GENERALIZED CYCLIC COHOMOLOGY
ASSOCIATED WITH DEFORMED COMMUTATORS

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Abstract. The generalized cyclic cohomology is introduced which is associated with $q$-deformed commutators $xy - qyx$. Some formulas related to the trace of the product of $q$-deformed commutators are established. The Chern character of odd dimension associated with $q$-deformed commutators is studied.

1. Introduction

In non-commutative differential geometry [1], the Chern character of a $p$-summable Fredholm module is expressed by the trace of some product of quantized differentials $df = [F, f]$, where $[F, f]$ is the commutator $Ff - fF$ and $F$ is a self-adjoint idempotent operator. In the odd dimension case, the Chern character $ch_{2n-1}$ is expressed as

$$
\text{tr}(\omega(a^0, a^1) \cdots \omega(a^{2n-2}, a^{2n-1}) - \omega(a^{2n-1}, a^0) \cdots \omega(a^{2n-3}, a^{2n-2}))
$$

where $\omega(x, y) = p(xy) - p(x)p(y)$ is the curvature of some mapping $p$ (see also [2], [6]).

Let $A$ be an algebra over $\mathbb{C}$, and let $C^n$ be the space of $n + 1$-linear functions on $A$. The basic operations in the cyclic cohomology are $b$, $b'$, $t$, etc. (cf. [1], [5]), where $b$ is the Hochschild boundary operation. Define $C^n_\lambda = \{ f \in C^n : tf = f \}$ and $Af = (1 + t + \cdots + t^n)f$, $f \in C^n$. Let $pf = \sum_{j=0}^n (n-j) t^j f$, $f \in C^n$. Define $S = bpb'$. The restriction of $2\pi i S$ at $Z^n_\lambda = \{ f \in C^n_\lambda : bf = 0 \}$ coincides with A. Connes’ $S$ operator (cf. [1], [12]).

Related to the Chern character, in the previous papers [11], [12], the author studied the cyclic cohomology associated with the product of commutators. Let $X$ and $Y$ be two subalgebras (or subgroups) of a unital algebra $A$. Let $C^{m,n}$ be the space of multi-linear functions (or functions) $f(x^0, \ldots, x^m; y^0, \ldots, y^n)$, $x^i \in X$ and $y^j \in Y$. Let $b_x, b'_x, t_x, A_x, S_x$ and $b_y, b'_y, t_y, A_y, S_y$ be the operators $b, b'$, $t$, $A$, $S$ with respect to the $x$’s and $y$’s respectively. Let $C^{m,n}_\lambda = \{ f \in C^{m,n} : t_x f = t_y f = 0 \}$.

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Suppose that there is a trace ideal $J$ in $\mathcal{A}$ with trace $\tau$. Assume that there is a natural number $k$ such that $[x^i,y^j] \cdots [x^k,y^k] \in J$, for $x^i \in X$ and $y^j \in Y$. Define

$$\phi_n(x^0,\ldots,x^n;y^0,\ldots,y^n) = \tau[x^0,y^0] \cdots [x^n,y^n], \quad \text{for } n \geq k-1$$

and

$$\psi_n(x^0,\ldots,x^n;y^0,\ldots,y^n) = \tau x^0 y^0 [x^1,y^1] \cdots [x^n,y^n], \quad \text{for } n \geq k.$$ 

Then $A_x \phi_n = A_y \phi_n$ and it is denoted by $A \phi_n$.

**Theorem A [11].** For $n \geq k$, there is a $\Theta_n \in C^{n,n}_\lambda$ such that $A \phi_n = b_x b_y \Theta_n + \phi_{n+1}$ where

$$\phi_{2m+p} = (-1)^m \frac{(p+1)!}{(p+2m)!} S_x^m S_y^m \phi_p, \quad p = k, k-1,$$

and there is a $\hat{\Theta}_n \in C^{n,n}_\lambda$ such that $A \phi_n = b_x b_y \hat{\Theta}_n + \hat{\phi}_{n+1}$ where

$$\hat{\phi}_{2m+p} = (-1)^m \frac{p!}{(p+2m)!} S_x^m S_y^m A \phi_p, \quad \text{for } p = k+1, k,$$

and the functions $\Theta_n$ and $\hat{\Theta}_n$ are expressed by $\psi_p, \ldots, \psi_n$.

By means of Theorem A (in the case of $k=1$) and the analytic model, the author proved that the function trace $\left[(\lambda^0 - S^*)^{-1}, (\mu^0 - S)^{-1}\right] \left[(\lambda_1 - S^*)^{-1}, (\mu_1 - S)^{-1}\right]$, $\lambda^i, \mu^i \in \text{sp}(S)$, is a complete unitary invariant for some subnormal operator $S$ with trace class commutator $[S^*,S]$ (cf. [14], [15]).

