PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 6, June 1996

ASYMPTOTIC BEHAVIOUR OF CERTAIN SETS OF PRIME IDEALS

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(Communicated by Wolmer V. Vasconcelos)

ABSTRACT. Let $I_1, \ldots, I_g$ be ideals of the commutative ring R, let M be a Noetherian R-module and let N be a submodule of M; also let A be an Artinian R-module and let B be a submodule of A. It is shown that, whenever $(a_m(1), \ldots, a_m(g))_{m \in \mathbb{N}}$ is a sequence of g-tuples of non-negative integers which is non-decreasing in the sense that $a_i(j) \leq a_{i+1}(j)$ for all $j = 1, \ldots, g$ and all $i \in \mathbb{N}$, then $\mathrm{Ass}_R\left(M/I_1^{a_n(1)} \ldots I_g^{a_n(g)} N\right)$ is independent of n for all large n, and also $\mathrm{Att}_R\left(B :_A I_1^{a_n(1)} \ldots I_g^{a_n(g)}\right)$ is independent of n for all large n. These results are proved without any regularity conditions on the ideals $I_1, \ldots, I_g$, and so (a special case of) the first answers in the affirmative a question raised by S. McAdam.

0. INTRODUCTION

Let R be a commutative ring (with identity) and I be an ideal of R. In [Ra] L. J. Ratliff, Jr., conjectured about the asymptotic behaviour of $\mathrm{Ass}_R(R/I^n)$ when R is a Noetherian domain. Subsequently, M. Brodmann [B] showed that if R is Noetherian and M is a finitely generated R-module, then $\mathrm{Ass}_R(M/I^n M)$ is ultimately constant for large $n \in \mathbb{N}$. (We use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers.) It is a very easy consequence of Brodmann's result that the same conclusion holds if we relax the hypotheses and assume only that R is a commutative ring and M is a Noetherian R-module. In [S1, Theorem 3.1(iii)] the second author proved that if A is an Artinian R-module, then $\mathrm{Att}_R(0 :_A I^n)$ is ultimately constant for large n; in a sense, this can be regarded as a dual of Brodmann's result. More recently in [Ru, Theorem 5.5] D. E. Rush further expanded upon this by showing that if N is an R-submodule of the Noetherian R-module M, then $\mathrm{Ass}_R(M/I^n N)$ is ultimately constant for large $n \in \mathbb{N}$, and if B is an R-submodule of the Artinian R-module A, then $\mathrm{Att}_R(B :_A I^n)$ is ultimately constant for large $n \in \mathbb{N}$.

On a different course of investigation, D. Katz, S. McAdam and L. J. Ratliff, Jr., have shown in [K-M-R, Corollary 1.8(c)] that if R is Noetherian, $I_1, \ldots, I_g$ are regular ideals of R, and $(a_m(1), \ldots, a_m(g))_{m \in \mathbb{N}}$ is a sequence of g-tuples of non-negative integers which is non-decreasing in the sense that $a_i(j) \leq a_{i+1}(j)$ for

Received by the editors December 13, 1994.
1991 *Mathematics Subject Classification.* Primary 13E05, 13E10.
Key words and phrases. Associated prime ideal, attached prime ideal, Noetherian module, Artinian module, asymptotic prime divisors.

all $j = 1, \dots, g$ and all $i \in \mathbb{N}$, then $\mathrm{Ass}_R\left(R/I_1^{a_n(1)} \dots I_g^{a_n(g)}\right)$ is independent of n for all large n. Further in [Mc, 13.5(i)] S. McAdam put forward the question: "Can we drop the assumption that $I_1, \dots, I_g$ are regular"; one of the main results of this paper is an affirmative answer to this question.

This paper is concerned with what might be considered a natural generalisation of the above-mentioned result of Katz, McAdam and Ratliff. Let $I_1, \dots, I_g$ be ideals of R and let $(a_m(1), \dots, a_m(g))_{m\in\mathbb{N}}$ be a sequence of g-tuples of non-negative integers which is non-decreasing. We shall show that if N is a submodule of the Noetherian R-module M and B is a submodule of the Artinian R-module A, then $\mathrm{Ass}_R\left(M/I_1^{a_n(1)} \dots I_g^{a_n(g)} N\right)$ and $\mathrm{Att}_R\left(B :_A I_1^{a_n(1)} \dots I_g^{a_n(g)}\right)$ are both ultimately constant for large n. We shall use techniques similar to those employed by Katz, McAdam and Ratliff to establish the first result when R is Noetherian, M is finitely generated and the ideals satisfy some regularity conditions. Then we shall extend the result to the general case by use of the Artin-Rees Lemma. In Section 2, we shall use Matlis duality and techniques investigated by the second author in [S3], which allow an Artinian module over a commutative ring to be considered as a faithful Artinian module over a complete semi-local Noetherian ring, to prove the second result. (Throughout this paper whenever we describe a semi-local Noetherian ring as "complete", we shall always mean it is complete with respect to the topology defined by its Jacobson radical.)

