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GENERALIZED CYCLIC COHOMOLOGY ASSOCIATED WITH DEFORMED COMMUTATORS

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Abstract. The generalized cyclic cohomology is introduced which is associated with q-deformed commutators $xy - qyx$. Some formulas related to the trace of the product of q-deformed commutators are established. The Chern character of odd dimension associated with q-deformed commutators is studied.

1. Introduction

In non-commutative differential geometry [1], the Chern character of a p-summable Fredholm module is expressed by the trace of some product of quantized differentials $df = [F, f]$, where $[F, f]$ is the commutator $Ff - fF$ and F is a self-adjoint idempotent operator. In the odd dimension case, the Chern character ch_{2n-1} is expressed as

$$\operatorname{tr}\big(\omega(a^0, a^1)\cdots\omega(a^{2n-2}, a^{2n-1}) - \omega(a^{2n-1}, a^0)\cdots\omega(a^{2n-3}, a^{2n-2})\big)$$

where $\omega(x, y) = p(xy) - p(x)p(y)$ is the curvature of some mapping p (see also [2], [6]).

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$, and let C^n be the space of $n+1$-linear functions on $\mathcal{A}$. The basic operations in the cyclic cohomology are b', b, t, etc. (cf. [1], [5]), where b is the Hochschild boundary operation. Define $C^n_\lambda = \{\, f \in C^n : tf = f \,\}$ and $Af = (1 + t + \cdots + t^n)f$, $f \in C^n$. Let $pf = \sum_{j=0}^n (n-j)t^j f$, $f \in C^n$. Define $S = bpb'$. The restriction of $2\pi i S$ at $\mathcal{Z}^n_\lambda = \{\, f \in C^n_\lambda : bf = 0 \,\}$ coincides with A. Connes' S operator (cf. [1], [12]).

Related to the Chern character, in the previous papers [11], [12], the author studied the cyclic cohomology associated with the product of commutators. Let X and Y be two subalgebras (or subgroups) of a unital algebra $\mathcal{A}$. Let $C^{m,n}$ be the space of multi-linear functions (or functions) $f(x^0, \ldots, x^m; y^0, \ldots, y^n)$, $x^i \in X$ and $y^j \in Y$. Let b_x, b'_x, t_x, A_x, S_x and b_y, b'_y, t_y, A_y, S_y be the operators b, b', t, A, S with respect to the x's and y's respectively. Let $C^{n,n}_\lambda = \{\, f \in C^{n,n} : t_x f = t_y f =$

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$f\}$. Suppose that there is a trace ideal J in $\mathcal{A}$ with trace τ. Assume that there is a natural number k such that $[x^1, y^1]\cdots[x^k, y^k] \in J$, for $x^i \in X$ and $y^j \in Y$. Define

$$\phi_n(x^0, \dots, x^n; y^0, \dots, y^n) = \tau[x^0, y^0]\cdots[x^n, y^n], \qquad \text{for } n \geq k-1 \tag{1}$$

and

$$\psi_n(x^0, \dots, x^n; y^0, \dots, y^n) = \tau x^0 y^0 [x^1, y^1]\cdots[x^n, y^n], \qquad \text{for } n \geq k. \tag{2}$$

Then $A_x\phi_n = A_y\phi_n$ and it is denoted by $A\phi_n$.

Theorem A [11]. *For $n \geq k$, there is a $\Theta_n \in C_\lambda^{n,n}$ such that $A\phi_n = b_x b_y \Theta_n + \hat{\phi}_{n+1}$ where*

$$\hat{\phi}_{2m+p} = (-1)^m \frac{(p+1)!}{(p+2m)!} S_x^m S_y^m \phi_p, \qquad p = k, k-1,$$

and there is a $\tilde{\Theta}_n \in C_\lambda^{n,n}$ such that $A\phi_n = b_x b_y \tilde{\Theta}_n + \tilde{\phi}_{n+1}$ where

$$\tilde{\phi}_{2m+p} = (-1)^m \frac{p!}{(p+2m)!} S_x^m S_y^m A\phi_p, \qquad \textit{for } p = k+1, k,$$

and the functions Θ_n and $\tilde{\Theta}_n$ are expressed by $\psi_p, \dots, \psi_n$.

By means of Theorem A (in the case of $k = 1$) and the analytic model, the author proved that the function trace$\big[(\bar{\lambda}^0 - S^*)^{-1}, (\mu^0 - S)^{-1}\big]\big[(\bar{\lambda}_1 - S^*)^{-1}, (\mu_1 - S)^{-1}\big]$, $\lambda^i, \mu^i \in \text{sp}(S)$, is a complete unitary invariant for some subnormal operator S with trace class commutator $[S^*, S]$ (cf. [14], [15]).

