APPROXIMATION FROM LOCALLY FINITE-DIMENSIONAL SHIFT-INVARIANT SPACES

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Abstract. After exploring some topological properties of locally finite-dimensional shift-invariant subspaces $S$ of $L^p(\mathbb{R}^s)$, we show that if $S$ provides approximation order $k$, then it provides the corresponding simultaneous approximation order. In the case $S$ is generated by a compactly supported function in $L_\infty(\mathbb{R})$, it is proved that $S$ provides approximation order $k$ in the $L^p(\mathbb{R})$-norm with $p > 1$ if and only if the generator is a derivative of a compactly supported function that satisfies the Strang-Fix conditions.

1. Introduction

Let $S$ be a linear space consisting of functions defined on $\mathbb{R}^s$. $S$ is said to be shift invariant if $f(\cdot + \alpha)$ lies in $S$ whenever $f$ does, for every $\alpha \in \mathbb{Z}^s$. $S$ is said to be locally finite-dimensional if the restriction of $S$ to any bounded subset of $\mathbb{R}^s$ is finite-dimensional. A typical example for such spaces is the linear span $S$ of a finite number of compactly supported functions and their shifts. That is,

$$S = S_0(\Phi) := \text{span}\{\varphi(\cdot + \alpha): \alpha \in \mathbb{Z}^s, \varphi \in \Phi\},$$

with $\Phi$ a finite family of compactly supported functions. When $\Phi$ consists of one function $\varphi$, we denote $S_0(\Phi)$ by $S_0(\varphi)$. $S_0(\varphi)$ is usually called a finitely generated shift-invariant space. It is clear that the shift-invariance and the local finite-dimension are purely algebraic properties. In this paper we shall show how to probe the (simultaneous) approximation order provided by $S$ by means of these two algebraic properties.

Let $m \geq 0$ be an integer and $k > 0$. A subspace $S$ of $L^p(\mathbb{R}^s)$ is said to provide simultaneous approximation order $(m,k)$ if

$$\inf_{g \in S} \sum_{j=0}^{m} \sum_{|\alpha|=j} h^j \|D^\alpha (f - g(\cdot/h))\|_p \leq C_f h^k$$

as $h \to 0+$, for every $f \in W^m_p(\mathbb{R}^s) \cap W^k_p(\mathbb{R}^s)$. Here, $C_f$ is a constant independent of $h$ and $D^\alpha$ is the $\alpha$-order differentiation operator. By convention, $S$ is said to provide approximation order $k$ when it provides simultaneous approximation order $(0,k)$. The simultaneous approximation order of shift-invariant subspaces generated by a
finite number of functions has been of interest in Approximation Theory and Finite Element Analysis for a long time and it is well known [1], [11], [12] that \( S_0(\varphi) \) provides approximation order \( k \) if \( \varphi \in L_p(\mathbb{R}^s) \) with \( p \geq 1 \) is compactly supported and satisfies the so-called Strang-Fix conditions of order \( k \):

\[
\begin{align*}
(\text{i}) & \quad \hat{\varphi}(0) \neq 0; \\
(\text{ii}) & \quad D^k_0 \varphi = 0 \text{ on } 2\pi \mathbb{Z}^s \setminus 0, \text{ for all } |\alpha| < k.
\end{align*}
\]

Here, \( \hat{\varphi} \) denotes the Fourier transform of \( \varphi \). The Strang-Fix conditions have been so well reputed because they enable us to determine the approximation order provided by \( S_0(\varphi) \), with \( \hat{\varphi}(0) \neq 0 \), by examining the single generator \( \varphi \), in spite of the fact that \( S_0(\varphi) \) is infinite-dimensional.

It is well known that the above-mentioned Strang-Fix conditions can also be described algebraically as follows. Denote by \( \varphi^{*'} \) the discrete convolution mapping with \( \varphi \). Namely,

\[
\varphi^{*'}: f \rightarrow \sum_{\alpha \in \mathbb{Z}^s} \varphi(\cdot - \alpha) f(\alpha).
\]

Since \( \varphi \) is assumed to be compactly supported, for each \( x \in \mathbb{R}^s \), \( \varphi^{*'} f(x) \) is a sum of finite number of terms. As shown in his paper [11], Shoenberg observed that \( S_0(\varphi) \) provides approximation order \( k \) if \( \varphi^{*'} \) maps \( \Pi_{k-1} \) onto \( \Pi_{k-1} \) in the univariate case, where \( \Pi_{k-1} \) is the linear space of all polynomials of degree \( < k \). As is well known now, the Strang-Fix conditions of order \( k \) are equivalent to that \( \varphi^{*'} \) maps \( \Pi_{k-1} \) onto \( \Pi_{k-1} \). Therefore, \( S_0(\varphi) \) provides approximation order \( k \) if \( \varphi \) has the algebraic property that \( \varphi^{*'} \) maps \( \Pi_{k-1} \) onto \( \Pi_{k-1} \). In his recent paper [8], Jia has shown the following interesting fact: For any function \( f \) defined on \( \mathbb{R}^s \), there exists \( \alpha: \mathbb{Z}^s \rightarrow \mathbb{C}^s \) such that \( f = \varphi^{*'} \alpha \) if and only if \( S_0(\varphi) \) locally contains \( f \). Here, \( S_0(\varphi) \) is said to locally contain \( f \) if the restriction of \( S_0(\varphi) \) to any compact subset \( B \) contains the restriction of \( f \) to \( B \). Therefore, if \( \hat{\varphi}(0) \neq 0 \), then \( S_0(\varphi) \) provides approximation order \( k \) if and only if \( S_0(\varphi) \) locally contains \( \Pi_{k-1} \) [8]. We note that \( S_0(\varphi) \) locally containing \( \Pi_{k-1} \) cannot guarantee it to provide approximation order \( k \) in general.