1. Associated prime ideals

1.1 *Notation.* Throughout this paper R will denote a commutative ring (with identity). For a fixed positive integer j, we shall denote by $\mathbb{N}_0^j$ the set of all j-tuples of non-negative integers. Throughout, let g be a fixed positive integer, and let $I_1, \dots, I_g$ be ideals of R; and for $n = (n_1, \dots, n_g) \in \mathbb{N}_0^g$ we shall denote $I_1^{n_1} \dots I_g^{n_g}$ by I^n. Let $j \in \mathbb{N}$. For $n \in \mathbb{N}_0^j$ we define $n(i)$ to be the ith component of n $(1 \le i \le j)$. Thus for $n, m \in \mathbb{N}_0^j$, by $n \le m$ (respectively $n < m$) we mean $n(i) \le m(i)$ (respectively $n(i) < m(i)$) for all $1 \le i \le j$. Addition, and multiplication by elements of $\mathbb{N}_0$, are defined on $\mathbb{N}_0^j$ in the obvious ways. We denote by $\mathbb{Z}^j$ the set of all j-tuples of integers, defining operations involving elements of $\mathbb{Z}^j$ in a similar fashion as for $\mathbb{N}_0^j$, with the convention that if $z(i) < 0$ for some $1 \le i \le j$, then we take $I_i^{z(i)}$ to be R.

1.2 **Lemma.** *Let R be a Noetherian ring, let M be a finitely generated R-module and let N be a submodule of M. Then the set $\bigcup_{n\in\mathbb{N}_0^g} \mathrm{Ass}_R(M/I^nN)$ is finite.*

Proof. For $z = (z_1, \dots, z_g) \in \mathbb{Z}^g$ we shall denote $t_1^{z_1} \dots t_g^{z_g}$ by t^z. Let $\mathfrak{R}$ denote $R[I_1t_1, \dots, I_gt_g, t_1^{-1}, \dots, t_g^{-1}]$, the extended Rees ring of $I_1, \dots, I_g$. We denote $R[t_1, \dots, t_g, t_1^{-1}, \dots, t_g^{-1}] \otimes_R M$ by $M[t_1, \dots, t_g, t_1^{-1}, \dots, t_g^{-1}]$ and denote its element $t^z \otimes m$ (for $z \in \mathbb{Z}^g$ and $m \in M$) by mt^z. Let $\mathfrak{M}$ denote $\bigoplus_{z\in\mathbb{Z}^g} I^zMt^z$, a $\mathbb{Z}^g$-graded $\mathfrak{R}$-submodule of $M[t_1, \dots, t_g, t_1^{-1}, \dots, t_g^{-1}]$. Similarly let $\mathfrak{N}$ denote $\bigoplus_{z\in\mathbb{Z}^g} I^zNt^z$, a $\mathbb{Z}^g$-graded $\mathfrak{R}$-submodule of $\mathfrak{M}$. Let $n \in \mathbb{N}_0^g$. It is clear that $t^{-n}\mathfrak{N}$ is a homogeneous $\mathfrak{R}$-submodule of $\mathfrak{M}$ with the $(0, \dots, 0)$-th component I^nN. Hence

$$\mathrm{Ass}_R(M/I^nN) \subseteq \{\mathfrak{P} \cap R : \mathfrak{P} \in \mathrm{Ass}_{\mathfrak{R}}(\mathfrak{M}/t^{-n}\mathfrak{N})\}.$$

