STABLE SPLITTINGS OF $BO(2n)$ AND $BU(2n)$

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ABSTRACT. Using the Snaith-Mitchell-Priddy splittings of $BO(2n)$ and $BU(2n)$, we can give new stable splittings of $BO(2n)$ and $BU(2n)$ respectively.

In [5] and [6] Snaith, Mitchell, and Priddy showed that the natural filtrations on $BO(n)$, $BU(n)$, and $BSP(n)$ stably split, respectively, and in [2] Henn and Mui showed the corresponding splitting for $BSO(2n+1)$ and $BSU(2n+1)$. The purpose of this note is to give new stable splittings of $BO(2n)$ and $BU(2n)$ respectively.

Let $g_{2n} : O(2n) \rightarrow SO(2n+1)$ be defined by $g_{2n}(\alpha) = det(\alpha) \oplus \alpha$. Then we have $B_{g_{2n}} : BO(2n) \rightarrow BSO(2n+1)$.

Let $Y_{2n}$ be the stable fibre of $B_{g_{2n}}$, that is,

$$Y_{2n} \xrightarrow{\delta_{2n}} BO(2n) \xrightarrow{B_{g_{2n}}} BSO(2n+1)$$

is a stable fibration.

**Theorem A.** Localized at the prime 2, the stable fibre $Y_{2n}$ of $B_{g_{2n}}$ is a stable summand of the stable fibre $Y_{2n+2}$ of $B_{g_{2n+2}}$.

**Corollary B.** Localized at the prime 2, for each $n$ there are 2-local spectra $F_{2i}$, $1 \leq i \leq n$, such that

$$F_2 \vee F_4 \vee F_6 \vee \cdots \vee F_{2n} \rightarrow BO(2n) \xrightarrow{B_{g_{2n}}} BSO(2n+1)$$

is a stable filtration and

$$BO(2n) \cong BSO(2n+1) \vee F_2 \vee F_4 \vee F_6 \vee \cdots \vee F_{2n}.$$

**Remark 1.** In [3], Mitchell and Priddy proved that the homotopy type of $F_2$ is $\sum_{-2}^{\infty} S^1 \wedge S^0 \vee P_1^\infty$ (see [3]).

**Remark 2.** We have analogous theorems for $BU(2n)$ and $BSU(2n+1)$.

From now on we will suppress details for $BU(2n)$, which can be obtained easily from those for $BO(2n)$, and all spaces or spectra are implicitly localized at prime 2.

**Lemma 1.** Let $Y_{2n} \xrightarrow{\delta_{2n}} BO(2n) \xrightarrow{B_{g_{2n}}} BSO(2n+1)$ be a stable fibration. Then $BO(2n) \cong BSO(2n+1) \vee Y_{2n}$.

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Proof. Since the fibre of
\[ Bg_2n : BO(2n) \to BSO(2n + 1) \]
is \( SO(2n + 1)/O(2n) \cong RP^{2n} \), the Becker-Gottlieb transfer [1] associated to this fibration is a stable map
\[ q_{2n} : BSO(2n + 1) \to BO(2n) \]
such that \((q_{2n})^* \circ (Bg_{2n})^*\) is multiplication by the Euler characteristic \(\chi(RP^{2n}) = 1(\text{mod } 2)\). Thus \(q_{2n}\) provides a stable splitting of \(Bg_{2n}\), and
\[ BO(2n) \cong BSO(2n + 1) \vee Y_{2n}. \]
This completes the proof.

Proof of Corollary B. This follows immediately from Theorem A and Lemma 1 by induction.

Lemma 2. There is a stable map \(\lambda_{2n} : BO(2n + 2) \to BO(2n)\) such that the composite map
\[ BO(2n) \xrightarrow{Bi_{2n}} BO(2n + 2) \xrightarrow{\lambda_{2n}} BO(2n) \]
is homotopic to the identity map, where the first map is the natural inclusion map.

Proof. This lemma follows immediately from [5] and [6].

Now we can prove Theorem A.

Proof of Theorem A. Let \(g_{2n}\) be the map \(g_{2n}(\alpha) = (\det \alpha) \oplus \alpha\). Then the diagram of groups
\[
\begin{array}{ccc}
O(2n) & \xrightarrow{g_{2n}} & SO(2n + 1) \\
\downarrow{i_{2n}} & & \downarrow{j_{2n+1}} \\
O(2n + 2) & \xrightarrow{g_{2n+2}} & SO(2n + 3)
\end{array}
\]
is strictly commutative, where \(i_{2n}\) and \(j_{2n+1}\) are the inclusions
\[(\det(\alpha) \oplus \alpha) \oplus 1 = \det(\alpha \oplus 1) \oplus (\alpha \oplus 1).\]
This yields a strictly commutative diagram of classifying spaces, and if \(Y_{2n}\) denotes the stable fibre of \(Bg_{2n}\), then we have a commutative diagram of stable maps:
\[
\begin{array}{ccc}
Y_{2n} & \xrightarrow{\delta_{2n}} & BO(2n) \\
\downarrow{k} & & \downarrow{Bi_{2n}} \\
Y_{2n+2} & \xrightarrow{\delta_{2n+2}} & BO(2n + 2)
\end{array}
\]
Now by Lemma 1, \(Bg_{2n}\) has a right inverse (in the stable category), that is, the fibration sequence splits, that is, \(\delta_{2n}\) has a left inverse, say \(\eta_{2n}\). By Lemma 2, \(Bi_{2n}\) has a left inverse, say \(\lambda_{2n}\). It follows that \(k\) has a left inverse
\[(\eta_{2n} \lambda_{2n} \delta_{2n+2})k = \eta_{2n} \lambda_{2n} Bi_{2n} \delta_{2n} = \eta_{2n} \delta_{2n} = 1.\]
This completes the proof of Theorem A.
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