When \( \hat{\varphi}(0) \neq 0 \), it is well known that there is a local approximation scheme that realizes the approximation order provided by \( S_0(\varphi) \) [1].

The condition that \( \hat{\varphi}(0) \neq 0 \) has been assumed in the past study of approximation order of \( S_0(\varphi) \). As shown by Strang and Fix [12], for any compactly supported \( \varphi \in L_2(\mathbb{R}^s) \), if \( S_0(\varphi) \) provides an approximation order \( k \) via a controlled approximation scheme, then this condition is also necessary. But \( S_0(\varphi) \) may provide a positive approximation order even if \( \hat{\varphi}(0) = 0 \). One well-known example is the function

\[
\varphi(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 1; \\
-1, & \text{if } 1 < x \leq 2; \\
0, & \text{else}.
\end{cases}
\]

It is known that \( S_0(\varphi) \) indeed provides approximation order \( 1 \) in the \( L_2(\mathbb{R}) \)-norm. In the recent paper by de Boor, DeVore, and Ron [4], the authors obtained a necessary and sufficient condition for \( S_0(\varphi) \) to provide approximation order \( k \) in the \( L_2(\mathbb{R}^s) \)-norm, for any generator \( \varphi \in L_2(\mathbb{R}^s) \). A corresponding result of a necessary and sufficient condition under which \( S_0(\varphi) \) provides simultaneous approximation order \((m, k)\) has been presented in [13], for any \( \varphi \in W_2^m(\mathbb{R}^s) \). As shown in [13], for
any compactly supported univariate function \( \varphi \in W_2^m(\mathbb{R}) \), \( S_0(\varphi) \) provides simultaneous approximation order \((m,k)\) in the \( L_2(\mathbb{R})\)-norm if and only if there exist a neighborhood \( \Omega_\alpha \) of the origin and a constant \( C_\alpha \) such that
\[
|\hat{\varphi}(x + 2\pi\alpha)| \leq C_\alpha |x|^k |\hat{\varphi}(x)|, \quad \forall x \in \Omega_\alpha,
\]
for all \( \alpha \in \mathbb{Z}\setminus\{0\} \). In particular, this implies that if \( S_0(\varphi) \) provides approximation order \( k \), then it also provides simultaneous approximation order \((m,k)\).

In the next sections we shall prove that, for any shift-invariant subspace \( S \subset W_\infty^m(\mathbb{R}^s) \) that is locally finite-dimensional, if \( S \) provides approximation order \( k \), then it also provides simultaneous approximation order \((m,k)\). In the univariate case, we shall prove that, for any nontrivial compactly supported \( \varphi \in L_p(\mathbb{R}) \cap L_{p'}(\mathbb{R}) \) with \( p > 1 \), \( S_0(\varphi) \) provides approximation order \( k \) if and only if \( \varphi \) satisfies (1.3) for all \( \alpha \in \mathbb{Z}\setminus\{0\} \), where \( p' = p/(p-1) \) is the conjugate number of \( p \). In other words, in this case, \( S_0(\varphi) \) provides approximation order \( k \) if and only if
\[
D^k \hat{\varphi} = 0 \quad \text{on } 2\pi\mathbb{Z}\setminus\{0\}, \forall 0 \leq \alpha < k + m,
\]
where \( m \) is the smallest integer such that \( D^m \hat{\varphi}(0) \neq 0 \). As we shall see, any compactly supported function \( \varphi \in L_p(\mathbb{R}) \) satisfies (1.4) if and only if it is the \( m \)-th order derivative of a compactly supported function that satisfies the Strang-Fix condition of order \( k + m \), with \( m \) the smallest integer such that \( D^m \hat{\varphi}(0) \neq 0 \). Therefore, among the compactly supported functions in \( L_p(\mathbb{R}) \) are only those that satisfy the Strang-Fix conditions or their derivatives the candidates for generating a shift-invariant space that may provide a positive approximation order.

2. Locally finite-dimensional shift-invariant spaces

In this section we show some nice topological properties owned by every shift-invariant subspace \( S \subset L_p(\mathbb{R}^s) \) that is locally finite-dimensional.