There are three possibilities for n: we have (i) $n = 0$, or (ii) $t^{-n} = t_i^{-1}$ for some $1 \le i \le g$, or (iii) there exists $k \in \mathbb{N}$ with $1 \le k \le g$ such that $n(k) \neq 0$ and $t^{-n} = t_k^{-1} t^{-n'}$ for some $0 \neq n' \in \mathbb{N}_0^g$. Then since t_k^{-1} is a non-zerodivisor on $\mathfrak{M}$, there is an exact sequence

$$0 \longrightarrow \mathfrak{M}/t^{-n'}\mathfrak{N} \longrightarrow \mathfrak{M}/t_k^{-1}t^{-n'}\mathfrak{N} \longrightarrow \mathfrak{M}/t_k^{-1}\mathfrak{M} \longrightarrow 0,$$

and so

$$\mathrm{Ass}_{\mathfrak{R}}\left(\mathfrak{M}/t^{-n}\mathfrak{N}\right) \subseteq \mathrm{Ass}_{\mathfrak{R}}\left(\mathfrak{M}/t_k^{-1}\mathfrak{M}\right) \cup \mathrm{Ass}_{\mathfrak{R}}\left(\mathfrak{M}/t^{-n'}\mathfrak{N}\right).$$

Repetition of this argument shows that, in cases (ii) or (iii),

$$\mathrm{Ass}_{\mathfrak{R}}\left(\mathfrak{M}/t^{-n}\mathfrak{N}\right) \subseteq \bigcup_{1\le k\le g}\left(\mathrm{Ass}_{\mathfrak{R}}\left(\mathfrak{M}/t_k^{-1}\mathfrak{M}\right) \cup \mathrm{Ass}_{\mathfrak{R}}\left(\mathfrak{M}/t_k^{-1}\mathfrak{N}\right)\right).$$

Hence

$$\bigcup_{n\in\mathbb{N}_0^g} \mathrm{Ass}_R\left(M/I^nN\right) \subseteq \left\{\mathfrak{P}\cap R : \mathfrak{P} \in \bigcup_{n\in\mathbb{N}_0^g}\mathrm{Ass}_{\mathfrak{R}}\left(\mathfrak{M}/t^{-n}\mathfrak{N}\right)\right\}$$

$$\subseteq \left\{\mathfrak{P}\cap R : \mathfrak{P} \in \left(\bigcup_{1\le k\le g}\left(\mathrm{Ass}_{\mathfrak{R}}\left(\mathfrak{M}/t_k^{-1}\mathfrak{M}\right) \cup \mathrm{Ass}_{\mathfrak{R}}\left(\mathfrak{M}/t_k^{-1}\mathfrak{N}\right)\right) \bigcup \mathrm{Ass}_{\mathfrak{R}}\left(\mathfrak{M}/\mathfrak{N}\right)\right)\right\},$$

which clearly is a finite set. □

1.3 **Lemma.** *Let R be a Noetherian ring and let M be a finitely generated R-module. Let h be a fixed positive integer and let $J_1, \ldots, J_h$ be ideals of R. Let J^n, for $n \in \mathbb{N}_0^h$, be defined in a similar fashion to $I^{n'}$, for $n' \in \mathbb{N}_0^g$, in 1.1. Also for fixed i, we define K^s for $s \in \mathbb{N}_0^{h-1}$ to be $J_1^{s(1)} \ldots J_{i-1}^{s(i-1)} J_{i+1}^{s(i)} \ldots J_h^{s(h-1)}$.*

(i) *For each i, with $1 \le i \le h$, if J_i is M-regular, then there exists $c_i \in \mathbb{N}_0$ such that $\left(J_i^{d_i+e}K^sM :_M J_i^e\right) = J_i^{d_i}K^sM$ for all $s \in \mathbb{N}_0^{h-1}$, $d_i, e \in \mathbb{N}_0$ with $d_i > c_i$.*

(ii) *Let H be an ideal of R and suppose that $J_1, \ldots, J_h$ are M-regular. Then there exists $t \in \mathbb{N}_0^h$ such that $(J^{n+m}HM :_M J^m) = J^nHM$ for all $n, m \in \mathbb{N}_0^h$ with $n > t$.*

Proof. (i) We may clearly assume that $M \neq 0$ (and $J_i \neq 0$). Let

$$M = m_1R + \ldots + m_rR.$$

Fix $i \in \mathbb{N}$ with $1 \le i \le h$, and assume that J_i is M-regular. Then there exist M-regular elements $a_1, \ldots, a_t \in R$ such that $J_i = a_1R + \ldots + a_tR$. For $j = 1, \ldots, t$, let $S_j = \{a_j^r : r \in \mathbb{N}_0\}$, and let N, L be the R-submodules of $S_1^{-1}M \oplus \ldots \oplus S_t^{-1}M$ given by $N = \{(m/1, \ldots, m/1) : m \in M\}$ and $L = \{(m_1'/a_1, \ldots, m_t'/a_t) : m_1', \ldots, m_t' \in M\}$. Clearly $N \subseteq L$, and L is finitely generated. Define $\phi : M \to N$ by $\phi(m) = (m/1, \ldots, m/1)$ for all $m \in M$, so that ϕ is an isomorphism. Now