For the unbounded operator case, the author (cf. [9], [10]) also studied the almost unperturbed Schrödinger pair of operators u and v which are self-adjoint operators on a Hilbert space $\mathcal{H}$ satisfying the condition that

$$e^{ius}e^{ivt} - e^{ist}e^{ivt}e^{ius} \in \mathcal{L}^1(\mathcal{H}), \qquad s, t \in \mathbb{R},$$

where $\mathcal{L}^1(\mathcal{H})$ is the trace class of operators on $\mathcal{H}$. If we denote e^{ius} and e^{ivt} by x and y respectively, then $q(x, y) = e^{ist}$ is a complex number determined by x and y. Therefore instead of the commutator we have to study the trace class q-deformed commutator $\{x, y\} \overset{\text{def}}{=} xy - q(x, y)yx$. By the way, now-a-day the study of the q-deformed (or q-twisted) commutators becomes an interesting subject (cf. [3], [4], [7], etc.). In [9] and [10], the author studied the form of cyclic one-cocycles associated with q-deformed commutators, which has some application for establishing the theory of principal distribution and others.

The first aim of the present paper is to generalize Theorem A to the q-deformed commutators case. Suppose that there is a function $q(x, y)$, $x \in X$, $y \in Y$, satisfying $q(1, y) = q(x, 1) = 1$ and $q(x^1x^2, y^1y^2) = \prod_{i,j=1}^2 q(x^i, y^j)$. Assume that there is a natural number k such that $\{x^1, y^1\}\cdots\{x^k, y^k\} \in J$, $x^i \in X, y^j \in Y$. We define new functions ϕ_n and ψ_n by changing the commutator to a q-deformed commutator in (1) and (2). In §4 of the present paper, we give Theorem 1 on these functions ϕ_n and ψ_n which is a generalization of Theorem A in this q-deformed commutator case.

This generalization provides a possibility to study some complete unitary invariants for some unbounded hyponormal operators or pseudo-differential operators $u + iv$ where the pair of u and v is an almost unperturbed Schrödinger pair of operators.

In Theorem 1, the formulas are established for the functions $\phi_n, \dots$ on the "manifolds"

$$M^{n,n} = \{ (x^0, \dots, x^n; y^0, \dots, y^n) \in X^n \times Y^n : \prod_{i,j=1}^{n} q(x^i, y^j) = 1 \}.$$

Off these manifolds $M^{m,m}$, the q-deformed commutator case is quite different from the commutator case. The second aim of the present paper is to study the structure (see Theorem 2 of §5) of the Chern character ch_{2n-1} of $2n-1$ dimension associated with the q-deformed commutator off the manifold $M^{m,m}$. In the lower dimension cases, it is calculated in the corollary of Theorem 2 that the Chern characters ch_1 and ch_3 are boundaries of some cyclic cochain off $M^{n,n}$. Further study will be needed to answer the question whether or not all the Chern characters ch_m of odd dimensions associated with q-deformed commutators are boundaries of some cyclic cochains off the manifold $M^{m,m}$. Theorem 2 may provide a basis for this study. We have to point out that the Chern characters of odd dimension have not been fully studied neither on the manifold $M^{n,n}$ (for the q-deformed commutator case), nor for the commutator case.

All of these studies we mentioned above are based on some new tools, the operations δ_x, δ'_x, τ_x and δ_y, δ'_y, τ_y, which are the generalizations of b_x, b'_x, t_x and b_y, b'_y, t_y respectively. The formulation of this study is given in §2, which is a set of modified definitions of Hochschild cohomology and cyclic cohomology. The setting is that of a semidirect product of groups, i.e. a group X acting on a group Y by automorphisms q_x. This returns to the ordinary case if q_x acts trivially, i.e. $q_x y = y$ for all x, y. Although in §2 the concept of a q-deformed commutator is not needed, the modified cyclic cohomology operations are introduced for obtaining the formulas of Lemma 1 in §3. These are formulas connecting ϕ_n, ψ_n, etc., which are the trace of products of some q-deformed commutators $\{x, y\} = xy - q_x(y)x$, $x \in X$, $y \in Y$. These formulas are neccessary for establishing Theorems 1 and 2. It was not easy to find out those definitions in §2. Although in Theorem 1 and the proof of Theorem 2, $q_x(y)$ is simply $q(x, y)y$, the formulation adopted in §2 and §3 is for general automorphisms q_x for two reasons. First, even if we restrict ourselves on the simpler case $q_x(y) = q(x, y)y$, we cannot simplify either notation or formulas in §2 and §3. The more important reason is that the present formulation may set a basis for further study. As a matter of fact, in [13], the author studied a special case about the perturbation of some partial differential operators in which $q_x(y)$ is not $q(x, y)y$ but the Chern character of dimension one is the boundary of a zero cyclic cochain off a lower dimensional manifold. That case is not covered in §5 of the present paper. Therefore the setting for general $q_x(\cdot)$ in §2 may provide a tool to calculate certain Chern characters of odd dimension associated with q-deformed commutators in which $q_x(\cdot)$ is not $q(x, y)y$.

In the statement of Theorem 2 and its Corollary, only cyclic cohomology is involved, but in their proof, the formulas in generalized cyclic cohomology are involved.

This paper is only an introduction to a circle of ideas whose natural continuation will be explained in subsequent papers.

