suspension of it given by a covering cocycle is isomorphic to a time change of the identity suspension.
In the case \( m = 1 \), we have already noted that up to coboundaries we can assume \( f > 0 \) (\( f < 0 \)). In this case, of course, \( h \) is increasing (decreasing) and so P-L invertible. Thus the conclusions of the above theorem and corollaries hold for \( m = 1 \) without any assumption on the size of \( \mathcal{D} \).

References

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School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455
E-mail address: keynes@math.umn.edu

Department of Mathematics, University of Maryland, College Park, Maryland 20742
E-mail address: ngm@glve.umd.edu

Department of Mathematics, University of the Witwatersrand, Johannesburg, South Africa
E-mail address: 036mis@cosmos.wits.ac.za