$$\begin{aligned} J^nL \cap N &= \{(m/1, \ldots, m/1) : m \in M, a_jm/a_j = l_j/a_j \text{ with } l_j \in J^nM \text{ for all } j = 1, \ldots, t\} \\ &= \{(m/1, \ldots, m/1) : m \in M, J_im \subseteq J^nM\} \\ &= \phi\left((J^nM :_M J_i)\right) \end{aligned}$$

for all $n \in \mathbb{N}_0^h$. Thus by [K-M-R, Proposition 1.4], there exists $c_i \in \mathbb{N}_0$ such that

$$\begin{aligned}\phi\left(\left(J_i^{d_i+f}K^sM :_M J_i\right)\right) &= J_i^{d_i+f}K^sL \cap N \\ &= J_i^f\left(J_i^{d_i}K^sL \cap N\right) \\ &= \phi\left(J_i^f\left(J_i^{d_i}K^sM :_M J_i\right)\right) \\ &\subseteq \phi\left(J_i^{d_i+f-1}K^sM\right)\end{aligned}$$

for all $s \in \mathbb{N}_0^{h-1}$, $d_i, f \in \mathbb{N}_0$ with $d_i > c_i$ and $f \geq 1$. Let $e \in \mathbb{N}_0$ with $e \geq 1$ and $x \in \left(J_i^{d_i+e}K^sM : J_i^e\right)$ for some $d_i > c_i$. Then $xJ_i^e \subseteq J_i^{d_i+e}K^sM$, so $xJ_i^{e-1} \subseteq \left(J_i^{d_i+e}K^sM :_M J_i\right) \subseteq J_i^{d_i+e-1}K^sM$, and continuing inductively in this manner we get $x \in J_i^{d_i}K^sM$. Hence $\left(J_i^{d_i+e}K^sM :_M J_i^e\right) = J_i^{d_i}K^sM$ for all $s \in \mathbb{N}_0^{h-1}$, $d_i, e \in \mathbb{N}_0$ with $d_i > c_i$, as required.

(ii) This follows easily by repeated application of the first part. □

1.4 **Theorem.** *Let R be a Noetherian ring and let M be a finitely generated R-module.*

(i) *Let $(a_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{N}_0^g$ with $a_i \leq a_{i+1}$ for all $i \in \mathbb{N}$. Assume that I_i is M-regular for each i with $1 \leq i \leq g$ such that $\{a_j(i) : j \in \mathbb{N}\}$ is infinite. Then $\operatorname{Ass}_R(M/I^{a_m}M)$ is ultimately constant for large m.*

(ii) *Furthermore, if I_i is M-regular for all $i = 1, \ldots, g$, then there exists $d \in \mathbb{N}^g$ such that $\operatorname{Ass}_R(M/I^nM)$ is independent of n for all $n \geq d$.*

Proof. (i) If $\{a_j(i) : j \in \mathbb{N}\}$ is finite for all $i = 1, \ldots, g$, then the result is clear. Therefore we suppose this is not the case and we may reorder the ideals so that for some $q \in \mathbb{N}$ with $1 \leq q \leq g$, $\{a_j(i) : j \in \mathbb{N}\}$ is infinite for $1 \leq i \leq q$, and finite for $q < i \leq g$. Define $b_m := (a_m(1), \ldots, a_m(q)) \in \mathbb{N}_0^q$ and denote $I_1^{b_m(1)} \ldots I_q^{b_m(q)}$ by J^{b_m} for all $m \in \mathbb{N}$. Then there exists an ideal H of R such that $I^{a_m} = J^{b_m}H$ for all sufficiently large m, and it suffices to show that $\operatorname{Ass}_R\left(M/J^{b_m}HM\right)$ is ultimately constant for large m.