**Proposition 2.1.** Let \( S \subset L_p(\mathbb{R}^s) \) be a shift-invariant subspace with \( 1 \leq p \leq \infty \) such that \( S \) is locally finite-dimensional. For any linear mapping \( A: S \to L_q(\mathbb{R}^s) \), with \( 1 \leq q \leq \infty \), if \( A \) commutes with integer translations, then there exist two positive constants \( C_1 \) and \( C_2 \) such that, for all \( f \in S \),
\[
C_1 \left( \sum_{\alpha \in \mathbb{Z}^s} \min_{g \in \ker A} \|f(\cdot + \alpha) - g\|_{L_p([0..1]^s)}^q \right)^{1/q} 
\leq \|Af\|_q \leq C_2 \left( \sum_{\alpha \in \mathbb{Z}^s} \min_{g \in \ker A} \|f(\cdot + \alpha) - g\|_{L_p([0..1]^s)}^q \right)^{1/q}
\]
when \( q < \infty \),
\[
C_1 \sup_{\alpha \in \mathbb{Z}^s} \min_{g \in \ker A} \|f(\cdot + \alpha) - g\|_{L_p([0..1]^s)} 
\leq \|Af\|_q \leq C_2 \sup_{\alpha \in \mathbb{Z}^s} \min_{g \in \ker A} \|f(\cdot + \alpha) - g\|_{L_p([0..1]^s)}
\]
when \( q = \infty \), where \( \ker A_1 := \{ f \in S: Af = 0 \text{ on } [0..1]^s \} \).

**Proof.** It is clear that \( \ker A_1 \) is a subspace of \( S \). For any \( f \in S \) and \( g \in S \), the restriction of \( Af \) to \([0..1]^s \) equals the restriction of \( Ag \) to \([0..1]^s \) if and only if
\[
A(f - g) = Af - Ag = 0 \quad \text{on } [0..1]^s.
\]
This proves that the restriction of the range of $A$ to $[0,1]^*$ is isomorphic to the restriction of the quotient space $S/\ker A_1$ to $[0,1]^*$. Since $S$ is locally finite-dimensional, the restrictions of $S/\ker A_1$ and $S$ to $[0,1]^*$ are finite-dimensional. As is well known, any linear mapping on a finite-dimensional normed space is bounded and any two norms on a finite-dimensional space are equivalent. Consequently, there exist two positive constants $C_1$ and $C_2$ such that

$$C_1 \min_{g \in \ker A_1} \| f - g \|_{L_p([0,1]^*)} \leq \| Af \|_{L_q([0,1]^*)} \leq C_2 \min_{g \in \ker A_1} \| f - g \|_{L_p([0,1]^*)}$$

for all $f \in S$. Since $S$ is shift invariant, $f \in S$ implies that $f(\cdot + \alpha) \in S$ for any $\alpha \in \mathbb{Z}^s$. Also we have that

$$(Af)(\cdot + \alpha) = A(f(\cdot + \alpha)), \quad \forall \alpha \in \mathbb{Z}^s$$

because $A$ commutes with integer translations. When $q = \infty$, (2.2) follows from (2.3) and the fact that $S$ is shift invariant. When $q < \infty$, it follows from (2.3) that, for any $f \in S$,

$$\int_{\mathbb{R}^s} |(Af)(x)|^q \, dx = \sum_{\alpha \in \mathbb{Z}^s} \int_{[0,1]^*} |(Af)(x + \alpha)|^q \, dx$$

$$= \sum_{\alpha \in \mathbb{Z}^s} \int_{[0,1]^*} |A(f(x + \alpha))|^q \, dx$$

$$\leq \sum_{\alpha \in \mathbb{Z}^s} C_2 \min_{g \in \ker A_1} \left( \int_{[0,1]^*} |f(x + \alpha) - g|^p \, dx \right)^{q/p}$$

$$= C_2 \sum_{\alpha \in \mathbb{Z}^s} \min_{g \in \ker A_1} \| f(\cdot + \alpha) - g \|_{L_p([0,1]^*)}.$$

Analogously we can establish the other inequality of (2.1). \hfill \Box

In the case where $p = q$ we obtain that $\| Af \|_p \leq C_2 \| f \|_p$.

**Corollary 2.2.** Let $1 \leq p \leq \infty$ and $S \subset L_p(\mathbb{R}^s)$ be a shift-invariant subspace that is locally finite-dimensional. If $A: S \to L_p(\mathbb{R}^s)$ is a linear mapping that commutes with integer translations, then $A$ is bounded.

For any polynomial of degree $n$: $p_n = \sum_{|\alpha| \leq n} a_\alpha(\cdot)^\alpha$, by $p_n(D)$ we mean the differential operator

$$p_n(D) := \sum_{|\alpha| \leq n} a_\alpha D^\alpha.$$

It is clear that $p_n(D)$ is a linear mapping from $W_p^n(\mathbb{R}^s)$ to $L_p(\mathbb{R}^s)$. For any sufficiently smooth function $f$ and any $t \in \mathbb{R}^s$,

$$D^\alpha(f(\cdot + t)) = (D^\alpha f)(\cdot + t).$$

This shows that $p_n(D)$ commutes with integer translations.

**Corollary 2.3.** Let $p_n$ be any polynomial in $s$-variable of degree $n$ and $S$ a shift-invariant subspace of $W_p^n(\mathbb{R}^s)$ that is locally finite-dimensional. Then there exists a constant $C$ such that $\| p_n(D)f \|_p \leq C \| f \|_p$ for all $f$ in $S$. 

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It is well known that
\[
\left( \sum_{\alpha \in \mathbb{Z}^s} |a(\alpha)|^q \right)^{1/q} \leq \left( \sum_{\alpha \in \mathbb{Z}^s} |a(\alpha)|^p \right)^{1/p}
\]
if \(1 \leq p \leq q \leq \infty\) and \(a \in L_q(\mathbb{Z}^s)\). From the proof of Proposition 2.1 we obtain

**Corollary 2.4.** Let \(1 \leq p < q \leq \infty\) and \(S \subseteq L_p(\mathbb{R}^s)\) be a shift-invariant subspace that is locally finite-dimensional and is locally contained in \(L_q(\mathbb{R}^s)\). Then, \(S \subseteq L_q(\mathbb{R}^s)\) and there exists a constant \(C\) such that \(\|f\|_q \leq C\|f\|_p\) for all \(f \in S\).

