MINIMAL PRIME IDEALS IN ENVELOPING ALGEBRAS
OF LIE SUPERALGEBRAS

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Abstract. Let \( g \) be a finite dimensional Lie superalgebra over a field of characteristic zero. Let \( U(g) \) be the enveloping algebra of \( g \). We show that when \( g = b(n) \), then \( U(g) \) is not semiprime, but it has a unique minimal prime ideal; it follows then that when \( g \) is classically simple, \( U(g) \) has a unique minimal prime ideal. We further show that when \( g \) is a finite dimensional nilpotent Lie superalgebra, then \( U(g) \) has a unique minimal prime ideal.

Throughout let \( g = g_0 \oplus g_1 \) be a finite dimensional Lie superalgebra over a field \( k \) of characteristic zero (see [S] or [K] for general definitions). Let \( U(g) \) be the enveloping algebra of \( g \). When \( g \) is a Lie algebra \( U(g) \) is a domain, but when \( g \) is a Lie superalgebra \( U(g) \) need not be semiprime. In this paper we will present some classes of Lie superalgebras for which we can prove that \( U(g) \) has a unique minimal prime ideal; we begin by noting that we know of no example of a finite dimensional Lie superalgebra \( g \) for which \( U(g) \) does not have a unique minimal prime ideal.

A. Bell gave a sufficient condition [Be, Theorem 1.5] for \( U \) to be a prime ring (so that 0 is the unique minimal prime ideal). Let \( ([y_i, y_j]) \) be the \( m \times m \) matrix of brackets of basis elements of \( g_1 \), and let \( d(g) \) be the determinant of that matrix, considered as a matrix with elements in the symmetric algebra \( S(g_1) \). If \( d(g) \neq 0 \), then \( U \) is prime. Using this criterion, Bell showed [Be, Corollary 3.6] that all the classically simple, finite dimensional Lie superalgebras \( g \) (except for one class \( b(n) \)) (called \( P(n-1) \) in the notation of [K]) have \( d(g) \neq 0 \), and hence \( U \) is prime. Since \( d(b(n)) = 0 \), Bell’s criterion could not determine whether \( U(b(n)) \) was prime. Behr [B] had shown \( U(b(2)) \) was not semiprime, but \( b(n) \) is classically simple only for \( n \geq 3 \). Bell [Be, Section 1.6] had further shown that when \( g_0 \) is supercentral in \( g \), then \( U \) has a unique minimal prime ideal, and he raised the question of whether \( U \) always has a unique minimal prime ideal. Here, in Section 2, we will show that \( U(b(n)) \) is not semiprime for all \( n \), but that \( U(b(n)) \) always has a unique minimal prime ideal. Hence the enveloping algebra \( U(g) \) of any classically simple Lie superalgebra \( g \) has a unique minimal prime ideal. This result has been used by Letzter and Musson in [LM]. Wilson [W] has shown that \( W(n) \), a class of simple Lie superalgebras of Cartan type, has a prime enveloping algebra when \( n \) is even and \( n \geq 4 \).
In Section 3 we consider a second class of Lie superalgebras, the nilpotent Lie superalgebras. We show that the enveloping algebra of a nilpotent Lie superalgebra always has a unique minimal prime ideal.

Both proofs make use of the fact, noted by Behr [B, Corollary, p. 21] that $U$ always has an Artinian quotient ring $Q$, obtained by inverting the nonzero elements of $U_0 = U(g_0)$. In Section 1 we note the fact that this quotient ring will always be a quasi-Frobenius ring; we will use this fact in Section 2 in the proof that $U(b(n))$ has a unique minimal prime ideal. This result is an easy consequence of results of Stafford and Zhang [SZ] on Auslander-Gorenstein rings. When $g$ is a Lie algebra, its enveloping algebra has many nice homological properties, properties not holding for general Noetherian rings; these properties include its finite global dimension, and the even stronger Auslander-regular property (which makes its homological properties more like those of a commutative regular ring). As was noted by Behr [B, Proposition 5], the enveloping algebras of Lie superalgebras need not have finite global dimension (e.g. only one class of classically simple Lie superalgebras $osp(1, n)$ has an enveloping algebra with finite global dimension). However, as noted in [KKS], the enveloping algebra of a Lie superalgebra always has finite injective dimension, and here we note the fact that it has the stronger property of being an Auslander-Gorenstein ring.

1. Homological properties of $U(g)$

In this section we will note some homological facts about $U$, which we will use to show that $Q$ is a quasi-Frobenius ring. This result will be used in Section 2 to show that $U(b(n))$ has a unique minimal prime ideal.

1.1. Let \{x_1, \ldots, x_n\} be a basis for $g_0$, and let \{y_1, \ldots, y_m\} be a basis for $g_1$. We shall make frequent use of the fact that $U$ has a PBW basis (see e.g. [B, Section 2]); namely, any element of $U$ can be written uniquely as a linear combination of the elements of the form $x_1^{e_1} \cdots x_n^{e_n} y_1^{f_1} \cdots y_m^{f_m}$, for nonnegative integers $e_i$ and $f_j = 0$ or 1. As noted in Behr [B], when $U$ is filtered by taking the $x$'s and $y$'s to have degree one, the associated graded ring $Gr(U) = \Lambda(y_1, \ldots, y_m)[x_1, \ldots, x_n]$ is a polynomial ring in $n$ indeterminates over the exterior algebra $\Lambda(y_1, \ldots, y_m)$. It then follows [KKS, Proposition 2.3] that $U$ has finite (right and left) injective dimension, $\text{injdim}(U) = \dim(g_0)$, the vector space dimension of $g_0$. We will use the homological properties of $Gr(U)$ to deduce further homological properties of $U$.