For the unbounded operator case, the author (cf. [9], [10]) also studied the almost unperturbed Schrödinger pair of operators $u$ and $v$ which are self-adjoint operators on a Hilbert space $\mathcal{H}$ satisfying the condition that

$$e^{iu_x} e^{iv_t} - e^{iut} e^{ivt} \in L^1(\mathcal{H}), \quad s, t \in \mathbb{R},$$

where $L^1(\mathcal{H})$ is the trace class of operators on $\mathcal{H}$. If we denote $e^{iu_x}$ and $e^{ivt}$ by $x$ and $y$ respectively, then $q(x,y) = e^{iut}$ is a complex number determined by $x$ and $y$. Therefore instead of the commutator we have to study the trace class $q$-deformed commutator $\{x,y\} \overset{\text{def}}{=} xy - q(x,y)yx$. By the way, now-a-day the study of the $q$-deformed (or $q$-twisted) commutators becomes an interesting subject (cf. [3], [4], [7], etc.). In [9] and [10], the author studied the form of cyclic one-cocycles associated with $q$-deformed commutators, which has some application for establishing the theory of principal distribution and others.

The first aim of the present paper is to generalize Theorem A to the $q$-deformed commutators case. Suppose that there is a function $q(x,y)$, $x \in X, y \in Y$, satisfying $q(1,y) = q(x,1) = 1$ and $q(x^1 x^2, y^1 y^2) = \prod_{i,j=1}^2 q(x^i, y^j)$. Assume that there is a natural number $k$ such that $\{x^1, y^1\} \cdots \{x^k, y^k\} \in J$, $x^i \in X, y^i \in Y$. We define new functions $\phi_n$ and $\psi_n$ by changing the commutator to a $q$-deformed commutator in (1) and (2). In §4 of the present paper, we give Theorem 1 on these functions $\phi_n$ and $\psi_n$ which is a generalization of Theorem A in this $q$-deformed commutator case.
This generalization provides a possibility to study some complete unitary invariants for some unbounded hyponormal operators or pseudo-differential operators $u + iv$ where the pair of $u$ and $v$ is an almost unperturbed Schrödinger pair of operators.

In Theorem 1, the formulas are established for the functions $\phi_n, \ldots$ on the "manifolds"

$$M^{n,n} = \{ (x^0, \ldots, x^n; y^0, \ldots, y^n) \in X^n \times Y^n : \prod_{i,j=1}^n q(x^i, y^j) = 1 \}.$$ 

Off these manifolds $M^{m,m}$, the $q$-deformed commutator case is quite different from the commutator case. The second aim of the present paper is to study the structure (see Theorem 2 of §5) of the Chern character $\chi_{2n-1}$ of $2n - 1$ dimension associated with the $q$-deformed commutator off the manifold $M^{m,m}$. In the lower dimension cases, it is calculated in the corollary of Theorem 2 that the Chern characters $\chi_1$ and $\chi_3$ are boundaries of some cyclic cochain off $M^{n,n}$. Further study will be needed to answer the question whether or not all the Chern characters $\chi_m$ of odd dimensions associated with $q$-deformed commutators are boundaries of some cyclic cochains off the manifold $M^{m,m}$. Theorem 2 may provide a basis for this study.

We have to point out that the Chern characters of odd dimension have not been fully studied neither on the manifold $M^{n,n}$ (for the $q$-deformed commutator case), nor for the commutator case.

All of these studies we mentioned above are based on some new tools, the operations $\delta_x$, $\delta'_x$, $\tau_x$ and $\delta_y$, $\delta'_y$, $\tau_y$, which are the generalizations of $b_x$, $b'_x$, $t_x$ and $b_y$, $b'_y$, $t_y$ respectively. The formulation of this study is given in §2, which is a set of modified definitions of Hochschild cohomology and cyclic cohomology. The setting is that of a semidirect product of groups, i.e. a group $X$ acting on a group $Y$ by automorphisms $q_x$. This returns to the ordinary case if $q_x$ acts trivially, i.e. $q_x y = y$ for all $x, y$. Although in §2 the concept of a $q$-deformed commutator is not needed, the modified cyclic cohomology operations are introduced for obtaining the formulas of Lemma 1 in §3. These are formulas connecting $\phi_n, \psi_n,$ etc., which are the trace of products of some $q$-deformed commutators $\{x, y\} = xy - q_x(y)x$, $x \in X, y \in Y$. These formulas are necessary for establishing Theorems 1 and 2. It was not easy to find out those definitions in §2. Although in Theorem 1 and the proof of Theorem 2, $q_x(y)$ is simply $q(x, y)y$, the formulation adopted in §2 and §3 is for general automorphisms $q_x$ for two reasons. First, even if we restrict ourselves on the simpler case $q_x(y) = q(x, y)y$, we cannot simplify either notation or formulas in §2 and §3. The more important reason is that the present formulation may set a basis for further study. As a matter of fact, in [13], the author studied a special case about the perturbation of some partial differential operators in which $q_x(y)$ is not $q(x, y)y$ but the Chern character of dimension one is the boundary of a zero cyclic cochain off a lower dimensional manifold. That case is not covered in §5 of the present paper. Therefore the setting for general $q_x(\cdot \cdot)$ in §2 may provide a tool to calculate certain Chern characters of odd dimension associated with $q$-deformed commutators in which $q_x(\cdot \cdot)$ is not $q(x, y)y$.