2. Basic definitions

Let X and Y be two groups. Let 1 be the identity of X and Y. Suppose that for every $x \in X$ there is an automorphism $q_x : Y \to Y$ satisfying $q_{x_1 x_2} = q_{x_1} q_{x_2}$, $q_1 =$ identity mapping. For $m \geq 0$ and $n \geq 0$, let $C^{m,n} = C^{m,n}(X, Y)$ be the space of functions

$$f_{m,n}(x^0, \dots, x^m; y^0, \dots, y^n), \quad x^i \in X,\ y^j \in Y.$$

Define δ'_x and $\delta_x : C^{m,n} \longrightarrow C^{m+1,n}$ in the following. For $f \in C^{m,n}$, $x = (x^0, \dots, x^{m+1})$ and $y = (y^0, \dots, y^n)$,

$$\begin{aligned}(\delta'_x f)(x; y) = &\sum_{j=0}^{m-n} (-1)^j f(x^0, \dots, x^j x^{j+1}, \dots, x^{m+1}; y) \\ &+ \sum_{j=m-n+1}^{m} (-1)^j f(x^0, \dots, x^j x^{j+1}, \dots, x^{m+1}; \\ &q_{x^{m-n+1}}(y^0), \dots, q_{x^j}(y^{j-m+n-1}), y^{j-m+n}, \dots, y^n)\end{aligned}$$

and

$$\begin{aligned}(\delta_x f)(x; y) = &(\delta'_x f)(x; y) \\ &+ (-1)^{m+1} f(x^{m+1} x^0, x^1, \dots, x^m; q_{x^{m-n+1}}(y^0), \dots, q_{x^{m+1}}(y^n)),\end{aligned}$$

if $m \geq n$; and

$$\begin{aligned}(\delta'_x f)(x; y) = &\sum_{j=0}^{m} (-1)^j f(x^0, \dots, x^j x^{j+1}, \dots, x^{m+1}; \\ &y^0, \dots, y^{n-m-2}, q_{x^0}(y^{n-m-1}), \dots, q_{x^j}(y^{n-m+j-1}), y^{n-m+j}, \dots, y^n)\end{aligned}$$

and

$$\begin{aligned}(\delta_x f)(x; y) = &(\delta'_x f)(x; y) + (-1)^{m+1} f(x^{m+1} x^0, \dots, x^m; \\ &q_{x^{m+1}}(y^0), \dots, q_{x^{m+1}}(y^{n-m-2}), q_{x^{m+1} x^0}(y^{n-m-1}), q_{x^1}(y^{n-m}), \dots, q_{x^{m+1}}(y^n)),\end{aligned}$$

if $0 \leq m < n$. Define δ'_y and $\delta_y : C^{m,n} \longrightarrow C^{m,n+1}$ as follows. For $f \in C^{m,n}$, $x = (x^0, \dots, x^m)$ and $y = (y^0, \dots y^{n+1})$,

$$\begin{aligned}(\delta'_y f)(x; y) = &\sum_{j=0}^{n-m-1} (-1)^m f(x; y^0, \dots, y^j y^{j+1}, \dots, y^{n+1}) \\ &+ \sum_{j=n-m}^{n} (-1)^j f(x; y^0, \dots, y^{n-m-1}, q_{x^0}^{-1}(y^{n-m}), \dots, q_{x^{j-n+m}}^{-1}(y^j) y^{j+1}, \dots, y^{n+1})\end{aligned}$$

and

$$\begin{aligned}(\delta_y f)(x; y) = &(\delta'_y f)(x; y) \\ &+ (-1)^{n-1} f(x; y^{n+1} y^0, \dots, y^{n-m-1}, q_{x^0}^{-1}(y^{n-m}), \dots, q_{x^m}^{-1}(y^n)),\end{aligned}$$

if $m < n$; and

$$(\delta'_y f)(x;y) = \sum_{j=0}^{n}(-1)^j f(x; q^{-1}_{x^{m-n}}(y^0), \ldots , q^{-1}_{x^{j+m-n}}(y^j)y^{j+1}, y^{j+2}, \ldots , y^{n+1})$$

and

$$\begin{aligned}(\delta_y f)(x;y) &= (\delta'_y f)(x;y)\\ &\quad + (-1)^{n+1} f(x; q^{-1}_{x^0\cdots x^{m-n}}(y^{n+1})q^{-1}_{x^{m-n}}(y^0), q^{-1}_{x^{m-n+1}}(y^1), \ldots , q^{-1}_{x^m}(y^n))\end{aligned}$$

if $m \geq n \geq 0$. It can be verified through calculation that $\delta'^2_x = \delta^2_x = 0$ and $\delta'^2_y = \delta^2_y = 0$. If $q_x =$ identity, then $\delta_x = b_x$ and $\delta_y = b_y$ in [11]. These δ_x and δ_y are the generalized Hochschild boundary operations in some sense.