Recall that a ring $R$ is called Auslander-Gorenstein if $R$ is a Noetherian ring of finite (right and left) injective dimension with the additional property that for every finitely generated $R$-module $M$ and every submodule $N \subseteq \text{Ext}_R^i(M, R)$, one has $\text{Ext}_R^i(N, R) = 0$ for all $i < j$. Let $j(M) = \min\{j : \text{Ext}_R^j(M, R) \neq 0\}$; then $R$ is called (GKdim) Cohen-Macaulay provided that $\text{GKdim}(R) < \infty$, and $j(M) + \text{GKdim}(M) = \text{GKdim}(R)$ holds for every finitely generated $R$-module $M$.

Using a result of Stafford-Zhang, it follows easily that $U$ is an Auslander-Gorenstein, Cohen-Macaulay ring.

**Theorem 1.2.** The enveloping algebra $U(g)$ of a finite dimensional Lie superalgebra $g$ is an Auslander-Gorenstein, (GKdim) Cohen-Macaulay ring.

**Proof.** Filtering $U$ as in 1.1, the associated graded ring $Gr(U)$ is a Noetherian PI ring of finite injective dimension, and the result follows from [SZ, Corollary 4.5].
The result below is a consequence of properties that Levasseur [L, Proposition 5.9] has shown hold for any Auslander-Gorenstein, Cohen-Macaulay $k$-algebra (see also [MR, Proposition 8.3.11]).

**Proposition 1.3.** Let $U$ be the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie superalgebra $\mathfrak{g}$. Then:

1. $\text{GKdim}$ is exact and finitely partitive for finitely generated $U$-modules.
2. $U$ is homogeneous for $\text{GKdim}$.

It now follows by an argument outlined in Stafford-Zhang [SZ, remarks following proof of Proposition 5.2], and the results of Levasseur [L, Proposition 5.9] that any Auslander-Gorenstein, Cohen-Macaulay $k$-algebra has a quotient ring that is a quasi-Frobenius ring. We include the argument for completeness.

**Theorem 1.4.** An Auslander-Gorenstein, Cohen-Macaulay $k$-algebra $A$ has a (right and left) quotient ring $Q$ that is a quasi-Frobenius ring.

**Proof.** Levasseur [L, Proposition 5.9(ii)] has shown that $A$ has an Artinian quotient ring $Q$. To show that $Q$ is self-injective it suffices to show that $\text{Ext}^i_A(Q, Q) = 0$ for all simple right $Q$-modules $S$ and all $i \geq 1$. Since $Q$ is Artinian, any simple $Q$-module will occur as a summand of $Q/M$, where $M$ is a maximal ideal of $Q$. Expansion and contraction are 1-1 correspondences between the prime ideals of $Q$ and the prime ideals of $A$ that do not intersect the set of regular elements of $A$ (see e.g. [MR, Proposition 2.1.16, page 47]), so we may assume that $M = PA$, for $P$ a prime ideal of $A$. By [SZ, Lemma 3.3] $\text{Ext}^i_A(Q/PQ, Q) \cong Q \otimes_A \text{Ext}^i_A(A/P, A)$. Let $i \geq 1$ and $q \otimes a \in Q \otimes_A \text{Ext}^i_A(A/P, A)$. Let $C = Aa$ be the cyclic left $A$-submodule of $\text{Ext}^i_A(A/P, A)$ generated by $a$. Since $A$ is Auslander-Gorenstein, we have $\text{Ext}^j_A(C, A) = 0$ for all $j < i$. It follows that $j(C) \geq 1$, and since $A$ is Cohen Macaulay, we have $\text{GKdim}(C) < \text{GKdim}(A)$. Let $C = A/I$ for $I$ a left ideal of $A$. Since $\text{GKdim}$ is an exact dimension function on finitely generated $A$-modules, $A$ is homogeneous for $\text{GKdim}$, and $N(A)$ is (weakly) ideal invariant under $\text{GKdim}$ (see e.g. [MR, Corollary 8.3.16]), then by [MR, Proposition 6.8.14(ii) and Theorem 6.8.15] there is a regular element $c \in I$, and hence $q \otimes a = qc^{-1} \otimes ca = 0$. It follows that $Q \otimes_A \text{Ext}^i_A(A/P, A) = 0$, and hence $\text{Ext}^i_A(Q, Q) = 0$. \hfill $\square$

**Corollary 1.5.** The quotient ring $Q$ of the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie superalgebra $\mathfrak{g}$ is a quasi-Frobenius ring.

2. **Minimal prime ideals in $U(b(n))$**

In this section we show that $U(b(n))$ is never semiprime, but has a unique minimal prime ideal for all $n$.

2.1. Throughout this section let $\mathfrak{g} = b(n) = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A^t \end{bmatrix} \right\} \oplus \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right\} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $A$ is an $n \times n$ matrix of trace zero, $B$ is an $n \times n$ symmetric matrix, and $C$ is an $n \times n$ skew-symmetric matrix, with bracket defined on homogeneous elements by $[x, y] = xy + (-1)^{ij}yx$ when $x \in \mathfrak{g}_i$ and $y \in \mathfrak{g}_j$. For $n \geq 3$ it has been shown that $b(n)$ is classically simple. Basis elements of $\mathfrak{g}_1$ are of two types:

**(Type I):** Let $\{y_1, y_2, \ldots, y_r\}$ be a basis for $\left\{ \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right\}$. Note that $r = \frac{n(n+1)}{2}$. The $k$-space spanned by the type I basis elements is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$.
(Type II): Let \( \{z_1, z_2, \ldots, z_s\} \) be a basis for \( \{ \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \} \). Note that \( s = \frac{n(n-1)}{2} \).

