ON THE FIXED POINT SETS OF SMOOTH INVOLUTIONS
ON THE PRODUCTS OF SPHERES

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Abstract. In this paper, we have, under some conditions on cohomology,
that the fixed point set of a smooth involution on a product of spheres is of
constant dimension.

1. Introduction

Throughout this paper, we assume $G = \mathbb{Z}_2$. Let $G$ act smoothly on a smooth
closed manifold $M$ with fixed point set $F$. Denote by $M_G$ the Borel construction
associated with a $G$ action on $M$, and by $p : M_G \to B(G) = \mathbb{RP}^\infty$ the fibre bundle
with fibre $M$. It is well known that if $F$ is nonempty, then it is a disjoint union
of finite number of smooth closed submanifolds of $M$. In this paper, we study the
relations between the dimensions of the components of $F$ and the cohomology of
$M$ or $M_G$. We will prove

Theorem 1.1. Let $M^n$ be a smooth closed manifold with a smooth involution $\tau$. Then the fixed point set $F$ is either empty or of constant dimension if one of the
following conditions is satisfied:

(i) $H^*(M_G; \mathbb{Z})$ has a generator set $\{1, y_j\}$ as an algebra over $H^*(\mathbb{RP}^\infty; \mathbb{Z})$ with
deg$(y_j)$ odd for all possible $j$;

(ii) $\tilde{H}^*(M^n; \mathbb{Z})$ has no 2-torsions and is algebraically generated by some elements
$\{x_i\}$ of odd degrees with deg$(x_i) + \text{deg}(x_j) > \text{deg}(x_l)$ for $i \neq j$, and $\tau$ induces a
trivial $\mathbb{Z}_2$ action on $\tilde{H}^*(M^n; \mathbb{Z})$.

Let $R$ be a principal ideal domain. Recall that $M^n$ is totally nonhomologous
to zero in $M_G$ with coefficient in $R$ if the fibre inclusion $j : M^n \to M_G$ induces a
surjection in cohomology $H^*(-; R)$ ([3, p373]). Thus by the Leray-Hirsch theorem
[3, Theorem 1.4, p372], the condition (i) of Theorem 1.1 is satisfied if $M^n$ is totally
nonhomologous to zero in $M_G$ with coefficient in $\mathbb{Z}$, and $\tilde{H}^*(M^n; \mathbb{Z})$ has no 2-
torsions, and is algebraically generated by some elements of odd degrees.

Let $X \sim_R Y$ denote two spaces $X$ and $Y$ such that $H^*(X; R)$ and $H^*(Y; R)$
are isomorphic as rings. Denote by $W(M)$ the total Stiefel-Whitney classes of $M$. Note that $W(M) = 1$ if $M$ is a product of some spheres. The statement (i) of
the next theorem is an immediate corollary of Theorem 1.1.

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Theorem 1.2. Let $M^n$ be a smooth closed manifold with a smooth involution $\tau$. Then $F$ is either empty or of constant dimension, if

(i) $\tau$ induces the trivial $\mathbb{Z}_2$ action on $H^*(M^n; \mathbb{Z})$, and $M^n \sim_{\mathbb{Z}} S^{2n_1+1} \times S^{2n_2+1} \times \ldots \times S^{2n_k+1}$ with $2n_i + 2n_j > 2n_1 - 1$ whenever $i \neq j$ (e.g. $M^n \sim_{\mathbb{Z}} (S^{2m+1})^n$), or

(ii) $\tau$ induces the trivial $\mathbb{Z}_2$ action on $H^*(M^n; \mathbb{Z}_2)$, $M \sim_{\mathbb{Z}_2} (S^1)^n$ and $W(M) = 1$.

Theorem 1.3. Let $M$ be a smooth closed manifold with $W(M) = 1$. Suppose $\tau$ is a smooth involution on $M$ which induces the trivial $\mathbb{Z}_2$ action on $H^*(M; \mathbb{Z}_2)$.

(i) If $M^{2n} \sim_{\mathbb{Z}} (S^2)^n$, then $F$ is nonempty and is of constant dimension. Let $k$ be the dimension of $F$. Then $k$ is even and $F$ has at most $2^{n-k/2}$ components $\{F_i\}$, and for each $F_i$, $H^*(F_i; \mathbb{Z}_2)$ is algebraically generated by some elements $\{b_{ij}\}_{1 \leq j \leq n}$ with $b_{ij} \in H^2(F_i; \mathbb{Z}_2)$ and $b_{ij}^2 = 0$ for all possible $j$. In particular, $H^*(F_i; \mathbb{Z}_2)$ contains a subring which is isomorphic to $H^*((S^2)^k/2; \mathbb{Z}_2)$.