**Corollary 2.5.** Let \(S \subseteq L_p(\mathbb{R}^s) \cap L_q(\mathbb{R}^s)\) be a shift-invariant subspace that is locally finite-dimensional, with \(1 \leq p < q \leq \infty\). Then the closure of \(S\) in \(L_p(\mathbb{R}^s)\) is contained in the closure of \(S\) in \(L_q(\mathbb{R}^s)\).

When \(S = S_0(\Phi)\) is generated by a finite number of compactly supported functions in \(L_p(\mathbb{R}^s)\), we denote by \(S_+(\Phi)\) the closure of \(S_0(\Phi)\) in the topology of pointwise convergence. Namely, \(f \in S_+(\Phi)\) if and only if there is a sequence \(s_j \in S_0(\Phi)\) such that \(\lim_{j \to \infty} s_j(x) = f(x)\) for almost every \(x \in \mathbb{R}^s\). One can verify that \(S_+(\Phi)\) consists of all functions of the form
\[
\sum_{\varphi \in \Phi} \varphi^* a_\varphi,
\]
for all \(a_\varphi : \mathbb{Z}^s \to \mathbb{C}^s\). As shown in [8], \(f\) is contained in \(S_+(\Phi)\) if and only if \(f\) is locally contained in \(S_0(\Phi)\). Since the topology of \(L_p(\mathbb{R}^s)\) is stronger than that of \(S_+(\Phi)\), we have the following corollary that was first observed by Jia.

**Corollary 2.6.** For any finite family \(\Phi\) of compactly supported functions in \(L_p(\mathbb{R}^s)\), \(S_+(\Phi) \cap L_p(\mathbb{R}^s)\) is a closed subspace of \(L_p(\mathbb{R}^s)\).

Since \(S_+(\Phi) \cap L_p(\mathbb{R}^s)\) is also shift-invariant and locally finite-dimensional, from Corollary 2.4 we obtain

**Proposition 2.7.** For any \(1 \leq p < q \leq \infty\) and any finite \(\Phi \subseteq L_p(\mathbb{R}^s) \cap L_q(\mathbb{R}^s)\) consisting of compactly supported functions, we have that \(S_+(\Phi) \cap L_p(\mathbb{R}^s) \subseteq S_+(\Phi) \cap L_q(\mathbb{R}^s)\).

For \(S_0(\Phi) \subseteq L_p(\mathbb{R}^s)\), denote by \(S_p(\Phi)\) the closure of \(S_0(\Phi)\) in \(L_p(\mathbb{R}^s)\). Let \(\varphi\) be a compactly supported function in \(L_2(\mathbb{R}^s)\). As proved by de Boor, DeVore, Ron [4], \(S_+(\varphi) \cap L_2(\mathbb{R}^s)\) is a subspace of \(S_2(\varphi)\). Since \(S_+(\varphi) \cap L_2(\mathbb{R}^s)\) contains \(S_0(\varphi)\) and is closed, it follows that \(S_2(\varphi) = S_+(\varphi) \cap L_2(\mathbb{R}^s)\). We shall show that this has an extension to the case where \(1 < p < 2\) and \(\varphi \in L_p(\mathbb{R}^s)\). Recall that \(p' = p/(p-1)\). For the proof we need

**Proposition 2.8.** Let \(1 < p < \infty\) and \(\Phi \subseteq L_p(\mathbb{R}^s) \cap L_{p'}(\mathbb{R}^s)\) be a finite family of compactly supported functions. Then, \(S_{p'}(\Phi)\) can be identified with the dual space \((S_p(\Phi))^*\) of \(S_p(\Phi)\).

**Proof.** As \(S_{p'}(\Phi)\) is a closed subspace of \(L_{p'}(\mathbb{R}^s)\), we know that the dual space of \(S_{p'}(\Phi)\) is isomorphic to the quotient space \(L_{p'}(\mathbb{R}^s)/(S_{p'}(\Phi))^\perp\) [10]. Since \(L_p(\mathbb{R}^s)\) and \(L_{p'}(\mathbb{R}^s)\) are reflexive and \(S_p(\Phi)\) is a closed subspace of \(L_p(\mathbb{R}^s)\), we have that
\[
(S_p(\Phi))^* = (L_p(\mathbb{R}^s)/(S_p(\Phi))^\perp)^* = (S_p(\Phi))^\perp = S_{p'}(\Phi).
\]
It suffices to prove the case where \(p \leq 2\) because, otherwise, \(p' \leq 2\) and \((S_p(\Phi))^* = (S_p(\Phi))^\perp = S_{p'}(\Phi)\).