Now by 1.3, for all sufficiently large m, $J^{b_m}HM = \left(J^{b_{m+1}}HM :_M J^{b_{m+1}-b_m}\right)$. Let $P \in \operatorname{Ass}_R\left(M/J^{b_m}HM\right)$ for such an m, so that $P = \left(0 :_R x + J^{b_m}HM\right)$ for some $x \in M \backslash J^{b_m}HM$. It follows that $r \in \left(0 :_R x + J^{b_m}HM\right)$ if and only if $r \in \left(0 :_R \left(J^{b_{m+1}-b_m}x + J^{b_{m+1}}HM\right)/J^{b_{m+1}}HM\right)$, and so $P \in \operatorname{Ass}_R\left(M/J^{b_{m+1}}HM\right)$. Now by 1.2,

$$\bigcup_{m\in\mathbb{N}} \operatorname{Ass}_R\left(M/J^{b_m}HM\right) \subseteq \bigcup_{n\in\mathbb{N}_0^q} \operatorname{Ass}_R\left(M/I_1^{n(1)}\ldots I_q^{n(q)}HM\right)$$

is finite, and hence it follows that $\operatorname{Ass}_R(M/I^{a_m}M)$ is ultimately constant for large m.

(ii) By 1.3(ii), there exists $k \in \mathbb{N}_0^g$ such that $I^nM = (I^{n+m}M :_M I^m)$ for all $n, m \in \mathbb{N}_0^g$ with $n > k$. Let $s' \geq t' > k$ and let $P \in \operatorname{Ass}_R\left(M/I^{t'}M\right)$, so that $P = \left(0 :_R x + I^{t'}M\right)$ for some $x \in M \backslash I^{t'}M$. It follows that $r \in \left(0 :_R x + I^{t'}M\right)$ if and only if $r \in \left(0 :_R \left(I^{s'-t'}x + I^{s'}M\right)/I^{s'}M\right)$, and so $P \in \operatorname{Ass}_R\left(M/I^{s'}M\right)$.

For each $j \in \mathbb{N}$, let $n_j = (j, \dots, j) \in \mathbb{N}^g$. Then, by part (i), there exists $l \in \mathbb{N}$ such that

$$\mathrm{Ass}_R(M/I^{n_l}M) = \mathrm{Ass}_R(M/I^{n_{l+i}}M) \qquad \text{for all } i \in \mathbb{N}.$$

Let $c = \max\{l, k(1)+1, \dots, k(g)+1\}$ and define $d := n_c$. Let $s \in \mathbb{N}^g$ with $s \geq d$. Then there exists $r \in \mathbb{N}$ such that $rd \geq s$, and hence, since $d > k$,

$$\mathrm{Ass}_R\left(M/I^dM\right) \subseteq \mathrm{Ass}_R(M/I^sM) \subseteq \mathrm{Ass}_R\left(M/I^{rd}M\right) = \mathrm{Ass}_R\left(M/I^dM\right). \quad \square$$

In the next theorem, there is no 'regularity' assumption on the ideals $I_1, \dots, I_g$. Thus the particular case of the theorem in which $M = R$ answers in the affirmative a question raised by McAdam in [Mc, 13.5(i)].

1.5 **Theorem.** *Let R be a Noetherian ring, let M be a finitely generated R-module and let N be a submodule of M.*

(i) *Let $(a_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{N}_0^g$ with $a_i \leq a_{i+1}$ for all $i \in \mathbb{N}$. Then $\mathrm{Ass}_R(M/I^{a_m}N)$ is ultimately constant for large m.*

(ii) *Furthermore, there exists $d \in \mathbb{N}^g$ such that $\mathrm{Ass}_R(M/I^nN)$ is independent of n for all $n \geq d$.*

Proof. (i) The result is obvious if $\{a_j(i) : j \in \mathbb{N}\}$ is finite for all $1 \leq i \leq g$. Hence we shall assume this is not the case and we may reorder the ideals so that there exists $q \in \mathbb{N}$ with $1 \leq q \leq g$ such that $\{a_j(i) : j \in \mathbb{N}\}$ is infinite for $1 \leq i \leq q$ and finite for $q < i \leq g$. Define $b_m := (a_m(1), \dots, a_m(q)) \in \mathbb{N}_0^q$ and denote $I_1^{b_m(1)} \dots I_q^{b_m(q)}$ by J^{b_m} for all $m \in \mathbb{N}$. Then there exists an ideal K of R such that $I^{a_m} = J^{b_m}K$ for all $m \geq l$, say, and it suffices to show that $\mathrm{Ass}_R\left(M/J^{b_m}KN\right)$ is ultimately constant for large m.

By applying the Artin-Rees Lemma to $(0 :_N I_i) \subseteq N$, we see that, for each $i = 1, \dots, q$, there exists $h_i \in \mathbb{N}_0$ such that

$$\left(0 :_{I_i^{h_i+1}N} I_i\right) = I_i^{h_i+1}N \cap (0 :_N I_i) = I_i\left(I_i^{h_i}N \cap (0 :_N I_i)\right) = 0.$$

Let $L = I_1^{h_1+1} \dots I_q^{h_q+1}K$. Then, for each $i = 1, \dots, q$, we have $(0 :_{LN} I_i) = 0$, so that I_i is LN-regular.