Define $\tau_x : C^{m,n} \longrightarrow C^{m,n}$ and $\tau_y : C^{m,n} \longrightarrow C^{m,n}$ as follows. For $f \in C^{m,n}, x = (x^0, \ldots , x^m)$ and $y = (y^0, \ldots , y^n)$,

$$\begin{aligned}(\tau_x f)(x;y) &= (-1)^m f(x^m, x^0, \ldots , x^{m-1}; q_{x^{m-n}}(y^0), \ldots , q_{x^m}(y^n)),\\ (\tau_y f)(x;y) &= (-1)^n f(x; q^{-1}_{x^0\cdots x^{m-n}}(y^n), q^{-1}_{x^{m-n+1}}(y^0), \ldots , q^{-1}_{x^m}(y^{n-1}))\end{aligned}$$

if $m \geq n$; and

$$\begin{aligned}&(\tau_x f)(x;y) =\\ &(-1)^m f(x^m, x^0, \ldots , x^{m-1}; q_{x^m}(y^0), \ldots , q_{x^m}(y^{n-m-1}), q_{x^0}(y^{n-m}), \ldots , q_{x^m}(y^n)),\\ &(\tau_y f)(x;y) =\\ &(-1)^n f(x; y^n, y^0, \ldots , y^{n-m-2}, q^{-1}_{x^0}(y^{n-m-1}), \ldots , q^{-1}_{x^m}(y^{n-1}))\end{aligned}$$

if $m < n$. If $q_x =$ identity, then $\tau_x = t_x$ and $\tau_y = t_y$ in [12]. Through complicated calculation it can be verified that the operations $\delta_x, \delta'_x, \tau_x$ commute with $\delta_y, \delta'_y, \tau_y$. We also have

$$\delta'_x(1 - \tau_x) = (1 - \tau_x)\delta_x, \quad \delta'_y(1 - \tau_y) = (1 - \tau_y)\delta_y. \tag{3}$$

These formulas are the generalizations of the formula $b'(1 - t) = (1 - t)b$ in [5]. Define

$$\alpha_x f = (1 + \tau_x + \cdots + \tau_x^m)f, \qquad \alpha_y f = (1 + \tau_y + \cdots + \tau_y^n)f$$

for $f \in C^{m,n}$ which are the generalizations of operators A_x and A_y in [12]. Then

$$\alpha_x\delta'_x = \delta_x\alpha_x, \qquad \alpha_y\delta'_y = \delta_y\alpha_y. \tag{4}$$

For $n \geq 0$ let $p_n(z) = \sum_{j=0}^{n}(n - j)z^j$. Define $\pi_x f = p_m(\tau_x)f$ and $\pi_y f = p_n(\tau_y)f$ for $f \in C^{m,n}$. Define $\sigma_x = \delta_x\pi_x\delta'_x$ and $\sigma_y = \delta_y\pi_y\delta'_y$. These are generalizations of Connes' operators S_x and S_y in [12]. Then σ_x commutes with σ_y. From (3) it is easy to see that $\delta'_x\delta_x\alpha_x = -(1 - \tau_x)\sigma_x$, $\delta'_y\delta_y\alpha_y = -(1 - \tau_y)\sigma_y$. From this, we may prove that σ_x commutes $\delta_x\alpha_x$ and σ_y commutes $\delta_y\alpha_y$.

3. Trace of product of deformed commutators

Suppose X and Y are also subgroups of an algebra $\mathcal{A}$ over $\mathbb{C}$. Define

$$\{x,y\} = xy - q_x(y)x, \qquad x \in X,\ y \in Y,$$

where q_x satisfies the conditions in §2. This $\{x,y\}$ is called the *q-deformed commutator* of x and y. Suppose that there is a trace ideal J of $\mathcal{A}$ with a trace τ on J, i.e. τ is a linear functional on J satisfying $\tau(ab) = \tau(ba)$ for $b \in J$ and $a \in \mathcal{A}$. Assume that there is a natural number k such that $\{x^1,y^1\}\cdots\{x^k,y^k\} \in J$. For $n \geq k$, define $\psi_n(x^0,\dots,x^n;y^0,\dots,y^n) = \tau x^0y^0\{x^1,y^1\}\cdots\{x^n,y^n\}$, for $x^j \in X$, $y^j \in Y$. Then $\psi_n \in C^{n,n}$. Define functions

$$\begin{aligned}
&\xi_n(x;y^0,\dots,y^{n-1}) = \psi_n(x;1,y^0,\dots,y^n), \qquad n \geq k,\\
&\eta_n(x^0,\dots,x^{n-1};y) = \psi(1,x^0,\dots,x^n;y), \qquad n \geq k,\\
&\phi_n(x;y) = \psi_{n+1}(1,x^0,\dots,x^n;1,y^0,\dots,y^n), \qquad n \geq k-1,
\end{aligned}$$

where $x = (x^0,\dots,x^n)$ and $y = (y^0,\dots,y^n)$. The following lemma gives the basic relations between ψ_n, ξ_n, η_n, and ϕ_{n-1}.