In the statement of Theorem 2 and its Corollary, only cyclic cohomology is involved, but in their proof, the formulas in generalized cyclic cohomology are involved.

This paper is only an introduction to a circle of ideas whose natural continuation will be explained in subsequent papers.
2. Basic definitions

Let \( X \) and \( Y \) be two groups. Let 1 be the identity of \( X \) and \( Y \). Suppose that for every \( x \in X \) there is an automorphism \( q_x : Y \to Y \) satisfying \( q_x : q_x = q_x \), \( q_1 = \text{identity mapping} \). For \( m \geq 0 \) and \( n \geq 0 \), let \( C^{m,n} = C^{m,n}(X,Y) \) be the space of functions

\[
f_{m,n}(x^0, \ldots, x^m; y^0, \ldots, y^n), \quad x^i \in X, \ y^j \in Y.
\]

Define \( \delta_x^e \) and \( \delta_x : C^{m,n} \to C^{m+1,n} \) in the following. For \( f \in C^{m,n}, \ x = (x^0, \ldots, x^{m+1}) \) and \( y = (y^0, \ldots, y^n) \),

\[
(\delta_x^e f)(x; y) = \sum_{j=0}^{m-n} (-1)^j f(x^0, \ldots, x^j x^{j+1}, \ldots, x^{m+1}; y)
\]

\[
+ \sum_{j=m-n+1}^{m} (-1)^j f(x^0, \ldots, x^j x^{j+1}, \ldots, x^{m+1}; q_{x^{m-n+1}}(y^0), \ldots, q_{x^{m+1}}(y^n))
\]

and

\[
(\delta_x f)(x; y) = (\delta_x^e f)(x; y)
\]

\[
+ (-1)^{m+1} f(x^{m+1}, x^0, x^1, \ldots, x^m; q_{x^{m-n+1}}(y^0), \ldots, q_{x^{m+1}}(y^n)),
\]

if \( m \geq n \); and

\[
(\delta_x^e f)(x; y) = \sum_{j=0}^{m-n} (-1)^j f(x^0, \ldots, x^j x^{j+1}, \ldots, x^{m+1}; y^0, \ldots, y^{n-m-2}, q_{x^0}(y^{n-m-1}), \ldots, q_{x^1}(y^{n-m+j-1}), y^{n-m+j}, \ldots, y^n)
\]

and

\[
(\delta_x f)(x; y) = (\delta_x^e f)(x; y) + (-1)^{m+1} f(x^{m+1}, x^0, \ldots, x^m; q_{x^{m+1}}(y^0), \ldots, q_{x^{m+1}}(y^{n-m-2}), q_{x^{m+1}}(y^{n-m-1}), q_{x^1}(y^{n-m}), \ldots, q_{x^{m+1}}(y^n)),
\]

if \( 0 \leq m < n \). Define \( \delta_y^e \) and \( \delta_y : C^{m,n} \to C^{m,n+1} \) as follows. For \( f \in C^{m,n}, \ x = (x^0, \ldots, x^m) \) and \( y = (y^0, \ldots, y^{n+1}) \),

\[
(\delta_y^e f)(x; y) = \sum_{j=0}^{n-m-1} (-1)^m f(x; y^0, \ldots, y^j y^{j+1}, \ldots, y^{n+1})
\]

\[
+ \sum_{j=n-m}^{n} (-1)^j f(x; y^0, \ldots, y^{n-m-1}, q_{x^0}(y^{n-m}), \ldots, q_{x^{n-m+m}}(y^j) y^{j+1}, \ldots, y^{n+1})
\]

and

\[
(\delta_y f)(x; y) = (\delta_y^e f)(x; y)
\]