First we prove that \(S_{p'}(\Phi)\) is dense in \((S_p(\Phi))^*\). For any
That is, \( f \in (S_p(\Phi))^\ast = S_p(\Phi) \) that annihilates \( S_{p'}(\Phi) \), \( \int_{R^n} |f|^2 = 0 \) because \( f \) lines in \( S_p(\Phi) \subset S_{p'}(\Phi) \). So \( f = 0 \). This proves that \( S_{p'}(\Phi) \) is dense in \((S_p(\Phi))^\ast \). Therefore, 
\[
S_p(\Phi) = (S_p(\Phi))^\ast = (S_{p'}(\Phi))^\ast.
\]

\[ (2.4) \quad S_{p'}(\Phi) = (S_p(\Phi))^\ast = (S_{p'}(\Phi))^\ast. \quad \square \]

**Theorem 2.9.** Let \( 1 < p \leq 2 \) and \( p' = p/(p-1) \). For any compactly supported \( \varphi \in L_{p'}(R^s) \), \( S_p(\varphi) = S_4(\varphi) \cap L_p(R^s) \).

**Proof.** By Corollary 2.5 and Proposition 2.7, 
\[
S_p(\varphi) \subset S_4(\varphi) \cap L_p(R^s) \subset S_4(\varphi) \cap L_2(R^s) = S_2(\varphi) \subset S_{p'}(\varphi).
\]

This implies the dual space of \( S_4(\varphi) \cap L_p(R^s) \) contains that of \( S_p(\varphi) \). As we know, \( S_p(\varphi) \) and \( S_4(\varphi) \cap L_p(R^s) \) both are closed in \( L_p(R^s) \). From the reflexity it follows that some closed subspace of \( S_{p'}(\varphi) \) can be identified with \((S_4(\varphi) \cap L_p(R^s))^\ast \). From Proposition 2.8 we know that \( S_{p'}(\varphi) \) can be identified with the dual space of \( S_p(\varphi) \). Hence, \( S_p(\varphi) \) and \( S_4(\varphi) \cap L_p(R^s) \) have the same dual space \( S_{p'}(\varphi) \). Thus we obtain that 
\[
S_p(\varphi) = (S_p(\varphi))^\ast = (S_4(\varphi) \cap L_p(R^s))^\ast = S_4(\varphi) \cap L_p(R^s). \quad \square
\]

3. Application 1: Multivariate approximation

In this section we apply the results obtained in Section 2 to obtain some results about multivariate approximation from shift-invariant spaces that are locally finite-dimensional.

**Theorem 3.1.** Let \( S \subset W^m_p(R^s) \) be a shift-invariant subspace that is locally finite-dimensional. If \( S \) provides approximation order \( k \) in the \( L_p(R^s) \)-norm, then \( S \) also provides simultaneous approximation order \((m,k)\).

**Proof.** For any \( f \in W^m_p(R^s) \), there exists \( s_h \in S \) such that 
\[
||f - s_h(\cdot/h)||_p \leq C_f h^k
\]

with some constant \( C_f \) independent of \( h \). Let \( \psi \) be any compactly supported function in \( W^m_p(R^s) \) such that \( S_0(\psi) \) provides simultaneous approximation order \((m,k)\). For instance, we can choose \( \psi \) as a tensor product of some univariate B-spline functions [5]. Let 
\[
S_+ := S + S_0(\psi).
\]

That is, \( S_+ \) is the space spanned by \( S \) and \( S_0(\psi) \). It is clear that \( S_+ \) is shift invariant and locally finite-dimensional. Therefore, by Corollary 2.3, there exists a constant \( C \) independent of \( h \) such that, for any \(|\alpha| \leq m\),
\[
(3.1) \quad ||D^\alpha g(\cdot/h)||_p \leq Ch^{-|\alpha|}||g(\cdot/h)||_p, \quad \forall g \in S_+.
\]

Since \( S_0(\psi) \) provides simultaneous approximation order \( k \), there exists \( g_h \) in \( S_0(\psi) \) such that 
\[
||D^\alpha (f - g_h(\cdot/h))||_p \leq B_f h^{k-|\alpha|}
\]
for all $|\alpha| \leq m$, where $B_f$ is some constant independent of $h$. As $g_h - s_h$ lies in $S_+$,

$$\|D^\alpha (f - s_h(\cdot/h))\|_p \leq \|D^\alpha (f - g_h(\cdot/h))\|_p + \|D^\alpha (g_h(\cdot/h) - s_h(\cdot/h))\|_p$$

$$\leq B_f h^{|\alpha|} + Ch^{-|\alpha|} \|g_h(\cdot/h) - s_h(\cdot/h)\|_p$$

$$\leq B_f h^{|\alpha|} + Ch^{-|\alpha|}(\|f - g_h(\cdot/h)\|_p + \|f - s_h(\cdot/h)\|_p)$$

$$\leq (B_f + CB_f + CC_f) h^{k - |\alpha|}.$$ 

From the above proof we obtain the following

**Corollary 3.2.** Let $S \subset W^m_p(\mathbb{R}^s)$ be a shift-invariant subspace that is locally finite-dimensional. For any $f \in W^m_p(\mathbb{R}^s) \cap W^k_p(\mathbb{R}^s)$ and any $s_h \in S_0(\Phi)$, if $\|f - s_h(\cdot/h)\|_p = O(h^k)$, then $\|D^\alpha (f - s_h(\cdot/h))\|_p = O(h^{k - |\alpha|})$ for every $|\alpha| \leq m$.

This shows that any approximation scheme having approximation order $k$ in the $L_p(\mathbb{R}^s)$-norm automatically has simultaneous approximation order $(m,k)$ in this case.