Lemma 1. *For $n \geq k$,*

$$\xi_{n+1} = \delta_x\psi_n, \qquad \eta_{n+1} = -\tau_x\delta_y\psi_n, \tag{5}$$

$$(1-\tau_x)\xi_n = \delta'_x\phi_{n-1}, \qquad (1-\tau_y)\eta_n = \delta'_y\phi_{n-1}, \tag{6}$$

$$(1-\tau_y^{-1})\xi_n = \delta_x\phi_{n-1}, \qquad (1-\tau_x^{-1})\eta_n = \delta_y\phi_{n-1}, \tag{7}$$

$$(1-\tau_y)\psi_n = \delta'_y\xi_n - \tau_y\phi_n, \qquad (1-\tau_x)\psi_n = \delta'_x\eta_n + \phi_n. \tag{8}$$

Proof. We only give the proofs of those formulas in which ξ's are involved. The others can be proved similarly. The basic formulas for deformed commutators are

$$\{x_1x_2,y\} = x_1\{x_2,y\} + \{x_1,q_{x_2}(y)\}x_2 \tag{9}$$

and

$$\{x,y_1y_2\} = \{x,y_1\}y_2 + q_x(y_1)\{x,y_2\}, \tag{10}$$

for $x,x_1,x_2 \in X$ and $y,y_1,y_2 \in Y$. To prove (5), by means of (9), we have

$$\begin{aligned}
\xi_{n+1}(x;y) &= \tau x^0x^1y^0\{x^2,y^1\}\cdots\{x^{n+1},y^n\} - \tau x^0q_{x^1}(y^0)x^1\{x^2,y^1\}\cdots\{x^{n+1},y^n\}\\
&= \psi_n(x^0x^1,x^2,\dots,x^{n+1};y) - \psi_n(x^0,x^1x^2,\dots,x^{n+1};q_{x^1}(y^0),y^1,\dots,y^n)\\
&\qquad + \tau x^0q_{x^1}(y^0)\{x^1,q_{x^2}(y)\}x^2\{x^3,y^2\}\cdots\{x^{n+1},y^n\},
\end{aligned}$$

for $x = (x^0,\dots,x^{n+1})$ and $y = (y^0,\dots,y^n)$. Continuing this process, we may prove (5).

To prove (6), by means of (9), we have

$$\xi_n(x;y) = \phi_{n-1}(x^0x^1,\dots,x^n;y) - \tau\{x^0,q_{x^1}(y^0)\}x^1\{x^2,y^1\}\cdots\{x^n,y^{n-1}\}.$$

The last term of the right-hand side of the above formula equals

$$-\phi_{n-1}(x^0, x^1x^2, \dots, x^n; q_{x^1}(y^0), y^1, \dots, y^{n-1}) \\ + \tau\{x^0, q_{x^1}(y^0)\}\{x^1, q_{x^1}(y^1)\}x^2\{x^3, y^2\}\cdots.$$

Continuing this process, we may prove (6). By the definitions and (9), it is easy to verify that

$$\tau_x\xi_n - \tau_y^{-1}\xi_n = \delta_x\phi_{n-1} - \delta'_x\phi_{n-1}.$$

Therefore (7) follows from (6). To prove (8), by means of (10), we observe that

$$\begin{aligned}\psi_n(x;y) - \phi_n(x;y) &= \tau x^0\{x^1, y^1\}\cdots\{x^n, y^n\}q_{x^0}(y^0) \\ &= \xi_n(x; y^1, \dots, y^{n-1}, y^n q_{x^0}(y^0)) \\ &\quad - \tau x^0\{x^1, y^1\}\cdots\{x^{n-1}, y^{n-1}\}q_{x^n}(y^n)\{x^n, q_{x^0}(y^0)\}.\end{aligned}$$

Continuing this process we may prove that $\psi_n(x;y) - \phi_n(x;y)$ equals

$$\sum_{j=2}^{n+1}(-1)^{n+1-j}\xi_n(x; y_1, \dots, y_{j-1}q_{x^j}(y^j), q_{x^{j+1}}(y^{j+1}), \dots, q_{x^{n+1}}(y^{n+1})) \\ + (-1)^n\psi_n(x; q_{x^1}(y^1), \dots, q_{x^{n+1}}(y^{n+1})),$$

where $x^{n+1} = x^0, y^{n+1} = y^0$. Thus $\psi_n - \phi_n = -\tau_y^{-1}\delta'_y\xi_n + \tau_y^{-1}\psi_n$ which proves (8). □