The \( k \)-space spanned by the type II basis elements is a \( g_0 \)-submodule of \( g_1 \).

The fact that there are more basis elements of type I than of type II will be used in our argument. In choosing the PBW basis for \( U = U(g) \) we will regard the \( x \)'s < \( y \)'s, and \( y \)'s < \( z \)'s, where the \( x \)'s are basis elements of \( g_0 \).

2.2. We next collect some facts about products in \( U \) of type I and type II elements:

1. Note that \( 2y^2 = y_i y_i + y_i y_i = [y_i, y_i] = 0 \). Similarly \( z^2 = 0 \).
2. Also \( y_i y_j + y_j y_i = [y_i, y_j] = 0 \). Hence the \( y \)'s anti-commute (and similarly the \( z \)'s anti-commute).
3. If \( Y = y_1 y_2 \cdots y_n \) is a monomial in \( U \) of elements \( y_i \)'s of type I, we call \( n \) the length of \( Y \), and we write \( \ell(Y) = n \). We similarly talk about the length of a monomial of \( x \)'s or of \( z \)'s.
4. Thus any product of \( y \)'s of length greater than \( r \) is zero (and any product of \( z \)'s of length greater than \( s \) is zero).

Lemma 2.3. In \( U(g) \):

(a) If \( Y = y_1 y_2 \cdots y_n \) is a product of type I elements of length \( n \), and if \( u \in U_0 = U(g_0) \), then \( Yu = \sum u_a Y_{\alpha} \), where \( u_a \in U_0 \), \( \ell(u_a) \leq \ell(u) \), and \( Y_{\alpha} \) is a product of \( y \)'s with \( \ell(Y_{\alpha}) = \ell(Y) = q \) for all \( \alpha \). If \( Y = y_1 \cdots y_r \) is the product of all the type I basis elements, then \( Yu = u'Y \) for some \( u' \in U_0 \). Similarly for \( Z = z_1 z_2 \cdots z_m \).

(b) If \( Y \) is a product of type I basis elements and \( Z \) is a product of type II basis elements, then \( ZY = \sum u_a Y_{\alpha} Z_{\alpha} \), where \( u_a \in U_0 \), \( Z_{\alpha} \) is a product of \( z \)'s, and \( \ell(Y_{\alpha}) - \ell(Z_{\alpha}) = \ell(Y) - \ell(Z) \) for all \( \alpha \).

Proof. For (a) note that in \( U(g) \) if \( x \in g_0 \), and if \( y \in g_1 \) is of type I, then \( xy - yx = [x, y] = \sum y_i y_i \in g_1 \). Then part (a) follows by induction. (The result for the \( z \)'s is similar.)

For (b) note that if \( y, z \) are of types I and II (respectively) in \( g_1 \), then in \( U(g) \) we have \( yz + zy = [y, z] \in g_0 \). Again induction gives the result.

Behr [B] showed that \( U(b(2)) \) is not semiprime (\( b(2) \) is not simple). We next show that all \( U(b(n)) \) are not semiprime. In fact it follows from the proof of the following proposition that if \( Y = y_1 \cdots y_r \) is the product of all \( y \)'s of type I, then \( (YU)^2 = 0 \), and hence \( U(b(n)) \) always has the nilpotent ideal \( YU \).

Proposition 2.4. The ideal generated by all products of at least \( s + 1 \) type I elements is in the nil radical, \( \langle \{y_1 y_2 \cdots y_{s+1}\} \rangle \subseteq N(U) \).

Proof. Let \( Y \) be such a product of at least \( s + 1 \) type I elements. In \( YUY \) elements are sums of terms of the form \( Ya Y_1 Z_1 Y \) where \( u \in U_0 \), \( Y_1 \) is a product of type I elements, and \( Z_1 \) is a product of type II elements. Note that by Lemma 2.3(a), \( YuY_1 Z_1 Y \) is a sum of terms of the form \( u'Y'Y_1 Z_1 Y \) for \( u' \in U_0 \) and \( Y' \) a product of type I elements with \( \ell(Y') = \ell(Y) \geq s + 1 \). Next note that by Lemma 2.3(b) \( Z_1 Y \) is a sum of terms of the form \( u_a Y_1 Z_1 Z_{\alpha} \), where \( u_a \in U_0 \) and \( \ell(Y_{\alpha}) - \ell(Z_{\alpha}) = \ell(Y) - \ell(Z_{\alpha}) \geq s + 1 - \ell(Z_1) \geq 1 \), and hence \( \ell(Y_{\alpha}) \geq 1 \). Thus elements of the form \( u'Y'Y_1 Z_1 Y \) are sums of elements of the form \( u''Y'Y_1 u_{\alpha} Y_1 Z_1 Z_{\alpha} \), and by Lemma 2.3(a) these elements are sums of elements of the form \( u''Y''Y'Y_1 Y_1 Z_1 Z_{\alpha} \), where \( \ell(Y'') \geq \ell(Y') \geq s + 1 \). Hence elements in \( YUY \) are of the form \( \sum u_{\beta} Y_{\beta} Z_{\beta} \), where \( \ell(Y_{\beta}) \geq \ell(Y'') \geq s + 1 \).
s + 2. It follows by induction that elements in $Y(UY)^k$ are of the form $\sum u_{ij}Y_{ij}Z_{ij}$, where $\ell(Y_{ij}) \geq s + 1 + k$. Consequently, $Y(UY)^{r-s} = 0$, and $Y \in N(U)$.