(ii) Suppose $M \sim_{\mathbb{Z}_2} (S^1)^n$ and $F$ nonempty. Then $F$ is of constant dimension. Let $k$ be the dimension of $F$. Then $F$ has at most $2^{n-k}$ components $\{F_i\}$, and for each $F_i$, $H^*(F_i; \mathbb{Z}_2)$ is algebraically generated by some elements $\{b_{ij}\}_{1 \leq j \leq n}$ with $b_{ij} \in H^1(F_i; \mathbb{Z}_2)$ and $b_{ij}^2 = 0$ for all possible $j$. In particular, $H^*(F_i; \mathbb{Z}_2)$ contains a subring which is isomorphic to $H^*((S^1)^k; \mathbb{Z}_2)$.

We point out, since the statement (i) in [5, Proposition 2.1] (there is a misprint there, $i^*c_k^{(m)} = c_k^{(m)}$ should be $i^*c_k^{(m)} = c_k$) is true if and only if the smooth involution $\tau$ induces the trivial $\mathbb{Z}_2$ action on $H^*((S^1)^n; \mathbb{Z}_2)$, the main theorem we proved there should be modified as follows.

Theorem. Any smooth involution on $(S^1)^n$ with the trivial induced $\mathbb{Z}_2$ action on $H^*((S^1)^n; \mathbb{Z}_2)$ has either empty or constant-dimensional fixed point set $F$.

2. PROOFS OF THE THEOREMS

Let $M^n$ be a smooth closed manifold with a smooth involution $\tau$. Then $\tau$ induces a $\mathbb{Z}_2$-equivariant vector bundle structure on the tangent bundle $T(M^n)$ of $M^n$. Let $S^\infty$ be the infinite-dimensional sphere with a $\mathbb{Z}_2$ action given by the antipodal involution. Consider the product space $S^\infty \times M^n$ with the $\mathbb{Z}_2$ diagonal action. Then projection $S^\infty \times M^n \to M^n$ is equivariant. Pulling back the $\mathbb{Z}_2$-equivariant vector bundle $T(M^n)$ by this projection, we obtain a $\mathbb{Z}_2$-equivariant vector bundle over $S^\infty \times M^n$, which defines a vector bundle over the Borel space $M_G = (S^\infty \times M^n)/\mathbb{Z}_2$ by [1, Proposition 1.6.1, p36]. Denote this vector bundle by $\mathcal{T}(M^n)$. Similarly, the diagonal action on $S^\infty \times M^n$, where the $\mathbb{Z}_2$ action on $S^\infty$ is given by the antipodal involution, defines a smooth closed manifold $R^m(\tau) = (S^m \times M^n)/\mathbb{Z}_2$. Let $p$ denote either projection $R^m(\tau) \to RP^m$ or $M_G \to RP^\infty$. Then $(R^m(\tau), p, RP^m)$ is a differentiable fibre bundle over $RP^m$ with fibre $M^n$. Consequently, the tangent bundle of $R^m(\tau)$ splits and

$$T(R^m(\tau)) \cong p^*T(RP^m) \oplus T_m(M^n),$$

where $T_m(M^n)$ is called the tangent bundle along the fibres [(2, p482)]. Actually, $T_m(M^n) = i^*(\mathcal{T}(M^n))$, where $i : R^m(\tau) \to M_G$ is the natural inclusion. Note that the restriction of $T_m(M^n)$ (or $\mathcal{T}(M^n)$) on a specific fibre is exactly the tangent bundle $T(M^n)$.

Suppose $F \neq \phi$. Given $x \in F$, define $d_x$ to be the codimension of the component of $F$ containing $x$, and $I(\tau)$ the set of numbers $d_x$. Let $\rho_x$ be the section of $p$ associated with $x \in F$. Consider the induced bundle $\eta_x^{(m)} = \rho_x^*T_m(M^n)$. Observe
that \(d_x\) is the number of the eigenvalues \((-1)\) of the local represetation of the group \(Z_2\) induced by tangent map \(d(\tau)\) on the tangent space \(T_x(M^n)\). This implies the induced bundle \(\eta_x^{(m)}\) is the Whitney sum of an \((n-d_x)\)-dimensional trivial bundle and \(d_x\) copies of the Hopf bundle. Therefore \(W(\eta_x^{(m)}) = (1 + a)^{d_x}\), where \(a \in H^1(RP(m); Z_2)\) is a generator. Thus
\[
I(\tau) = \{d_x|x \in F, W(\eta_x^{(m)}) = (1 + a)^{d_x}, 0 \leq d_x \leq n\}
\]
for every \(m > n\).