4. A basic theorem

Let us consider a very important special case. Suppose that there is a function $q(\cdot\,,\cdot)$ on $X \times Y$ satisfying

$$q(x^1x^2; y^1y^2) = \prod_{i,j=1}^{2} q(x^i, y^j), \quad x^i \in X,\ y^i \in Y, \tag{11}$$

and

$$q(1, y) = q(x, 1) = 1, \qquad x \in X, y \in Y. \tag{12}$$

Let $Q = \{q(x,y) : x \in X, y \in Y\} \subset \mathbb{C}$; then Q is a group. Let $\tilde{Y} = \{cy : c \in Q, y \in Y\}$. Define $q_x(cy) = c^2q(x,y)y$, $c \in Q$, $y \in Y$. Then $q_x : \tilde{Y} \longrightarrow \tilde{Y}$ is an isomorphism satisfying condition (1). Let $\tilde{C}^{m,n} = \tilde{C}^{m,n}(X, \tilde{Y})$ be the space of the restriction of functions f in $C^{m,n}(X, \tilde{Y})$ satisfying the condition that

$$f(x; c^0y^0, \dots, c^ny^n) = \prod_{j=0}^{n} c^j f(x; y^0, \dots, y^n)$$

for $c^j \in Q, y^j \in Y$. For example, $\psi_n \in \tilde{C}^{m,n}$. Then $\tilde{C}^{m,n}$ is invariant with respect to the set $D = \{\delta'_x, \delta_x, \tau_x, \delta'_y, \delta_y, \tau_y\}$ of operations defined in $C^{m,n}(X, \tilde{Y})$.

Let us consider the restriction on $\tilde{C}^{m,n}$ of those operations in D only. Then these operations in D possess all the properties described in §2.

Let $M^{m,n} = \{(x^0, \dots, x^m; y^0, \dots, y^n) : x^i \in X, y^j \in Y, \prod_{i=0}^m \prod_{j=0}^n q(x^i, y^j) = 1\}$ and $\hat{C}^{m,n}$ be the restriction of the functions in $\tilde{C}^{m,n}$ on $M^{m,n}$. An important property of $\hat{C}^{m,n}$ is that $\tau_x \alpha_x f = \alpha_x f$, $\tau_y \alpha_y f = \alpha_y f$, $f \in \hat{C}^{m,n}$. Define $C_\lambda^{m,n} = \{f \in \hat{C}^{m,n} : \tau_x f = \tau_y f = f\}$. It is easy to see that $\alpha_x \phi_n(x; y) = \alpha_y \phi_n(x, y)$ for $(x, y) \in M^{n,n}$. Define $\alpha\phi_n = \alpha_x \phi_n = \alpha_y \phi_n$ on $M^{n,n}$. Then $\alpha\phi_n \in C_\lambda^{m,n}$. By the same method in [12], using Lemma 1 and the formulas in §2 and §3, we may prove the following theorem and omit the proof.

Theorem 1. *For $n \geq k$, there is a $\Theta_n \in C_\lambda^{n,n}$ such that*

$$\alpha\phi_{n+1} = \delta_x \delta_y \Theta_n + \hat{\phi}_{n+1}, \qquad \text{on } M^{n,n},$$

where

$$\hat{\phi}_{2m+p} = (-1)^m \frac{(p+1)!}{(p+2m)!} \sigma_x^m \sigma_y^m \phi_p,$$

for $p = k, k-1$. For $n \geq k$, there is a $\tilde{\Theta}_n \in C_\lambda^{n,n}$ such that

$$\alpha\phi_{n+1} = \delta_x \delta_y \tilde{\Theta}_n + \tilde{\phi}_{n+1}, \qquad \text{on } M^{n,n},$$

where

$$\tilde{\phi}_{2m+p} = (-1)^m \frac{p!}{(p+2m)!} \sigma_x^m \sigma_y^m \alpha\phi_p$$

for $p = k+1, k$. Besides, the functions Θ_n and $\tilde{\Theta}_n$ are expressed by $\psi_n, \dots, \psi_p$.

5. Chern character of odd dimension

As in the previous sections, assume that $\mathcal{A}$ is an algebra over $\mathbb{C}$, and J is a trace ideal in $\mathcal{A}$ with trace τ. Let X and Y be subgroups of $\mathcal{A}$. Assume that there is a function $q(x, y)$, $x \in X$, $y \in Y$, satisfying conditions (11) and (12). Assume that there is a natural number k such that $\{x^1, y^1\} \cdots \{x^k, y^k\} \in J$. Define $\Delta = \Delta_{m,n}(x^0, \dots, x^m; y^0, \dots, y^n) = \prod_{i=0}^m \prod_{j=0}^n q(x^i, y^j)$. Let $\nu = \tau_x \tau_y$; then $(\nu f)(x^0, \dots, x^n; y^0, \dots, y^n) = f(x^n, x^0, \dots, x^{n-1}; y^n, y^0, \dots, y^{n-1})$ for $f \in C^{n,n}$.