**Theorem 3.3.** Let $S \subset L_p(\mathbb{R}^s) \cap L_q(\mathbb{R}^s)$ be a shift-invariant subspace with $1 \leq p < q \leq \infty$, such that $S$ is locally finite-dimensional. If $S$ provides approximation order $k$ in the $L_p(\mathbb{R}^s)$-norm, then $S$ provides approximation order larger than or equal to $k - s(1/p - 1/q)$ in the $L_q(\mathbb{R}^s)$-norm.

**Proof.** Let $\psi$ be a compactly supported function in $L_\infty(\mathbb{R}^s)$ such that $\psi$ satisfies the Strang-Fix conditions of order $k$. Then $S_0(\psi)$ provides approximation order $k$ in any $L_r(\mathbb{R}^s)$-norm for all $1 \leq r \leq \infty$. Moreover [1], there is a local approximation scheme

$$Q_h : W^k_p(\mathbb{R}^s) \rightarrow S(\psi) := \psi^{s'} \ell_r(\mathbb{Z}^s)$$

independent of $r$ such that $\|f - (Q_hf)(\cdot/h)\|_r = O(h^k)$.

Let $S_+$ be the space spanned by $S$ and $S(\psi)$. It is clear that $S_+$ is also shift invariant and locally finite-dimensional. By Corollary 2.4, there exists a constant $C$ such that $\|f\|_q \leq C\|f\|_p$, for all $f \in S_+$. For $f \in W^k_p(\mathbb{R}^s) \cap W^k_q(\mathbb{R}^s)$ and some $s_h \in S$,

$$\|f - s_h(\cdot/h)\|_q \leq \|f - (Q_hf)(\cdot/h)\|_q + \|(Q_hf)(\cdot/h) - s_h(\cdot/h)\|_q$$

$$= h^{s/q} \|Q_hf - s_h\|_q + O(h^k)$$

$$\leq h^{s/q} C \|Q_hf - s_h\|_p + O(h^k)$$

$$= h^{s/q} (1 - p/q) C \|Q_hf)(\cdot/h) - s_h(\cdot/h)\|_p + O(h^k)$$

$$\leq C h^{-s(1/p - 1/q)}(\|f - (Q_hf)(\cdot/h)\|_p + \|f - s_h(\cdot/h)\|_p) + O(h^k)$$

$$= O(h^{k - s(1/p - 1/q)})$$

because $S$ provides approximation order $k$ in the $L_p(\mathbb{R}^s)$-norm. 

When $s(1/p - 1/q) < 1$, $k$ is an integer, and $S$ provides an integral approximation order in the $L_q(\mathbb{R}^s)$-norm, it follows that $S$ also provides approximation order $k$ in the $L_q(\mathbb{R}^s)$-norm. In particular, if $S$ is known to provide an integral approximation order in the $L_q(\mathbb{R}^s)$-norm for all $q \geq p$, $S$ providing approximation order $k$ in the $L_p(\mathbb{R}^s)$-norm with $k$ an integer implies $S$ providing (at least) approximation...
order $k$ in the $L_q(\mathbb{R}^s)$-norm, for all $q \geq p$. As we shall see, in the univariate case, the approximation order of any shift-invariant space generated by a compactly supported function must be an integer, provided $p \geq 2$.

4. Application 2: Univariate approximation

In the univariate case, there is an algebraic characterization for $S$ to provide approximation order $k$ even if $\hat{\varphi}(0) = 0$ for every $\varphi \in S \cap L_1(\mathbb{R})$, where $S$ is a shift-invariant subspace of $L_p(\mathbb{R})$ that is locally finite-dimensional. Let $\Phi \subset L_p(\mathbb{R})$ be a finite family of compactly supported functions. Recall that $S_\ast(\Phi)$ is the closure of $S_0(\Phi)$ in the topology of pointwise convergence. As proved by Jia [7], $S_\ast(\Phi)$ provides approximation order $k$ if and only if there is a compactly supported $\psi \in S_\ast(\Phi) \cap L_p(\mathbb{R})$ such that $\psi^{*t}$ maps $\Pi_{k-1}$ onto $\Pi_{k-1}$. Namely, $S_\ast(\Phi)$ provides approximation order $k$ if and only if there exist sequences $a_\varphi : \mathbb{Z} \to \mathbb{C}$ such that

$$\psi = \sum_{\varphi \in \Phi} \varphi^{*t} a_\varphi \in L_p(\mathbb{R})$$

is compactly supported and satisfies the Strang-Fix conditions of order $k$. This result reveals an intrinsic property of $S_\ast(\varphi)$ that provides a positive approximation order, as well as of $\varphi$ because $S_\ast(\varphi)$ is the closure of $S_0(\varphi)$ in the topology of pointwise convergence. As pointed out in [7], this follows that the approximation order provided by $S_\ast(\Phi)$ is an integer. Of another interest is that, when $\Phi \subset L_p(\mathbb{R}) \cap L_q(\mathbb{R})$ with $p < q$, $S_\ast(\Phi)$ provides approximation order $k$ in the $L_q(\mathbb{R})$-norm if and only if it provides approximation order $k$ in the $L_p(\mathbb{R}^s)$-norm because, as we proved in Section 2, $S_\ast(\Phi) \cap L_p(\mathbb{R}^s) \subset S_\ast(\Phi) \cap L_q(\mathbb{R}^s)$ and a compactly supported function in $L_q(\mathbb{R}^s)$ lies in $L_p(\mathbb{R})$.