**Remark 2.1.** Let \(W_j(-)\) be the \(j\)-th Stiefel-Whitney class. Then whenever \(m > n\), we have \(d_x = \max\{j|W_j(\eta_x^{(m)}) \neq 0\} = \max\{j|W_j(\eta_t) \neq 0\}\), where \(\eta_t = \rho_x^* \bar{T}(M^n)\).

Let \(C(-)\) and \(C_j(-)\) be the total Chern classes and the \(j\)-th Chern class respectively. Since \(\eta_x^{(m)} \otimes C\) is isomorphic to \(\rho_x^* (\bar{T}_m(M^n) \otimes C)\) as complex bundle and
\[
\rho C(\eta_x^{(m)} \otimes C) = (W(\eta_x^{(m)}))^2,
\]
where \(\rho\) is the mod 2 reduction homomorphism, we have
\[
d_x = \max\{j|C_j(\eta_x^{(m)} \otimes C) \neq 0\}
\]
whenever \(m > n\)
\[
= \max\{j|C_j(\eta_x \otimes C) \neq 0\}
\]
\[
= \max\{j|\rho C_j(\eta_x \otimes C) \neq 0\}.
\]

Thus \(I(\tau)\) can be computed by using either Stiefel-Whitney or Chern classes.

Let \(j\) be the inclusion \(M^n \to M_G\) or \(M^n \to \tau M^n\). The following theorem shows some relations between \(I(\tau)\) and the algebraic structure of \(H^*(M_G; Z)\) or \(H^*(M_G; Z_2)\).

**Theorem 2.2.** Let \(M\) be a smooth closed manifold with a smooth involution \(\tau\). Suppose there is a generator set \(\{c_i\}\) of \(H^*(M_G; Z)\) (resp. \(H^*(M_G; Z_2)\)) as an algebra over \(H^*(RP^\infty; Z)\) (resp. \(H^*(RP^\infty; Z_2)\)). If there is an \(n_i\) such that
\[
(c_i)^{n_i} \in p^* H^*(RP^\infty; Z) \quad (\text{resp. } p^* H^*(RP^\infty; Z_2))
\]
for each \(c_i\), then \(\tau\) has either empty or constant-dimensional fixed point set \(F\).

**Proof.** Suppose \(F \neq \phi\). In the case of coefficient \(Z_2\), consider the homomorphism \(\rho_x^*: H^*(M_G; Z) \to H^*(RP^\infty; Z_2)\). Then for each \(c_i\),
\[
\rho_x^*(c_i) = \begin{cases} 
0 & \text{if } c_i^{n_i} = 0, \\
a^{m_i} & \text{if } c_i^{n_i} \neq 0,
\end{cases}
\]
which is independent of the choices of \(x \in F\), where \(m_i\) is the degree of \(c_i\) and \(a \in H^1(RP^\infty; Z_2)\) is a generator. By Remark 2.1, \(d_x = \max\{j|W_j(\eta_t) = \rho_x^* W_j(\bar{T}(M)) \neq 0\}\) is independent of the choices of \(x \in F\). So \(F\) is of constant dimension. In the case of coefficient \(Z\), consider the complex bundle \(\eta_t \otimes C\) and the homomorphism \(\rho_x^*: H^*(M_G; Z) \to H^*(RP^\infty; Z)\). Just as the preceding case, \(\rho_x^*\) is independent of the choices of \(x \in F\). By Remark 2.1 again, \(d_x = \max\{j|C_j(\eta_t^{(m)} \otimes C) = \rho_x^* C_j(\bar{T}(M) \otimes C) \neq 0\}\) is independent of the choices of \(x \in F\). So \(F\) is also of constant dimension.

**Remark 2.3.** If \(H^*(M_G; Z_2)\) (resp. \(H^*(M_G; Z)\)) has a generator set \(\{1, x_1, x_2, \ldots, x_k\}\) as an algebra over \(H^*(RP^\infty; Z_2)\) (resp. \(H^*(RP^\infty; Z)\)), then there are at most \(2^k\) number of different maps \(\rho_x^*\) for \(x \in F\). Therefore \(F\) has at most \(2^k\) number of components which are of different dimensions.
Proof of Theorem 1.1. For (i), \( \rho^*_i(y_j) = 0 \) for all possible \( j \), since \( H^\text{odd}(RP^n; Z) = 0 \). Thus the homomorphism \( \rho^*_i \) is independent of the choices of \( x \in F \) and (i) follows from Remark 2.1.