We can now prove the main result of this section. Throughout the following we will use $Y_i$ to denote a monomial in the $y$’s and $Z_j$ to denote a monomial in the $z$’s.

**Theorem 2.5.** The enveloping algebra $U(b(n))$ has a unique minimal prime ideal.

**Proof.** By Behr [B, Corollary, page 21] the elements of $U_0^*$ form a two-sided Ore set of regular elements of $U$, and localizing at $U_0^*$ yields an Artinian two-sided classical quotient ring $Q$. Let $Q_0$ denote the division ring of quotients of $U_0$; then the set of monomials in the basis elements of $g_1$ form a basis for $Q$ over $Q_0$. As noted in Corollary 1.5, $Q$ is quasi-Frobenius. By [MR, Proposition 2.1.16, page 47] expansion and contraction are 1-1 correspondences between the prime ideals of $Q$ and the prime ideals of $U$ that do not intersect $U_0^*$; furthermore, $N(Q) = N(U) \cap Q = QN(U)$ and $N(U) = N(Q) \cap U$. It is sufficient to show that $N(U)$ is prime, for since $N(U)$ is nilpotent, it would then be the unique minimal prime ideal of $U$; hence it is sufficient to show that $N(Q)$ is prime. Since $Q$ is Artinian, this is the case exactly when $Q/N(Q)$ is simple Artinian, or equivalently, when $Q$ has exactly one simple module up to isomorphism. Since $Q$ is quasi-Frobenius, every simple right $Q$-module is isomorphic to a minimal right ideal of $Q$ (see, for example, [AF, Corollary 31.4, page 340]); hence it will be sufficient to show that all minimal right ideals of $Q$ are isomorphic.

Let $Y$ be the product in $U$ of all the type I basis elements of $g_1$ and let $Z$ be the product of all the type II basis elements of $g_1$. By Lemma 2.3(a) if $u \in U_0$, then $Yu = u’Y$ for some $u’ \in U_0$, and thus $Yu^{-1} = (u’)^{-1}Y$; it follows that if $q \in Q_0$, then $Yq = q’Y$ for some $q’ \in Q_0$. The same property holds for $Z$. Consider the right ideal $YZQ$ of $Q$; by 2.2.2 the right ideal $YZQ$ does not depend on the order of the factors of $Y$ and $Z$.

**Claim 1.** The right ideal $YZQ$ is a minimal right ideal of $Q$.

**Proof of Claim.** Take an arbitrary nonzero element $YZq$ of $YZQ$. We can write $YZq = \sum t_{ij}Y_{ij}Z_{ij}$, where each $t_{ij}$ is a nonzero element of $Q_0$, and thus have $YZq = \sum t_{ij}Y_{ij}Z_{ij} = \sum t’_{ij}Y_{ij}Z_{ij}$ where $t’_{ij} \in Q_0$ for each $i, j$. Since $\ell(Y_{ij}) > 0$, we can write $YZq$ in the form $YZq = Y \sum s_jZ_{ij}$ where $s_j \in Q_0$ for all $j$. Let $j_0$ be such that $Z_{j_0}$ has no proper subproduct that occurs in the sum $\sum s_jZ_{ij}$, and let $Z’_{j_0}$ be the product of all $z$’s not occurring as a factor of $Z_{j_0}$. Since the $z$’s anti-commute, and since the square of any $z$ is zero, we have $Z_{j_0}Z’_{j_0} = 0$ for $j \neq j_0$. Consequently, $YZqZ’_{j_0} = Y \sum s_jZ_{j_0}Z’_{j_0} = Ys_{j_0}Z_{j_0}Z’_{j_0} = \pm Ys_{j_0}Z = YZs_{j_0}$ for some $s_{j_0} \in Q_0$. Thus $YZq = YZqQ$ for every $q$, and the right ideal $YZQ$ is minimal.

**Claim 2.** If $I$ is a minimal right ideal of $Q$, then $I$ is isomorphic to $YZQ$.

**Proof of Claim.** Take $I = qQ$. We can write $q = \sum t_{ij}Y_{ij}Z_{ij}$ where for each $i, j$ we have that $t_{ij}$ is a nonzero element of $Q_0$. As in the proof of Claim 1, take $j_0$ so that $Z_{j_0}$ has no proper subproduct that occurs in the sum, and let $Z’_{j_0}$ be the product of the $z$’s not occurring as a factor of $Z_{j_0}$. Multiplying by $Z’_{j_0}$ yields $q’ = qZ’_{j_0} = \sum t_{i,j_0}Y_{ij}Z = \sum \pm s_iY_{i,j_0}Z$ where $s_i = \pm t_{i,j_0}$ for each $i$. Observe that $q’ \neq 0$ since the terms in the sum are distinct basis elements for $Q$ over $Q_0$; since $I$ is simple, we have $I = q’Q$. Take a common denominator $s$ for the $s_i$’s, and write $s_i = s^{-1}u_i$ for $s, u_i \in U_0$. Then $I \cong sI = q’Q'$ where $q’ = s = \sum u_iY_{ij}Z$. By Lemma 2.3(a) we can write $q’ = \sum v_jZ = (\sum v_jZ_j)Z$ where each $v_j$ is a
nonzero element of $U_0$. Again take $j_0$ so that no proper subproduct of $Y_{j_0}$ occurs in the sum, and let $Y_{j_0}'$ be the product of all the $y$’s not occurring as a factor of $Y_{j_0}$. Multiplying on the left by $Y_{j_0}'$ yields $Y_{j_0}'q'' = Y_{j_0}'Y_{j_0}v_{j_0}Z = \pm Yv_{j_0}Z = YZv_{j_0}$ for some $v_{j_0} \in U^*_0$. Observing that $Y_{j_0}'q'' \neq 0$, we have that $Y_{j_0}'q''Q = YZQ$. Hence $I \cong sI \cong Y_{j_0}'sI = YZQ$ as desired.