Lemma 2. *For $n \geq k$ and $\Delta \neq 1$,*

$$\psi_n = (1 - \Delta)^{-1} \alpha_x \phi_n - (1 - \Delta)^{-2} \Delta \delta_x \delta_y \alpha_x \alpha_y \phi_{n-1} \tag{13}$$

and

$$(1 - \nu)\psi_n = (1 - \Delta)^{-1} \left[-\delta'_x (\delta_y - \delta'_y) \tau_x + (\delta_x - \delta'_x) \delta'_y \right] \alpha_x \phi_{n-1}. \tag{14}$$

Proof. It is easy to see that $\alpha_x(1 - \tau_x) = 1 - \Delta$ and $\alpha_y(1 - \tau_y) = 1 - \Delta^{-1}$. Applying α_x to both sides of the right equality in (8), we have $(1 - \Delta)\psi_n = \delta_x \alpha_x \eta_n + \alpha_x \phi_n$, since $\alpha_x \delta'_x = \delta_x \alpha_x$ (see (4)). Similarly, from (6), we get $(1 - \Delta^{-1})\eta_n = \delta_y \alpha_y \phi_{n-1}$. Thus we obtain (13).

From (2) and $(1-\nu)=(1-\tau_x)\tau_y+(1-\tau_y)$ it is easy to calculate that

$$(1-\nu)\delta_x\alpha_x\delta_y\alpha_y=(1-\Delta)\delta_x'\tau_y\delta_y\alpha_y+(1-\Delta^{-1})\delta_x\alpha_x\delta_y'. \tag{15}$$

On the other hand, it is easy to see that if $\nu f=f$, then $\Delta\alpha_y f=\tau_x\alpha_x f$. From (13), (15) and the fact that $(1-\nu)\alpha_x\phi_{n-1}=0$, we obtain

$$(1-\nu)\psi_n=(1-\Delta)^{-1}(-\delta_x'\nu\delta_y+\delta_x\delta_y')\alpha_x\phi_{n-1}. \tag{16}$$

From (3), it is easy to calculate that

$$\delta_x'\nu\delta_y-\delta_x\delta_y'=\delta_x'(\delta_y-\delta_y')\tau_x-(\delta_x-\delta_x')\delta_y'+\delta_x'\delta_y'(\nu-1). \tag{17}$$

From (16) and (17), (14) follows. □

Let $\mathcal{W}=\{\,(x,y,c):x\in X,y\in Y,c\in Q\,\}$. Define the product

$$(x^0,y^0,c^0)(x^1,y^1,c^1)=\big(x^0x^1,y^0y^1,c^0c^1q(x^0,y^1)\big)$$

in $\mathcal{W}$; then $\mathcal{W}$ is a group. Define a mapping from $\mathcal{W}$ to $\mathcal{A}$ as $p(x,y,c)=cyx$. Then the "curvature" of this mapping p is defined as $\omega(w^0,w^1)=p(w^0w^1)-p(w^0)p(w^1)$, $w^0,w^1\in\mathcal{W}$. For $n\geq k$, define the Chern character of dimension $2n-1$ (see [1] and [6]) as

$$\begin{aligned}&\mathrm{ch}_{2n-1}(w^0,\dots,w^{2n-1})\\&\quad=\tau\left(\omega(w^0,w^1)\cdots\omega(w^{2n-2},w^{2n-1})-\omega(w^{2n-1},w^0)\cdots\omega(w^{2n-3},w^{2n-2})\right).\end{aligned}$$

A function $f(w^0,\dots,w^n)$ is said to be homogeneous if $f\left((u^0,c^0),\dots,(u^n,c^n)\right)=\prod_{j=0}^n c^jf\left((u^0,1),\dots,(u^n,1)\right)$. For the homogeneous function f, we always rewrite $f\big((u^0,1),\dots,(u^n,1)\big)$ as $f(u^0,\dots,u^n)$ or $f(x^0,\dots,x^n;y^0,\dots,y^n)$ for $u^j=(x^j,y^j)$, $x^j\in X$, $y^j\in Y$. It is obvious that ch_{2n-1} is homogeneous. Thus we only have to study $\mathrm{ch}_{2n-1}(u^0,\dots,u^n)$ for $(u^j,1)\in\mathcal{W}$. The Hochschild boundary bf of a homogeneous function f is

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^jq(x^j,y^{j+1})f(x^0,\dots,x^jx^{j+1},\dots,x^n;y^0,\dots,y^jy^{j+1},\dots,y^n)\\&\qquad+(-1)^{n+1}q(x^{n+1},y^0)f(x^{n+1}x^0,\dots,x^n;y^{n+1}y^0,\dots,y^n).\end{aligned}$$

A function $F(f_k,\dots,f_l)(x^0,\dots,x^m;y^0,\dots,y^m)$, $f_j\in C^{j,j}, j=k,\dots,l$ and $x^r\in X$, $y^s\in Y$, is said to be a linear functional if it is expressed as $\sum_{s=1}^N c_sh_s$, where c_s is a function of $q(x^i,y^j)$, $i,j=0,\dots,m$, and $h_s(x^0,\dots,x^m;y^0,\dots,y^m)$ is of the form

$$f_j\left(x^{l_{01}}\cdots x^{l_{0s_0}},\dots,x^{l_{j1}}\cdots x^{l_{js_j}};y^{r_{01}}\cdots y^{r_{0t_0}},\dots,y^{r_{j1}}\cdots y^{r_{jt_j}}\right)$$

for certain $j\in\{k,\dots,l\}$ where $(l_{01},\dots,l_{0s_0},\dots,l_{j1},\dots,l_{js_j})$ is $(a,a+1,\dots,a+m)$ for some a, $(r_{01},\dots,r_{0t_0},\dots,r_{j1},\dots,r_{jt_j})$ is $(c,c+1,\dots,c+m)$ for some c, and $x^n=x^{n-m-1}$, $y^n=y^{n-m-1}$ for $n>m$.