\[
+ (-1)^{n+1} f(x; y^{n+1}, y^0, \ldots, y^{n-m-1}, q_{x^0}(y^{n-m}), \ldots, q_{x^m}(y^n)),
\]
if \( m < n \); and
\[
(\delta'_y f)(x; y) = \sum_{j=0}^{n} (-1)^j f(x; q_{x}^{-1}(y^0), \ldots, q_{x}^{-1}(y^{j+1}), y^{j+2}, \ldots, y^{n+1})
\]
and
\[
(\delta_y f)(x; y) = (\delta'_y f)(x; y) + (-1)^{n+1} f(x; q_{x}^{-1}(y^0), q_{x}^{-1}(y^1), \ldots, q_{x}^{-1}(y^n))
\]
if \( m \geq n \geq 0 \). It can be verified through calculation that \( \delta_x^2 = \delta_x = 0 \) and \( \delta_y^2 = \delta_y = 0 \). If \( q_x \) is identity, then \( \delta_x = b_x \) and \( \delta_y = b_y \) in [11]. These \( \delta_x \) and \( \delta_y \) are the generalized Hochschild boundary operations in some sense.

Define \( \tau_x : C^{m,n} \to C^{m,n} \) and \( \tau_y : C^{m,n} \to C^{m,n} \) as follows. For \( f \in C^{m,n}, x = (x^0, \ldots, x^m) \) and \( y = (y^0, \ldots, y^n) \),
\[
(\tau_x f)(x; y) = (-1)^m f(x^m, x^0, \ldots, x^{m-1}, q_{x}^{-1}(y^0), \ldots, q_{x}^{-1}(y^n)),
\]
\[
(\tau_y f)(x; y) = (-1)^n f(x; q_{x}^{-1}(y^0), q_{x}^{-1}(y^1), \ldots, q_{x}^{-1}(y^n))
\]
if \( m \geq n \); and
\[
(\tau_x f)(x; y) =
(-1)^m f(x^m, x^0, \ldots, x^{m-1}, q_{x}^{-1}(y^0), \ldots, q_{x}^{-1}(y^n), q_{x}^{-1}(y^{n-m}), \ldots, q_{x}^{-1}(y^n)),
\]
\[
(\tau_y f)(x; y) =
(-1)^n f(x; y^n, y^0, \ldots, y^{n-m-2}, q_{x}^{-1}(y^{n-m-1}), \ldots, q_{x}^{-1}(y^n))
\]
if \( m < n \). If \( q_x = \text{identity} \), then \( \tau_x = t_x \) and \( \tau_y = t_y \) in [12]. Through complicated calculation it can be verified that the operations \( \delta_x, \delta'_x, \tau_x \) commute with \( \delta_y, \delta'_y, \tau_y \).

We also have
\[
(3) \quad \delta'_x (1 - \tau_x) = (1 - \tau_x) \delta_x, \quad \delta'_y (1 - \tau_y) = (1 - \tau_y) \delta_y.
\]
These formulas are the generalizations of the formula \( b'(1 - t) = (1 - t)b \) in [5]. Define
\[
\alpha_x f = (1 + \tau_x + \cdots + \tau_x^m)f, \quad \alpha_y f = (1 + \tau_y + \cdots + \tau_y^n)f
\]
for \( f \in C^{m,n} \) which are the generalizations of operators \( A_x \) and \( A_y \) in [12]. Then
\[
(4) \quad \alpha_x \delta'_x = \delta_x \alpha_x, \quad \alpha_y \delta'_y = \delta_y \alpha_y.
\]
For \( n \geq 0 \) let \( p_n(z) = \sum_{j=0}^{n} (n - j) z^j \). Define \( \pi_x f = p_m(\tau_x)f \) and \( \pi_y f = p_n(\tau_y)f \) for \( f \in C^{m,n} \). Define \( \sigma_x = \delta_x \pi_x \delta'_x \) and \( \sigma_y = \delta_y \pi_y \delta'_y \). These are generalizations of Connes’ operators \( S_x \) and \( S_y \) in [12]. Then \( \sigma_x \) commutes with \( \sigma_y \). From (3) it is easy to see that \( \delta'_x \delta_x \alpha_x = -(1 - \tau_x)\sigma_x, \delta'_y \delta_y \alpha_y = -(1 - \tau_y)\sigma_y \). From this, we may prove that \( \sigma_x \) commutes \( \delta_x \alpha_x \) and \( \sigma_y \) commutes \( \delta_y \alpha_y \).
3. Trace of product of deformed commutators