For (ii), we consider the spectral sequence \( \{ E^p,q, d_r \} \) with

\[
E^{2,0}_2 = H^0(R^\infty; H^0(M^n; Z))
\]

([3, p370]), which converges to \( H^*(M_G; Z) \). Here the coefficient \( H^0(M^n; Z) \) is a local system which becomes constant because of the trivial induced \( Z_2 \) action on \( H^* (M^n; Z) \). Let \( \{ x_i \} \) be the generator set of the ring \( H^*(M^n; Z) \) with \( \deg(x_i) \) odd for all \( i \). First note that for each \( x_i, x_i^2 \) must be of order \( \leq 2 \), since the degree of \( x_i \) is odd. Thus we have \( x_i^2 = 0 \), since \( H^0(M^n; Z) \) has no 2-torsions. The multiplicative property implies this spectral sequence collapses, since all elements in \( E^{0,2n+1}_2 \) and \( E^{0,0}_2 \) are permanent cocycles, where \( 2n_i + 1 \) is the degree of some \( x_i \). Here note that the only possible nontrivial target for the differential \( d_2 \) on an element of \( E^{0,2n_i+1}_2 \) is in \( E^{2r,2n_i-2r+2}_2 \). Since \( \deg(x_j) + \deg(x_i) > \deg(x_i) \) for \( j \neq i \), we see \( E^{2r,2n_i-2r+2}_2 = 0 \) for \( r \geq 1 \). Therefore every element of \( E^{0,2n_i+1}_2 \) is a permanent cocycle. Now the edge homomorphism \( H^0(M_G; Z) \rightarrow E^{0,0}_2 \rightarrow H^0(M^n; Z) \), which is precisely the \( j^*: H^*(M_G; Z) \rightarrow H^*(M^n; Z) \) ([3, p374]), is surjective, we see that \( H^*(M_G; Z) \) is an algebra over \( H^*(RP^\infty; Z) \) with generator set \( \{ 1, y_i \} \) and \( \deg(y_i) \) odd for all \( i \), and (ii) follows just as (i).

Proof of Theorem 1.2. We only need to prove (ii). Consider the spectral sequence which converges to \( H^*(M_G; Z_2) \) with \( E^{2,q}_2 = H^p(RP^\infty; H^q(M; Z_2)) \). Here the local coefficient system \( H^*(M; Z_2) \) again becomes constant because of the trivial induced \( Z_2 \) action on \( H^*(M; Z_2) \). By the multiplicative property, this spectral sequence collapses since all elements of \( E^{2,0}_2 \) and \( E^{1,0}_2 \) are permanent cocycles. By [3, Theorem 1.6, p374], \( M \) is totally nonhomologous to zero in \( M_G \) and hence in \( R^m(\tau) \) with coefficient in \( Z_2 \) for any \( m \geq 0 \). Thus \( H^*(R^m(\tau); Z_2) \) is a free \( H^*(RP^m; Z_2) \) module with a module basis \( \{ x_i \} \), where \( x_i \) runs through all the possible products \( (c_1)(c_2)^2...\langle c_n \rangle^n \). Here \( e_i = 0 \) or 1, and \( \{ c_1, c_2, ..., c_n \} \) are the elements of \( H^1(M_G; Z_2) \) such that \( \{ j^*(c_1), ..., j^*(c_k) \} \) make up a basis of the \( Z_2 \)-vector space \( H^1(M; Z_2) \).

Let \( V_i \) be the \( i \)-th Wu class of the tangent bundle \( T(R^m(\tau)) \) ([4, p132]). If \( V_i \in p^*H^i(RP(m); Z_2) \) for all \( i \), then \( W_k \in p^*H^k(RP(m); Z_2) \) for all \( k \) by the formula

\[
W_k = \sum_{i+j=k} Sq^i(V_j),
\]

where \( W_k \) is the \( k \)-th Stiefel-Whitney class of \( T(R^m(\tau)) \). This implies \( W_k(\bar{T}_m(M)) \in p^*H^k(RP(m); Z_2) \) for all \( k \) by the facts

\[
W(T(R^m(\tau))) = p^*W(T(RP(m)))W(\bar{T}_m(M))
\]

and \( W(T(RP(m))) = (1 + a)^{m+1} \), where \( a \in H^1(RP(m); Z_2) \) is a generator. Therefore by Remark 2.1, \( F \) must be of constant dimension if not empty.