As $S_\ast(\Phi)$ is an infinite-dimensional space unless it is trivial, apparently it is non-trivial to determine if $S_\ast(\Phi)$ contains a compactly supported function that satisfies the Strang-Fix conditions of order $k$, even if $\Phi$ consists of a single compactly supported function in $L_p(\mathbb{R})$. For any given compactly supported function $\varphi \in L_p(\mathbb{R})$, it is more practically interesting to have a necessary and sufficient condition on the generator $\varphi$ itself for determining the approximation order provided by $S_0(\varphi)$. In the following we shall show that when $S = S_0(\varphi)$ with $\varphi \in L_p(\mathbb{R}) \cap L_p'(\mathbb{R})$ compactly supported then $S_\ast(\varphi)$ and $S_0(\varphi)$ provide the same approximation order in the $L_p(\mathbb{R})$-norm for $p > 1$. In particular, we prove that, for any compactly supported $\varphi \in L_p(\mathbb{R}) \cap L_p'(\mathbb{R})$, with $p > 1$, $S_0(\varphi)$ provides approximation order $k$ if and only if $\varphi = D^m \psi$ for some compactly supported $\psi \in W^m_p(\mathbb{R})$ that satisfies the Strang-Fix conditions of order $k+m$. In other words, $S_0(\varphi)$ provides approximation order $k$ if and only if there exists an integer $m \geq 0$ such that $D^m \hat{\varphi}(0) \neq 0$, and $D^\alpha \hat{\varphi} = 0$ on $2\pi \mathbb{Z} \setminus \{0\}$ for all $0 \leq \alpha < k + m$.

**Theorem 4.1.** Let $1 < p \leq \infty$ and $\varphi \in L_p(\mathbb{R}) \cap L_p'(\mathbb{R})$ be compactly supported. Denote by $S_p(\varphi)$ and $S_\ast(\varphi)$ the closure of $S_0(\varphi)$ in the $L_p(\mathbb{R})$-norm and in the topology of pointwise convergence, respectively. Then $S_0(\varphi)$ provides approximation order $k$ in the $L_p(\mathbb{R})$-norm if and only if there exists a compactly supported function $\psi \in S_2(\varphi) \cap S_p(\varphi) \cap S_\ast(\varphi)$ such that $\psi$ satisfies the Strang-Fix conditions of order $k$.

**Proof.** We only need to prove the necessity because $S_0(\varphi)$ and $S_\ast(\varphi)$ provide the same approximation order. It is clear that if $S_0(\varphi)$ provides approximation order $k$ in the $L_p(\mathbb{R})$-norm, then so does $S_\ast(\varphi) \cap L_p(\mathbb{R})$. Therefore, $S_\ast(\varphi)$ contains a

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corresponding compactly supported function $\psi \in L_p(\mathbb{R})$ that satisfies the Strang-Fix conditions of order $k$. By Theorem 2.16 in [4], $\psi$ lies in $S_2(\varphi)$ if $p \geq 2$. By Corollary 2.5, $S_2(\varphi)$ is contained in $S_p(\varphi)$ when $p \geq 2$. In the case $p < 2$, by Theorem 2.9 and Corollary 2.5, $S_p(\varphi) = S_\alpha(\varphi) \cap L_\psi(\mathbb{R}) \subset S_2(\varphi)$.

When $p \geq 2$, we have that $p' \leq p$. Hence, if $p \geq 2$, then the condition that the compactly supported function $\varphi$ lies in $L_p(\mathbb{R})$ is automatically satisfied. As an immediate consequence, we obtain that the approximation order provided by $S_0(\varphi)$ is an integer if $\varphi \in L_p(\mathbb{R}) \cap L_\psi(\mathbb{R})$ is compactly supported and $p > 1$.

**Corollary 4.2.** Let $\varphi \in L_p(\mathbb{R}) \cap L_\psi(\mathbb{R})$ be a compactly supported function, with $1 < p \leq \infty$. Then, $S_0(\varphi)$ provides approximation order $k$ in the $L_\psi(\mathbb{R})$-norm if and only if $S_0(\varphi)$ provides approximation order $k$ in the $L_2(\mathbb{R})$-norm.

**Proof.** It suffices to prove that $S_0(\varphi)$ providing approximation order $k$ in the $L_2(\mathbb{R})$-norm implies $S_0(\varphi)$ providing approximation order $\geq k$ in the $L_p(\mathbb{R})$-norm.

If $S_2(\varphi)$ provides approximation order $k$, then $S_2(\varphi) = S_\alpha \cap L_\psi(\mathbb{R})$ contains a compactly supported function $\psi$ that satisfies the Strang-Fix conditions of order $k$. When $p > 2$, from that $S_2(\varphi) \subset S_p(\varphi)$ it follows that $S_p(\varphi)$ also provides approximation order $k$. In the case $p < 2$, we have that $\psi \in S_\alpha(\varphi) \cap L_p(\mathbb{R}^*) = S_p(\varphi)$, because $\psi$ is compactly supported. So $S_p(\varphi)$ provides approximation order $k$. \[\square\]