Suppose $X$ and $Y$ are also subgroups of an algebra $A$ over $\mathbb{C}$. Define

$$\{x, y\} = xy - q_x(y)x, \quad x \in X, \ y \in Y,$$

where $q_x$ satisfies the conditions in §2. This $\{x, y\}$ is called the $q$-deformed commutator of $x$ and $y$. Suppose that there is a trace ideal $J$ of $A$ with a trace $\tau$ on $J$, i.e. $\tau$ is a linear functional on $J$ satisfying $\tau(ab) = \tau(ba)$ for $b \in J$ and $a \in A$. Assume that there is a natural number $k$ such that $\{x^1, y^1\} \cdots \{x^k, y^k\} \in J$. For $n \geq k$, define $\psi_n(x^0, \ldots, x^n; y^0, \ldots, y^n) = \tau x^0y^0\{x^1, y^1\} \cdots \{x^n, y^n\}$, for $x^j \in X$, $y^j \in Y$. Then $\psi_n \in C^{n,n}$. Define functions

$$\xi_n(x)^0, \ldots, y^{n-1}) = \psi_n(x; 1, y^0, \ldots, y^n), \quad n \geq k,$$

$$\eta_n(x^0, \ldots, x^{n-1}; y) = \psi(1, x^0, \ldots, x^n; y), \quad n \geq k,$$

$$\phi_n(x; y) = \psi_{n+1}(1, x^0, \ldots, x^n; 1, y^0, \ldots, y^n), \quad n \geq k - 1,$$

where $x = (x^0, \ldots, x^n)$ and $y = (y^0, \ldots, y^n)$. The following lemma gives the basic relations between $\psi_n, \xi_n, \eta_n,$ and $\phi_{n-1}$.

**Lemma 1.** For $n \geq k$,

$$\begin{align*}
(5) \quad & \xi_{n+1} = \delta_x \psi_n, \\
(6) \quad & (1 - \tau_x)\xi_n = \delta'_x \phi_{n-1}, \\
(7) \quad & (1 - \tau_y)^{-1}\xi_n = \delta_x \phi_{n-1}, \\
(8) \quad & (1 - \tau_y)\psi_n = \delta'_x \eta_n, \\
& (1 - \tau_y)\psi_n = \delta'_x \eta_n + \phi_n.
\end{align*}$$

**Proof.** We only give the proofs of those formulas in which $\xi$’s are involved. The others can be proved similarly. The basic formulas for deformed commutators are

$$\begin{align*}
(9) \quad & \{x_1x_2, y\} = x_1\{x_2, y\} + \{x_1, q_{x_2}(y)\}x_2 \\
\quad & \{x, y_1y_2\} = \{x, y_1\}y_2 + q_x(y_1)\{x, y_2\},
\end{align*}$$

for $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$. To prove (5), by means of (9), we have

$$\begin{align*}
\xi_{n+1}(x; y) &= \tau x^0y^0\{x^1, y^1\} \cdots \{x^{n+1}, y^n\} - \tau x^0q_x(y^0)\{x^1, y^1\} \cdots \{x^{n+1}, y^n\} \\
&= \psi_n(x^0, x^1, \ldots, x^{n+1}; y) - \psi_n(x^0, x^1, \ldots, x^{n+1}; q_x(y^0), y^1, \ldots, y^n) \\
&\quad + \tau x^0q_x(y^0)\{x^1, q_{x^2}(y)\}x^2\{x^3, y^2\} \cdots \{x^{n+1}, y^n\},
\end{align*}$$

for $x = (x^0, \ldots, x^{n+1})$ and $y = (y^0, \ldots, y^n)$. Continuing this process, we may prove (5).

To prove (6), by means of (9), we have

$$\xi_n(x; y) = \phi_{n-1}(x^0, \ldots, x^n; y) - \tau\{x^0, q_{x^1}(y^0)\}x^1\{x^2, y^1\} \cdots \{x^n, y^{n-1}\}.$$
The last term of the right-hand side of the above formula equals

\[- \phi_{n-1}(x^0, x^1, x^2, \ldots, x^n, q_x(y^0), y^1, \ldots, y^{n-1}) + \tau \{ x^0, q_x(y^0) \} \{ x^1, q_x(y^1) \} x^2 \{ x^3, y^2 \} \ldots.\]

Continuing this process, we may prove (6). By the definitions and (9), it is easy to verify that

\[\tau x \xi_n - \tau y^{-1} \xi_n = \delta_x \phi_{n-1} - \delta_y \phi_{n-1}.\]

Therefore (7) follows from (6). To prove (8), by means of (10), we observe that

\[\psi_n(x; y) - \phi_n(x; y) = \tau x^0 \{ x^1, y^1 \} \cdots \{ x^n, q_x(y^0) \} = \xi_n(x; y^1, \ldots, y^{n-1}, y^n, q_x(y^0)) - \tau x^0 \{ x^1, y^1 \} \cdots \{ x^{n-1}, y^{n-1} \} q_x(y^n) \{ x^n, q_x(y^0) \}.\]