Now suppose \( V_k \) contains a nontrivial summand of the form \( a^{n} c_1 c_2...c_j \), \( j \geq 1 \). Then we claim \( j = n \). To see this, we notice \( j^*W(\bar{T}_m(M)) = W(\bar{T}_m(M)) = 1 \) and that \( \bar{T}_m(M) \) is \( (nl) \)-dimensional. Thus there is no such summands as \( a^{n} c_1 c_2...c_n \) in \( W(\bar{T}_m(M)) \) neither in \( W(T(R^m(\tau))) \). By using the formula (1), we can write \( V_k \)
as the sum
\[ V_k = \sum Sq^i_1 Sq^i_2 \ldots Sq^i_j (W_j), \quad j_1 + j_2 + \ldots + j_i + j = k. \]
Since for all \( i \), \((c_i)^2 = \sum b_j c_j + b_0 \), where \( b_0 \) and \( b_j \) are elements in \( p^*H^*(RP(m); Z_2) \), we have
\[ (2) \quad Sq^{m'}(c_i) = \sum b'_j c_j + b'_0 \]
for some elements \( b'_j \) and \( b'_0 \) in \( p^*H^*(RP(m); Z_2) \). Then by (2), there is no such term \( a^{m'}c_1c_2\ldots c_n \) in \( Sq^{m'}(W_j) \) neither in \( V_k \). Therefore, we may assume \( V_k \) contains a nontrivial summand \( a^{m'}c_1c_2\ldots c_j, \quad 0 < j < n \). Then by the definition of Wu class ([4]),
\[ 1 = \langle V_k(a^{m-n'}c_{j+1}c_{j+2}\ldots c_n), \sigma \rangle \]
\[ = \langle Sq^{k}(a^{m-n'}c_{j+1}c_{j+2}\ldots c_n), \sigma \rangle \]
\[ = 0, \]
since \( Sq^k(a^{m-n'}c_{j+1}c_{j+2}\ldots c_n) \) must be zero by the formula (2). Here \( \sigma \in H_{m+n}(R^m(\tau); Z_2) \) is the homology fundamental class of the closed manifold \( R^m(\tau) \). This contradiction shows \( V_k \in p^*H^*(RP^\infty; Z_2) \) for all \( k \), and (ii) follows. \( \square \)

**Proposition 2.4.** Let \( M \) be a smooth closed manifold with a smooth involution \( \tau \) which induces the trivial \( Z_2 \) action on \( H^*(M; Z_2) \). Suppose \( M \sim_{Z_2} (S^2)^n \) and \( W(M) = 1 \). Let \( F \neq \varnothing \) and \( k \) be the constant dimension of \( F \). If \( Sq^1(x) \in p^*H^*(RP^\infty; Z_2) \) for all \( x \in H^*(M_G; Z_2) \), then \( k \) is even and \( F \) has at most \( 2^{n-k/2} \) components \( \{F_i\} \), and for each \( F_i \), \( H^*(F_i; Z_2) \) is algebraically generated by some elements \( \{b_{ij}\}_{1 \leq j \leq n} \) with \( b_{ij} \in H^2(F_i; Z_2) \) and \( b_{ij}^2 = 0 \) for all \( j \). In particular, \( H^*(F_1; Z_2) \) contains a subring which is isomorphic to \( H^*((S^2)^{k/2}; Z_2) \).

**Proof.** First the spectral sequence which converges to \( H^*(M_G; Z_2) \) with \( E_2^{p,q} = H^p(RP^\infty; H^q(M; Z_2)) \) collapses. By [3, Theorem 1.6, p374], \( M \) is totally nonhomologous to zero with coefficient in \( Z_2 \). Let \( \{c_i\}_{i=1,2,\ldots,n} \) be elements of \( H^2(M_G; Z_2) \) such that their restrictions on \( M \) form a basis of the \( Z_2 \) vector space \( H^2(M; Z_2) \). Let \( j_1 : (F_i)_G = RP^\infty \times F_i \to M_G \) be the inclusion. Then \( j^*_1 : H^k(M_G; Z_2) \to H^k((F_i)_G; Z_2) \) is a surjection for \( k > 2n \) ([3, Theorem 1.5, p374]). Note that \( H^*((F_i)_G; Z_2) \approx H^*(RP^\infty; Z_2) \otimes H^*(F_i; Z_2) \). Let
\[ j^*_1(c_t) = a \otimes b_{1t1} + 1 \otimes b_{1t2} + a^2 \otimes 1, \]
where \( b_{1tj} \in H^2(F_i; Z_2) \) for \( j = 1, 2 \) and \( a \in H^1(RP^\infty; Z_2) \) is a generator. Since \( j^*_1 \) is onto in high degrees, \( H^*(F_i; Z_2) \) is algebraically generated by the set \( \{1, b_{1t1}, b_{1t2}\}_{t=1,2,\ldots,n} \).