As we know, for a nontrivial compactly supported $\varphi \in L_2(\mathbb{R})$, $S_0(\varphi)$ provides approximation order $k$ if and only if, for each $\alpha \in \mathbb{Z} \setminus 0$, there exist a neighborhood $\Omega_\alpha$ of the origin and a constant $C_\alpha$ such that (1.3) holds. So we have

**Theorem 4.3.** For any nontrivial compactly supported function $\varphi \in L_p(\mathbb{R}) \cap L_\psi(\mathbb{R})$, with $p$ satisfying $1 < p \leq \infty$, $S_0(\varphi)$ provides approximation order $k$ if and only if, for each $\alpha \in \mathbb{Z} \setminus 0$, there exist a neighborhood $\Omega_\alpha$ of the origin and a constant $C_\alpha$ such that

$$|\hat{\varphi}(x + 2\pi\alpha)| \leq C_\alpha|x|^k|\hat{\varphi}(x)|, \quad \forall x \in \Omega_\alpha.$$  

(4.1)

For any nontrivial compactly supported function $\varphi \in L_1(\mathbb{R})$, there exists an integer $m \geq 0$ such that $D^m\hat{\varphi}(0) = 0$ for all nonnegative integers $\alpha < m$ but $D^m\hat{\varphi}(0) \neq 0$. As one can verify, (4.1) is equivalent to that $D^\beta\hat{\varphi}(2\pi\alpha) = 0$ for all integers $0 \leq \beta < k + m$, where $m$ is the smallest integer such that $D^m\hat{\varphi}(0) \neq 0$. When $m > 0$, it is clear that

$$\varphi_1(x) := \int_{-\infty}^x \varphi(t) \, dt$$

is a compactly supported continuous function and for almost every $x \in \mathbb{R}$ we have that $D\varphi_1(x) = \varphi(x)$. Thus we obtain

$$\hat{\varphi}_1(x) = \frac{\hat{\varphi}(x)}{ix}, \quad \forall x \neq 0$$

and $\hat{\varphi}_1(0) = -iD\hat{\varphi}(0)$. When $m \geq 1$, define

$$\varphi_m = \int_{-\infty}^x \frac{(x-t)^{m-1}}{(m-1)!} \varphi(t) \, dt.$$ 

by induction we can prove that $\varphi_m$ is compactly supported, $D^m\varphi_m = \varphi$, and

$$\hat{\varphi}_m(x) = \frac{\hat{\varphi}(x)}{(ix)^m}, \quad \forall x \neq 0.$$ 

(4.2)
Note that $\lim_{x \to 0} \hat{\varphi}_m(x) = (-i)^m D^m \hat{\varphi}(0) \neq 0$.

**Corollary 4.4.** Let $p > 1$, $\varphi \in L_p(\mathbb{R}) \cap L'_p(\mathbb{R})$ be a compactly supported function, and $m$ be the smallest integer such that $D^m \varphi(0) \neq 0$. Then, $S_0(\varphi)$ provides approximation order $k$ in the $L_p(\mathbb{R})$-norm if and only if $\varphi = D^m \psi$ for some compactly supported function $\psi \in W^m_p(\mathbb{R})$ that satisfies the Strang-Fix conditions of order $k + m$.

**Corollary 4.5.** For $p > 1$ and any compactly supported function $\varphi \in L_p(\mathbb{R}) \cap L'_p(\mathbb{R})$, $S_0(\varphi)$ provides approximation order $k \geq 1$ if and only if it locally contains $\Pi_{k-1}$.

**Proof.** We only need to prove the sufficiency because the necessity has been proved by Jia [8]. Since $S_0(\varphi)$ locally contains a nontrivial subspace $\Pi_{k-1}$, $\varphi$ is not trivial. So, $\varphi = D^m \psi$ for some compactly supported $\psi \in W^m_p(\mathbb{R})$ that satisfies $\psi(0) \neq 0$. Note that $D^m S_0(\psi) = S_0(\varphi)$. It follows that $S_0(\psi)$ locally contains $\Pi_{k+m-1}$. Since $\psi(0) \neq 0$, we know that $S_0(\psi)$ locally contains $\Pi_{k+m-1}$ if and only if $\psi$ satisfies the Strang-Fix conditions of order $k + m$. Therefore, $S_0(\varphi)$ provides approximation order $k$.

**Example.** Let $\varphi$ be the function defined by (1.2). It is clear that $\varphi$ is bounded and is the first-order derivative of the following B-spline:

$$
\psi(x) = \begin{cases} 
  x, & \text{if } 0 \leq x \leq 1; \\
  2 - x, & \text{if } 1 < x \leq 2; \\
  0, & \text{else}.
\end{cases}
$$

(4.3)

One can verify that $\psi$ satisfies the Strang-Fix conditions of order 2. Thus we know that $S_0(\varphi)$ does provide approximation order 1 in the $L_p(\mathbb{R})$-norm for $p > 1$.

As we know, if $\varphi(0) = 0$, then $S_0(\varphi)$ cannot provide any positive approximation order in the $L_1(\mathbb{R})$-norm. When $\varphi \in L_1(\mathbb{R})$ is compactly supported and $\varphi(0) \neq 0$, it is well known that $S_0(\varphi)$ provides approximation order $k > 0$ if and only if $\varphi$ satisfies the Strang-Fix conditions of order $k$. So the approximation order provided by $S_0(\varphi)$ in the $L_1(\mathbb{R})$-norm is an integer.

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**References**


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