By the remarks at the beginning of the proof, the result now follows from Claims 1 and 2.

3. Minimal prime ideals in enveloping algebras of nilpotent Lie superalgebras

In this section we show that if $\mathfrak{g}$ is nilpotent, then $U(\mathfrak{g})$ has a unique minimal prime ideal.

3.1. The center of $\mathfrak{g}$, $Z(\mathfrak{g})$, is defined by $Z(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}$; the higher centers are defined by $Z_i(\mathfrak{g})/Z_{i-1}(\mathfrak{g}) = Z(\mathfrak{g}/Z_{i-1}(\mathfrak{g}))$. Recall that $\mathfrak{g}$ is called nilpotent if $Z_\ell(\mathfrak{g}) = \mathfrak{g}$ for some $\ell$. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for $\mathfrak{g}_0$, and let $\{y_1, y_2, \ldots, y_m\}$ be a basis for $\mathfrak{g}_1$. In the enveloping algebra $U = U(\mathfrak{g})$ call a monomial $x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}y_1^{f_1}y_2^{f_2}\cdots y_m^{f_m}$ in the PBW basis even if $f_1 + f_2 + \cdots + f_m$ is even; recall that each $f_i$ is either 0 or 1 so that a monomial is even if the number of ‘$y$-factors’ is even. Call all other monomials odd. If $a \in U$, group even and odd monomials together and write $a = a_0 + a_1$; this decomposition of elements yields a $\mathbb{Z}_2$-grading of $U$. Let $\sigma : U \rightarrow U$ be defined by $\sigma(a) = \sigma(a_0 + a_1) = a_0 - a_1$; the map $\sigma$ is an automorphism of $U$. Furthermore, an ideal $I$ is graded if and only if $\sigma(I) = I$.

Letzter has noted that the following theorem follows from [Le, Remark 3.11, page 254].

**Theorem 3.2.** Let $\mathfrak{g}$ be a solvable Lie superalgebra over an algebraically closed field of characteristic 0. If $U(\mathfrak{g})$ does not have a unique minimal prime, then it has exactly two minimal primes $P$ and $\sigma(P)$. Then $P \cap \sigma(P)$ is the unique minimal graded prime ideal and is $N(U)$, the nil radical of $U$.

Since $U$ has an Artinian two-sided classical quotient ring $Q$ that is obtained by inverting the nonzero elements of $U_0 = U(\mathfrak{g}_0)$, the monomials in the $y$’s form a basis for $Q$ over $Q_0 = Q(U_0)$. We can extend the grading to $Q$; the automorphism $\sigma$ extends to $Q$ by $\sigma(s^{-1}a) = s^{-1}\sigma(a)$ where $s \in U^*_0$ and $a \in U$.

While it is possible for $Q$ to have proper idempotents, this is not the case for $U$.

**Proposition 3.3.** Let $\mathfrak{g}$ be a nilpotent Lie superalgebra with enveloping algebra $U = U(\mathfrak{g})$. Then 0 and 1 are the only idempotents of $U$.

**Proof.** Induct on the dimension of $\mathfrak{g}$. The result is true if $\mathfrak{g}$ has dimension 1. Suppose the result is true for all nilpotent Lie superalgebras with dimension less than that of $\mathfrak{g}$. Consider $Z(\mathfrak{g})$, which is nonzero since $\mathfrak{g}$ is nilpotent. Since $Z(\mathfrak{g})$ is a graded Lie ideal, it must contain a homogeneous element.

**Case 1.** Suppose that $Z(\mathfrak{g})$ contains an odd homogeneous element $y$. Let $e$ be a nonzero idempotent of $U$. Then $e + yU$ is a nonzero element of $U/yU$ since $yU$ is nilpotent. Thus if $e$ is a proper nonzero idempotent of $U$, both $e$ and $1 - e$ are nonzero idempotents of $U/yU$. This cannot be for $U/yU \cong U(\mathfrak{g}/ky)$ has no proper idempotents by induction.
Case 2. Suppose that \( Z(\mathfrak{g}) \) contains an even homogeneous element \( z \) and let \( e \) be a proper idempotent of \( U \). We may assume that \( z \) is a basis element of \( \mathfrak{g} \). Suppose that \( e \in zU \). From the PBW basis of \( U \) we see that we can write \( e = z'u \) where \( u \) is not in \( zU \). Then \( zu = e = e^2 = z^2u^2 \) and we have \( u = zu^2 \in zU \), which is a contradiction. Hence \( e + zU \) must be a nonzero idempotent of \( U/\zU \). Thus \( e + zU \) and \( (1-e)+zU \) are proper idempotents of \( U/\zU \cong U(\mathfrak{g}/kz) \); this is a contradiction by induction.