Next by the assumed condition, \( Sq^1(c_t) \in p^*H^3(RP^\infty; Z_2) \); thus \( j^*_1Sq^1(c_t) \in p^*H^*(RP^\infty; Z_2) \). We claim \( b_{1t1} = 0 \). Otherwise,
\[ j^*_1Sq^1(c_t) = Sq^1j^*_1(c_t) \]
\[ = Sq^1(1 \otimes b_{1t2} + a \otimes b_{1t1} + a^2 \otimes 1) \]
\[ = 1 \otimes Sq^1b_{1t2} + a^2 \otimes b_{1t1} + a \otimes (b_{1t1})^2 \]
\[ \notin p^*H^*(RP^\infty; Z_2). \]
This is a contradiction. Now we claim \( (b_{1t2})^2 = 0 \) for each \( t \). Let \( j : M \to M_G \),
Theorem 1.1, and that the structure of \( H \) algebraically the
Here we use the notation of Proposition 2.4. Then
\[
\{ \text{Sq} \}
\]
We claim \( H \) must be even and \( H^*(F_1; Z_2) \) has a subring which is isomorphic
to \( H^*((S^2)^{k/2}; Z_2) \). This together with the equation \( \sum_{j \geq 0} \text{rank } H^3(M; Z_2) = 2^n \)
\([3, \text{Theorem 1.6, p374}]\) shows the number of the components of \( F \) is at most
\( 2^{2n-k/2} \).

Proof of Theorem 1.3. First we prove the statement (i). By the Lefschetz fixed
point theorem, \( F \) is nonempty. By Theorem 1.2 (ii), \( F \) is of constant dimension. Note that \( S^q \) is \( \rho \beta \), where \( \rho \) and \( \beta \) fit into the Bockstein exact sequence

\[
\rightarrow H^2(M_G; Z_2) \xrightarrow{\beta} H^3(M_G; Z) \xrightarrow{2} H^3(M_G; Z) \xrightarrow{\beta} H^3(M_G; Z_2) \rightarrow.
\]

We claim \( H^\text{odd}(M_G; Z) = 0 \). Indeed, the spectral sequence which converges to
\( H^*(M_G; Z) \) with \( E_2^{p,q} = H^p(RP^\infty; H^q(M_G; Z)) \) collapses. Since
\[ H^3(M; Z) = \begin{cases} 0 & \text{if } q \text{ is odd,} \\ \text{free abelian} & \text{if } q \text{ is even,} \end{cases} \]
and \( H^\text{odd}(RP^\infty; Z) = 0 \), \( H^\text{odd}(M_G; Z) \) must be trivial. This implies \( \beta = 0 \) and
\( S^q(c_i) = \rho \beta(c_i) = 0 \), where \( \{c_i\} \) are as those in the proof of Proposition 2.4. Finally, \( \tau \) induces the trivial action on \( H^*(M; Z) \) implies the triviality of the induced
\( Z_2 \) action on \( H^*(M; Z_2) \). So (i) follows from Proposition 2.4.

Next we consider (ii). Similarly to the proof of Proposition 2.4, let \( \{c_1\}_{1 \leq j \leq n} \)
be the elements of \( H^1(M_G; Z_2) \) such that \( \{j^*(c_1)\} \) is a basis of the \( Z_2 \) vector space
\( H^1(M^n; Z_2) \), and let
\[
j^*_1(c_i) = 1 \otimes b_{it} + a \otimes 1, \quad a \in H^1(RP^\infty; Z_2), \ b_{it} \in H^1(F_1; Z_2).
\]
Here we use the notation of Proposition 2.4. Then \( \{1, b_{1t}, b_{2t}, \ldots, b_{nt}\} \) generate
algebraically the \( Z_2 \) algebra \( H^*(F_1; Z_2) \), and just as in the proof of Proposition 2.4, we have \( (b_{it})^2 = 0 \), and (ii) follows as (i). \( \square \)

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ON FIXED POINT SETS

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