Continuing this process we may prove that \(\psi_n(x; y) - \phi_n(x; y)\) equals

\[- \sum_{j=2}^{n+1} (-1)^{n+1-j} \xi_n(x; y_1, \ldots, y_{j-1} q_x(y^j), q_{x+1}(y^{j+1}), \ldots, q_{x+n}(y^{n+1})) + (-1)^n \psi_n(x; q_x(y^1), \ldots, q_{x+n}(y^{n+1})),\]

where \(x^{n+1} = x^0, y^{n+1} = y^0\). Thus \(\psi_n - \phi_n = -\tau_y^{-1} \delta_y \xi_n + \tau_y^{-1} \psi_n\) which proves (8).

\[\square\]

4. A BASIC THEOREM

Let us consider a very important special case. Suppose that there is a function \(q(\cdot, \cdot)\) on \(X \times Y\) satisfying

\[(11) \quad q(x^j, x^j; y^1, y^j) = \prod_{i, j=1}^{2} q(x^i, y^j) \quad x^j \in X, \ y^j \in Y,\]

and

\[(12) \quad q(1, y) = q(x, 1) = 1, \quad x \in X, y \in Y.\]

Let \(Q = \{ q(x, y) : x \in X, y \in Y \} \subset \mathbb{C}\); then \(Q\) is a group. Let \(\tilde{Y} = \{ cy : c \in Q, y \in Y \}\). Define \(q_x(cy) = c^2 q(x, y), c \in Q, y \in Y\). Then \(q_x : \tilde{Y} \rightarrow \tilde{Y}\) is an isomorphism satisfying condition (1). Let \(\tilde{C}^{m,n} = \tilde{C}^{m,n}(X, \tilde{Y})\) be the space of the restriction of functions \(f\) in \(C^{m,n}(X, \tilde{Y})\) satisfying the condition that

\[f(x; c^0 y^0, \ldots, c^n y^n) = \prod_{j=0}^{n} c^j f(x; y^0, \ldots, y^n)\]

for \(c^j \in Q, y^j \in Y\). For example, \(\psi_n \in \tilde{C}^{m,n}\). Then \(\tilde{C}^{m,n}\) is invariant with respect to the set \(D = \{ \delta_x, \delta_x, \tau_x, \delta_y, \delta_y, \tau_y \}\) of operations defined in \(C^{m,n}(X, \tilde{Y})\).
Let us consider the restriction on $\tilde{C}^{m,n}$ of those operations in $D$ only. Then these operations in $D$ possess all the properties described in §2.

Let $M^m = \{(x^0, \ldots, x^m; y^0, \ldots, y^n) : x^i \in X, y^j \in Y, \prod_{i=0}^m \prod_{j=0}^n q(x^i, y^j) = 1\}$ and $\tilde{C}^{m,n}$ be the restriction of the functions in $\tilde{C}^{m,n}$ on $M^m$. An important property of $\tilde{C}^{m,n}$ is that $\tau_x \alpha_1 f = \alpha_x f$, $\tau_y \alpha_y f = \alpha_y f$, $f \in \tilde{C}^{m,n}$. Define $C^{m,n}_\lambda = \{f \in \tilde{C}^{m,n} : \tau_x f = \tau_y f = f\}$. It is easy to see that $\alpha_x \phi_n(x; y) = \alpha_y \phi_n(x, y)$ for $(x, y) \in M^{m,n}$. Define $\alpha \phi_n = \alpha_x \phi_n = \alpha_y \phi_n$ on $M^{m,n}$. Then $\alpha \phi_n \in C^{m,n}_\lambda$. By the same method in [12], using Lemma 1 and the formulas in §2 and §3, we may prove the following theorem and omit the proof.

**Theorem 1.** For $n \geq k$, there is a $\Theta_n \in C^{m,n}_\lambda$ such that

\[
\alpha \phi_n+1 = \delta_x \delta_y \Theta_n + \phi_n+1, \quad \text{on } M^{m,n},
\]

where

\[
\hat{\phi}_{2m+p} = (-1)^m \frac{(p+1)!}{(p+2m)!} \sigma^m \sigma^m \phi_p.
\]

for $p = k, k-1$. For $n \geq k$, there is a $\Theta_n \in C^{m,n}_\lambda$ such that

\[
\alpha \phi_n+1 = \delta_x \delta_y \Theta_n + \phi_n+1, \quad \text{on } M^{m,n},
\]

where

\[
\hat{\phi}_{2m+p} = (-1)^m \frac{p!}{(p+2m)!} \sigma^m \sigma^m \alpha \phi_p
\]

for $p = k+1, k$. Besides, the functions $\Theta_n$ and $\Theta_n$ are expressed by $\psi_n, \ldots, \psi_p$.