Theorem 2. *There is a linear functional* $F_m(\phi_{k-1}, \dots, \phi_{2m-3})$ *such that*

$$ch_{2m-1}\big((x^0, y^0), \dots, (x^{2m-1}, y^{2m-1})\big) = b(1-\Delta)^{-1}(-1)t_x\phi_{2m-2} + F_m(\phi_{k-1}, \dots, \phi_{2m-3}).$$

Proof. Let $n = 2m-1$ and $f = -t_x^{-1}\,\mathrm{ch}_n$. Then

$$f(x^0, \dots, x^n; y^0, \dots, y^n) = (1-\nu)\tau(v^0 \cdots v^{m-1})$$

where $v^j = x^{2j}y^{2j}\left\{x^{2j+1}, y^{2j+1}\right\}$. From (9) and (10), through a complicated calculation, we can prove that $\tau(v^0 \cdots v^{m-1}) = \psi_n + R_n(\psi_{n-1}) + G_n(\psi_k, \dots, \psi_{n-2})$, where $G_n(\psi_k, \dots, \psi_{n-2})$ is a linear functional,

$$R_n(f) = \sum_{i=2}^{n-1}\sum_{j=0}^{2[\frac{i}{2}]-1}(-1)^{i+j-1}c_{ij}f(x^0, \dots, x^ix^{i+1}, \dots, x^n; y^0, \dots, y^jy^{j+1}, \dots, y^n) + \sum_{j=0}^{n-2}(-1)^jc_{nj}f(x^{n+1}x^0, x^1, \dots, x^n; y^0, \dots, y^jy^{j+1}, \dots, y^n),$$

and $c_{ij} = \prod_{l=j+1}^{i} q(x^l, y^l)$. We can prove that if $f \in C^{n-1,n-1}$, then

$$(18)\quad (1-\nu)R_n(f) + (-\delta_x'(\delta_y - \delta_y')\tau_x + (\delta_x - \delta_x')\delta_y')f = Q_n\big((1-\nu)f\big) + S_n(f),$$

where

(19)

$$Q_n(g) = \sum_{i=2}^{n-1}\sum_{j=0}^{i-2}(-1)^{i+j-1}c_{ij}g(x^0, \dots, x^ix^{i+1}, \dots, x^n; y^0, \dots, y^jy^{j+1}, \dots, y^n)$$

and $S_n(f) = -t_x^{-1}bt_x(f)$, if $\nu f = f$. From (13), $G_n(\psi_k, \dots, \psi_{n-2})$ can be expressed as a linear functional $H_n(\phi_{k-1}, \dots, \phi_{n-3})$. From (13), (14), (18) and (19), we have

$$(1-\nu)\psi_n + (1-\nu)R_n(\psi_{n-1}) = -t_x^{-1}b(1-\Delta)^{-1}t_x\alpha_x\phi_{n-1} - S_n\big((1-\Delta)^{-2}\Delta\delta_x\delta_y\alpha_x\alpha_y\phi_{n-2}\big),$$

which proves the theorem, where $F_m = t_xS_n\big((1-\Delta)^{-2}\Delta\delta_x\delta_y\alpha_x\alpha_y\phi_{n-2}\big) - t_xH_m$. □

Corollary. *ch_1 and ch_3 are boundaries of cyclic cochains.*

Proof. By the proof of Theorem 2, $F_1 = 0$. Thus ch_1 is the boundary of a cyclic zero-cochain. By formula (19), through calculation we may prove that $F_2 = bA(1-\Delta)^{-2}G$, where $A = (1+\nu+\nu^2)$,

$$G = \frac{1}{2}f_1(x^0, x^1x^2; y^0, y^1, y^2)q(x^0, y^0)^{-1}q(x^1, y^1)^{-1}q(x^2, y^1y^2)^{-1} - \frac{1}{2}f_1(x^0, x^1x^2; y^1y^2, y^0)q(x^1, y^2) + f_1(x^0, x^1x^2; y^1, y^2y^0)q(x^2, y^2)^{-1} - f_1(x^0, x^1x^2; y^0y^1, y^2)q(x^0, y^0)^{-1}q(x^2, y^2)^{-1}$$

and $f_1 = \alpha_x^2\tau_x\phi_1$. Therefore ch_3 is also the boundary of some cyclic 2-cochain. □

Remark. Although the Chern character defined here is based on the mapping from the group $\mathcal{W}$ to the algebra $\mathcal{A}$, it is not difficult to prove that Theorem 2 and its corollary for the Chern character defined through the mapping form a suitable group algebra of $\mathcal{W}$ to $\mathcal{A}$.

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