The following proposition gives consequences of \( U \) having two minimal prime ideals. Let \( \tilde{Q} = Q/N(Q) \) and let \( \tilde{a} = a + N(Q) \). Since \( J(Q) = N(Q) = N(U)Q = QN(U) \) (for example, see [CH, Lemma 9.2, page 125]), we have \( \sigma(N(Q)) = \sigma(N(U)Q) = N(U)Q = N(Q) \) and hence \( N(Q) \) is \( \sigma \)-stable. Thus \( \sigma \) induces an automorphism \( \sigma \) on \( Q \).

**Proposition 3.4.** Let \( U = U(\mathfrak{g}) \) for a nilpotent finite dimensional Lie superalgebra \( \mathfrak{g} \) over an algebraically closed field \( k \) of characteristic 0. If \( U \) has two minimal prime ideals, then there exist elements \( c \) and \( s \) in \( U \) such that

1. \( c \) is an odd regular element of \( U \),
2. \( s \in U_0 \),
3. \( \bar{c}^2 = \frac{1}{4} \), and
4. \( \bar{c}u\bar{s} = \bar{s}u\bar{c} \) for all \( u \in U \). Note that \( s \) and \( c \) commute modulo \( N(Q) \).

Conversely, given such elements \( c \) and \( s \), \( \frac{1}{2} + c^{-1} + N(Q) \) is a central idempotent of \( Q/N(Q) \).

**Proof.** Suppose that \( U \) has two minimal prime ideals, \( P \) and \( \sigma(P) \). Since \( Q \) is obtained by classically localizing at the two-sided Ore set \( S = U_0 \) all of whose elements are regular in \( U \), there is a correspondence between the prime ideals of \( Q \) and the prime ideals of \( U \) that do not intersect \( S \) [MR, Proposition 2.1.16, page 47]. Since \( Q \) is Artinian, \( Q \) has Krull dimension 0 and every prime ideal of \( Q \) is the expansion of one of the two minimal prime ideals of \( U \). Since \( U \) is Noetherian, neither of the minimal prime ideals \( P \) nor \( \sigma(P) \) can contain a regular element. Consequently \( Q \) also has exactly two minimal prime ideals (which are the only prime ideals of \( Q \)), namely \( PQ \) and \( \sigma(P)Q \). The above correspondence shows that if \( Q \) does not have a unique minimal prime, then neither can \( U \).

Consider \( \tilde{Q} = Q/N(Q) = Q/J(Q) \). Then \( Q \) has more than one prime ideal if and only if \( \tilde{Q} \) does, and since \( \tilde{Q} \) is semisimple Artinian, this is the case if and only if \( \tilde{Q} \) has a proper central idempotent. In this case \( \tilde{Q} \) will have exactly two proper central idempotents corresponding to the prime ideals of \( Q \); call them \( \bar{c} \) and \( 1 - \bar{c} \). Furthermore \( \sigma(\bar{c}) = 1 - \bar{c} \). Write \( \bar{e} = e + J \) where \( e \) is an idempotent of \( Q \) and write \( e = a + b \) where \( a \) is even and \( b \) is odd. Then \( \sigma(\bar{e}) = 1 - \bar{e} = 1 - \bar{a} - \bar{b} = \bar{a} + \bar{b} \) and hence \( \bar{a} = \frac{1}{2} \). Since \( \bar{a} \) is central, so are \( \bar{a} \) and \( \bar{b} \). Then \( \bar{a}^2 + 2\bar{ab} + \bar{b}^2 = \bar{a} + \bar{b} \) which implies that \( \frac{1}{4} + \bar{b}^2 = \frac{1}{2} + \bar{b} \) and consequently that \( \bar{b}^2 = \frac{1}{4} \). Write \( \bar{b} = \bar{c}\bar{s}^{-1} \) where \( s \in U_0 \) and \( c \) is an odd element of \( U \). Then \( \bar{c}^2 = \frac{1}{4} \bar{s}^2 \). Note that \( c \) must be a regular element of \( U \). For any \( u \in U \) we have that \( \bar{c}\bar{s}^{-1}u = \bar{u}\bar{c}\bar{s}^{-1} \) and in particular \( \bar{c} \) and \( \bar{s} \) commute. Thus we have \( \bar{c}\bar{s} = \bar{s}\bar{c} \) for all \( u \in U \).

Calculation shows that the converse follows.

The following lemma appeared in Bell-Musson [BM, Lemma 1.10, page 406].
Lemma 3.5. Let \( \mathfrak{g} \) be a nilpotent Lie superalgebra with center \( Z(\mathfrak{g}) = kz \) where \( z \) is even. Let \( y \) be a homogeneous element of \( Z_2(\mathfrak{g}) - Z(\mathfrak{g}) \) where \( Z_2(\mathfrak{g})/Z(\mathfrak{g}) = Z(\mathfrak{g}/Z(\mathfrak{g})) \). Then there exists a homogeneous element \( x \) of the same parity as \( y \) and a graded ideal \( \mathfrak{h} \) of codimension 1 in \( \mathfrak{g} \) such that

1. \([y, x] = z\),
2. \( \mathfrak{h} \) is the centralizer of \( y \) in \( \mathfrak{g} \), and
3. \( \mathfrak{g} = \mathfrak{h} \oplus kx \).

We need the following simple lemma.