5. Chern character of odd dimension

As in the previous sections, assume that $A$ is an algebra over $C$, and $J$ is a trace ideal in $A$ with trace $\tau$. Let $X$ and $Y$ be subgroups of $A$. Assume that there is a function $q(x, y)$, $x \in X$, $y \in Y$, satisfying conditions (11) and (12). Assume that there is a natural number $k$ such that $\{x^1, y^1\} \cdots \{x^k, y^k\} \in J$. Define $\Delta = \Delta_{m,n}(x^0, \ldots, x^n; y^0, \ldots, y^n) = \Pi_{i=0}^m \Pi_{j=0}^n q(x^i, y^j)$. Let $\nu = \tau_x \tau_y$, then $(\nu f)(x^0, \ldots, x^n; y^0, \ldots, y^n) = f(x^n, x^0, \ldots, x^{n-1}; y^n, y^0, \ldots, y^{n-1})$ for $f \in C^{m,n}$.

**Lemma 2.** For $n \geq k$ and $\Delta \neq 1$,

\[
\psi_n = (1 - \Delta)^{-1} \alpha_x \phi_n - (1 - \Delta)^{-2} \Delta \delta_x \delta_y \alpha_x \alpha_y \phi_n-1
\]

and

\[
(1 - \nu) \psi_n = (1 - \Delta)^{-1} \left[ -\delta_x' \delta_y' \tau_x + (\delta_x - \delta_x') \delta_y' \right] \alpha_x \phi_n-1.
\]

**Proof.** It is easy to see that $\alpha_x (1 - \tau_x) = 1 - \Delta$ and $\alpha_y (1 - \tau_y) = 1 - \Delta^{-1}$. Applying $\alpha_x$ to both sides of the right equality in (8), we have $(1 - \Delta) \psi_n = \delta_x \alpha_x \eta_n + \alpha_x \phi_n$, since $\alpha_x \delta_x' = \delta_x \alpha_x$ (see (4)). Similarly, from (6), we get $(1 - \Delta^{-1}) \eta_n = \delta_y \alpha_y \phi_n-1$. Thus we obtain (13).
A function of a homogeneous function $f$ and $(16)$ as

From (16) and (17), (14) follows.

For certain $j$, we have to study the "curvature" of this mapping $W$ in (17).

From (3), it is easy to calculate that

From (16), (17), (14) follows.

Let $W = \{ (x, y, c) : x \in X, y \in Y, c \in Q \}$. Define the product

in $W$; then $W$ is a group. Define a mapping from $W$ to $A$ as $p(x, y, c) = cyx$. Then the "curvature" of this mapping $p$ is defined as $\omega (w^0, w^1) = p (w^0, w^1) - p (w^0) p (w^1)$, $w^0, w^1 \in W$. For $n \geq k$, define the Chern character of dimension $2n - 1$ (see [1] and [6]) as

A function $f (w^0, \ldots, w^m)$ is said to be homogeneous if $f ((u^0, c^0), \ldots, (u^n, c^n)) = \prod_{j=0}^m c^j f ((u^0, c^0), \ldots, (u^n, c^n))$. For the homogeneous function $f$, we always rewrite $f ((u^0, 1), \ldots, (u^n, 1))$ as $f (u^0, \ldots, u^n)$ or $f (x^0,\ldots,x^n; y^0,\ldots,y^n)$ for $w^j = (x^i, y^j)$, $x^j \in X$, $y^j \in Y$. It is obvious that $\text{ch}_{2n-1}$ is homogeneous. Thus we only have to study $\text{ch}_{2n-1} (w^0, \ldots, w^n)$ for $(w^1, 1) \in W$. The Hochschild boundary $bf$ of a homogeneous function $f$ is

A function $F (f_k, \ldots, f_l) (x^0, \ldots, x^m; y^0, \ldots, y^n)$, $f_j \in C^{i,j}$, $j = k, \ldots, l$ and $x^r \in X$, $y^s \in Y$, is said to be a linear functional if it is expressed as $\sum_{s=1}^{N} c_s h_s$, where $c_s$ is a function of $q (x^i, y^j)$, $i, j = 0, \ldots, m$, and $h_s (x^0, \ldots, x^m; y^0, \ldots, y^n)$ is of the form

for certain $j \in \{k, \ldots, l\}$ where $(l_{00}, \ldots, l_{000}, \ldots)$. The $j$th is $(a, a + 1, \ldots, a + m)$ for some $a$, $(r_1, r_2, \ldots, r_m)$ is $(c, c + 1, \ldots, c + m)$ for some $c$, and $x^n = x^{n-m-1}$, $y^n = y^{n-m-1}$ for $n > m$. 