Lemma 3.6. Let \( \mathfrak{g} \) be a Lie superalgebra with \( \dim (Z(\mathfrak{g}) \cap \mathfrak{g}_0) \geq 2 \); say \( z \) and \( z_1 \) are linearly independent such elements. If \( u \) is a nonzero element of \( U \), then there are only finitely many \( \lambda \)'s in \( k \) such that \( u \in (z + \lambda z_1)U \).

Proof. Let \( z_\lambda \) denote \( z + \lambda z_1 \). If \( \mu \neq \lambda \), then \( z_\lambda \) is a regular element of \( U/z_\mu U \cong U(\mathfrak{g}/kz_\mu) \). Hence if \( u = z_\lambda u_1 = z_\mu u_2 \) for \( u_1, u_2 \in U \), then \( u_1 = z_\mu u_3 \) for some \( u_3 \in U \). Consequently, \( u_1 \in z_\mu U \) for every \( z_\mu \) such that \( u \in z_\mu U \) with \( \mu \neq \lambda \). Since the high term of \( u_1 \) is less than the high term of \( u \) in the length lexicographic ordering of monomials, an induction argument yields the result.

We need the following lemma of Bell and Musson [BM, Lemma 1.5, page 404].

Lemma 3.7. Let \( A = A_0 \oplus A_1 \) be a \( Z_2 \)-graded \( k \)-algebra with \( \sigma : A \rightarrow A \) defined by \( \sigma(a_0 + a_1) = a_0 - a_1 \). Let \( \delta \) be a \( \sigma \)-derivation on \( A \), and suppose that there is an even \( h \in A \) with \( \delta(h) = 0 \). Let \( y' \) be an odd supercentral element of \( A \) with \( \delta(y') = 1 \), and let \( S \) be the Ore extension \( A[t; \sigma, \delta] \). Set \( B = \delta(y'A) \). Then \( B \) is a subalgebra of \( A \) such that \( A = B \oplus y'A \), and so \( A/y'A \cong B \). Moreover, \( t^2 - h \) is central in \( S \) and \( S/(t^2 - h)S \cong A/y'A \otimes M_2(k) \).

We can now prove the main result of this section.

Theorem 3.8. Let \( \mathfrak{g} \) be a nilpotent Lie superalgebra. Then \( U(\mathfrak{g}) \) has a unique minimal prime ideal.

Proof. First assume that \( k \) is algebraically closed. The proof will be by induction on the dimension of \( \mathfrak{g} \). The result is clearly true if \( \mathfrak{g} \) has dimension 1. Suppose that the result is true for all nilpotent Lie superalgebras with dimension less than that of \( \mathfrak{g} \). Furthermore, assume that \( U \) has two minimal prime ideals.

Case 1. There is an odd element \( y \in Z(\mathfrak{g}) \). Since \( yU \) is a nilpotent ideal of \( U \), there is a correspondence between the minimal prime ideals of \( U \) and those of \( U/yU \). This contradicts the induction hypothesis, for \( U/yU \cong U(\mathfrak{g}/(ky)) \).

Case 2. We have \( Z(\mathfrak{g}) \subset \mathfrak{g}_0 \) and \( \dim \mathfrak{g}(2) \geq 2 \). Let \( c \) and \( s \) be the elements of \( U \) given by Proposition 3.4. Take a nonzero \( z \in Z(\mathfrak{g}) \). It follows from Proposition 3.4 that \( \bar{e} = \frac{1}{2} + cs^{-1} \) is a central idempotent of \( Q(U/zU)/N(Q(U/zU)) \) provided \( s \) is not an element of \( 2U \), for \( U/zU \cong U(\mathfrak{g}/kz) \) would contain \( s + zU \) as an element of the Ore set \( U_0(\mathfrak{g}/kz) \). By Lemma 3.6, \( z \) can be chosen so that \( s \) is not an element of \( U \). It is conceivable that this could be the zero idempotent; that is, \( \frac{1}{2} + cs^{-1} + zU \in N(Q(U/zU)) \). In this case \( s + 2c + zU \) is an element of the nilpotent ideal \( N(U/zU) \). Recall that \( Q(U(\mathfrak{g})) \) is spanned by all the monomials in the basis elements of \( \mathfrak{g}_1 \) over the division ring \( Q(U_0) \), and that \( N(Q)^i = N(U)^i Q \); hence \( N(U/zU) \) must be nilpotent of index less than or equal to \( p = 2^n \) \( (n = \dim \mathfrak{g}_1) \). Note that \( (s + 2c)^p \) and \( (s - 2c)^p \) are nonzero since \( \bar{e} \) and \( 1 - \bar{e} \) are proper idempotents of \( \bar{Q} \). Again by Lemma 3.6 and the fact that \( k \) is infinite, we can
choose \( z \) so that \( s \) is not in \( zU \), \((s + 2c)p\) is not in \( zU \), and \((s - 2c)p\) is not in \( zU \). Thus \( c \) is a proper nonzero central idempotent of \( Q(U/zU)/N(Q(U/zU)) \). Since 
\[
U/zU \cong U(g/kz),
\]
this contradicts the induction hypothesis.

**Case 3.** We have \( Z(g) \subseteq g_0 \) and \( \dim Z(g) = 1 \). Let \( x, y \) be as in Lemma 3.5 and write \( g = h \oplus kx \) where \( h \) is the centralizer of \( y \). We have three subcases.

**Case 3a.** Suppose that \( x \) and \( y \) are both even. Then we have \( U(g) \cong U(h)[x; \delta] \).
Since \( h \) is again nilpotent, the induction hypothesis implies that \( U(h) \) has a unique minimal prime ideal. The theorem now follows by [MR, Theorem 1.2.9, page 16 and Proposition 14.2.3, page 495].

**Case 3b.** Suppose that \( x \) and \( y \) are both odd and \( [y, y] = 0 \). Let \( h = [x, x]/2 \). Then we have \( U(g) = U(h)[x; \sigma, \delta]/(x^2 - h) \); localizing at the powers of \( z \) yields \( U(g)(z_1) = U(h)(z_1)[x; \sigma, \delta]/(x^2 - h) \) where \( \sigma \) and \( \delta \) have been extended to all of \( U(h)(z_1) \). Hence letting \( A = U(h)(z_1) \) and applying Lemma 3.7 we have \( U(g(z_1)) \cong A/(y, A) \). By induction \( U(h) \) has a unique minimal prime ideal and hence so does \( U(g(z_1)) \). Since \( y \) is a supercentral odd element of \( h \), the ideal \( yA = yz^{-1}A \) is a nilpotent ideal of \( A \), and thus \( A/yA \) has a unique minimal prime. Hence so does \( U(g(z_1)) \). Since \( z \) cannot be contained in any minimal prime of \( U(g) \) (see the proof of Proposition 3.4), \( U \) has a unique minimal prime ideal.

**Case 3c.** Suppose that \( [y, y] \neq 0 \) for all \( y \in Z_2(g) - Z(g) \). Let \( x \) be as in Lemma 3.5. Since \( y \in Z_2(g) \), \([y, y] = \lambda z \) for some scalar \( \lambda \). Suppose that \( y \) and \( y' \) represent linearly independent cosets in \( Z_2(g)/Z(g) = Z_2(g)/kz \); by Case 3a we may assume that \( y' \) is also odd. Let \([y', y'] = \mu x \) and \([y, y'] = \rho z \). Consider \([y + ay', y + ay'] = (\lambda + 2\rho a + \mu a^2)z \). Since \( k \) is algebraically closed, there exists \( a \in k \) such that \( \lambda + 2\rho a + \mu a^2 = 0 \). In this case \([y + ay', y + ay'] = 0 \), which is a contradiction. Thus we may assume that \( \dim Z_2(g)/Z(g) = 1 \). Then in Lemma 3.5 we may take \( x = y \) and write \( g = h \oplus ky \); recall that \( h \) is the centralizer of \( y \). If \( w \in Z_2(h) \), then \([w, y] = 0 \) and \( w \in Z_2(g) \). Since \( \dim Z_2(g)/Z(g) = 1 \), we have that \( w = \alpha z \) for some scalar \( \alpha \). The nilpotence of \( h \) implies that \( h = kz \) and \( g = kx \). Hence \( d(g) \neq 0 \) and it follows from Bell [Be, Theorem 1.5] that \( U(g) \) is prime.

If \( k \) is not algebraically closed, let \( K \) be the algebraic closure of \( k \) and let \( B = \{ b_\alpha : \alpha \in A \} \) be a basis for \( K \) over \( k \). Then \( B \) is a central free basis for \( U_K(g) \) over \( U_k(g) \). Since \( K \) is algebraically closed, the preceding part of the proof implies that \( U_k(g) \) has a unique minimal prime ideal \( P \) with \( P = N(U_k(g)) \). Let \( p = P \cap U_k(g) \). Suppose that \( uU_k(g) \) for \( u \in U_k(g) \) with \( u \) not an element of \( p \). Since \( ub_\alpha U_k(g) \subseteq P \) for all \( \alpha \in A \), we have \( U_k(g) \subseteq P \); consequently, \( v \subseteq P \cap U_k(g) = p \), and \( p \) is a prime ideal. Since \( p \) is a nilpotent prime ideal, \( p \) is the unique minimal prime ideal of \( U_k(g) \).

The following example shows that, while \( U \) has no nontrivial idempotents, it is possible that \( Q \) contains nontrivial idempotents.

3.9. Let \( g = g_0 \oplus g_1 \) be the subalgebra of \( 4 \times 4 \) matrices, where \( g_0 \) is the vector space spanned by the matrix units \( \{ x_1 = e_{1,2}, x_2 = e_{1,3}, x_3 = e_{2,3} \} \), \( g_1 \) is the vector space spanned by the matrix units \( \{ y_1 = e_{1,4}, y_2 = e_{4,3}, y_3 = e_{4,2} \} \), and the bracket is defined on basis elements as \( [g_i, g_j] = g_i g_j + (-1)^{ij+1} g_j g_i \). In the notation of \([S]\), \( g \) is a subalgebra of \( sl(3, 1) \). The only nonzero brackets involving basis elements are: \([x_1, x_3] = x_2, [x_3, y_3] = -y_2, [y_1, y_2] = x_2, \) and \([y_1, y_3] = x_1 \). It is easy to check that:

\( Z_4 = g, \) and so \( g \) is a nilpotent Lie superalgebra.
(2) $y_2y_3Uy_2y_3 = 0$, so $U$ is not semiprime.
(3) $x_2 \in Z(g)$.
(4) $e = y_1y_2x_2^{-1}$ is a nontrivial idempotent of $Q$.

Let $\mathfrak{h}$ be the $k$-span of $\{x_2, x_3, y_1, y_2\}$. Then it can be shown that $\mathfrak{h}$ is a nilpotent Lie superalgebra, $U(\mathfrak{h})$ is prime, and $e = y_1y_2x_2^{-1}$ is a proper idempotent of $Q(U(\mathfrak{h}))$.

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