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Theorem 2. There is a linear functional $F_m(\phi_{k-1}, \ldots, \phi_{2m-3})$ such that 
\[
ch_{2m-1}((x^0, y^0), \ldots, (x^{2m-1}, y^{2m-1})) = b(1-\Delta)^{-1}(-1)t_x\phi_{2m-2} + F_m(\phi_{k-1}, \ldots, \phi_{2m-3}).
\]

Proof. Let $n = 2m-1$ and $f = -t_x^{-1}ch_n$. Then 
\[
f(x^0, \ldots, x^n; y^0, \ldots, y^n) = (1-\nu)^{n-1} \nu^{n-1}
\]
where $\nu^j = x^{2j}y^{2j} \{x^{2j+1}, y^{2j+1}\}$. From (9) and (10), through a complicated calculation, we can prove that $\tau(\nu^0 \cdots \nu^{n-1}) = \psi_n + R_n(\psi_{n-1}) + G_n(\psi_k, \ldots, \psi_{n-2})$, where $G_n(\psi_k, \ldots, \psi_{n-2})$ is a linear functional, 
\[
R_n(f) = \sum_{i=2}^{n-1} \sum_{j=0}^{n-2} (-1)^{i+j-1} c_{ij}f(x^0, \ldots, x^i x^{i+1}, \ldots, x^n; y^0, \ldots, y^j y^{j+1}, \ldots, y^n) + \sum_{j=0}^{n-2} (-1)^j c_{nj}f(x^{n-1} x^0, x^1, \ldots, x^n; y^0, \ldots, y^j y^{j+1}, \ldots, y^n),
\]
and $c_{ij} = \prod_{l=i+j} g(x^l, y^l)$. We can prove that if $f \in C^{n-1,n-1}$, then
\[
(1-\nu)R_n(f) + (-\delta_x^y(\delta_y^y - \delta_y^x)\tau_x + (\delta_x^y - \delta_y^x)\delta_y^x)f = Q_n((1-\nu)f) + S_n(f),
\]
where
\[
Q_n(g) = \sum_{i=2}^{n-1} \sum_{j=0}^{n-2} (-1)^{i+j-1} c_{ij}g(x^0, \ldots, x^i x^{i+1}, \ldots, x^n; y^0, \ldots, y^j y^{j+1}, \ldots, y^n)
\]
and $S_n(f) = -t_x^{-1}bt_x(f)$, if $\nu f = f$. From (13), $G_n(\psi_k, \ldots, \psi_{n-2})$ can be expressed as a linear functional $H_n(\phi_{k-1}, \ldots, \phi_{n-3})$. From (13), (14), (18) and (19), we have 
\[
(1-\nu)\psi_n + (1-\nu)R_n(\psi_{n-1}) = -t_x^{-1}b(1-\Delta)^{-1}t_x\alpha_x\phi_{n-1} - S_n((1-\Delta)^{-2}\Delta\delta_x^x\delta_y^y\alpha_x\phi_{n-2}),
\]
which proves the theorem, where $F_m = t_x S_n((1-\Delta)^{-2}\Delta\delta_x^x\delta_y^y\alpha_x\phi_{n-2}) - t_x H_n$. \(\Box\)

Corollary. $ch_1$ and $ch_3$ are boundaries of cyclic cochains.

Proof. By the proof of Theorem 2, $F_1 = 0$. Thus $ch_1$ is the boundary of a cyclic zero-cochain. By formula (19), through calculation we may prove that $F_2 = bA(1-\Delta)^{-2}G$, where $A = (1+\nu+\nu^2)$,
\[
G = \frac{1}{2} f_1(x^0, x^1 x^2; y^0, y^1, y^2)q(x^0, y^0)^{-1}q(x^1, y^1)^{-1}q(x^2, y^2)^{-1} - \frac{1}{2} f_1(x^0, x^1 x^2; y^1, y^0)q(x^1, y^2)^{-1} + f_1(x^0, x^1 x^2; y^1, y^2 y^0)q(x^2, y^2)^{-1} - f_1(x^0, x^1 x^2; y^0 y^1, y^2)q(x^0, y^0)^{-1}q(x^2, y^2)^{-1}
\]
and $f_1 = \alpha_x^2 \tau_x \phi_1$. Therefore $ch_3$ is also the boundary of some cyclic 2-cochain. \(\Box\)

Remark. Although the Chern character defined here is based on the mapping from the group $W$ to the algebra $A$, it is not difficult to prove that Theorem 2 and its corollary for the Chern character defined through the mapping form a suitable group algebra of $W$ to $A$. 

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