Now there exists $l' \in \mathbb{N}$ with $l' \geq l$ such that $b_m(i) \geq h_i + 1$ for all $m \geq l'$ and $1 \leq i \leq q$. Then for $m \geq l'$ we define

$$c_m := (b_m(1) - h_1 - 1, \dots, b_m(q) - h_q - 1) \in \mathbb{N}_0^q,$$

and J^{c_m} in a way similar to J^{b_m}. Then $I^{a_m} = J^{c_m}L$ for all $m \geq l'$. Now for all $m \geq l'$, there is an exact sequence

$$0 \longrightarrow J^{c_m}LN/J^{c_{m+1}}LN \longrightarrow M/I^{a_{m+1}}N \longrightarrow M/I^{a_m}N \longrightarrow 0,$$

and so

$$(*) \qquad \mathrm{Ass}_R(M/I^{a_{m+1}}N) \subseteq \mathrm{Ass}_R(J^{c_m}LN/J^{c_{m+1}}LN) \cup \mathrm{Ass}_R(M/I^{a_m}N).$$

By 1.4(i), we know that $\mathrm{Ass}_R\left((LN)/J^{c_{m+1}}(LN)\right)$ is ultimately constant for large m. Thus for all sufficiently large m,

$$\begin{aligned}\mathrm{Ass}_R\left(J^{c_m}LN/J^{c_{m+1}}LN\right) &\subseteq \mathrm{Ass}_R\left(LN/J^{c_{m+1}}LN\right) = \mathrm{Ass}_R\left(LN/J^{c_m}LN\right)\\ &\subseteq \mathrm{Ass}_R\left(M/I^{a_m}N\right),\end{aligned}$$

so that, by $(*)$,

$$\mathrm{Ass}_R\left(M/I^{a_{m+1}}N\right) \subseteq \mathrm{Ass}_R\left(M/I^{a_m}N\right).$$

Hence $\mathrm{Ass}_R\left(M/I^{a_m}N\right)$ is ultimately constant for large m.

(ii) For each $j \in \mathbb{N}$, let $n_j = (j, \ldots, j) \in \mathbb{N}^g$. Then by part (i), there exists $k \in \mathbb{N}$ such that

$$\mathrm{Ass}_R\left(M/I^{n_k}N\right) = \mathrm{Ass}_R\left(M/I^{n_{k+i}}N\right) \qquad \text{for all } i \in \mathbb{N}.$$

As in the proof of part (i), we can apply the Artin-Rees Lemma to $(0 :_N I_i) \subseteq N$ to see that, for each $i = 1, \ldots, g$, there exists $x_i \in \mathbb{N}_0$ such that $\left(0 :_{I_i^{x_i+1}N} I_i\right) = 0$. Let $y = \max\{x_1, \ldots, x_g\} + 1$, and let $v = n_y$. Then, for each $i = 1, \ldots, g$, we have $(0 :_{I^vN} I_i) = 0$, so that I_i is I^vN-regular. Thus by 1.4(ii), there exists $c \in \mathbb{N}_0^g$ such that

$$\mathrm{Ass}_R\left(I^vN/I^aI^vN\right) = \mathrm{Ass}_R\left(I^vN/I^cI^vN\right)$$

for all $a \in \mathbb{N}_0^g$ with $a \geq c$.

Let $w = \max\{k, c(1) + y, \ldots, c(g) + y\}$. Define $d := n_w$, and let $s, t \in \mathbb{N}^g$ with $s \geq t \geq d$. Then there is an exact sequence

$$0 \longrightarrow I^tN/I^sN \longrightarrow M/I^sN \longrightarrow M/I^tN \longrightarrow 0,$$

and so

$$\mathrm{Ass}_R\left(M/I^sN\right) \subseteq \mathrm{Ass}_R\left(I^tN/I^sN\right) \cup \mathrm{Ass}_R\left(M/I^tN\right).$$

Now since $s - v \geq t - v \geq d - v \geq c$,

$$\begin{aligned}\mathrm{Ass}_R\left(I^tN/I^sN\right) &\subseteq \mathrm{Ass}_R\left(I^vN/I^{s-v}I^vN\right) = \mathrm{Ass}_R\left(I^vN/I^{t-v}I^vN\right)\\ &\subseteq \mathrm{Ass}_R\left(M/I^tN\right).\end{aligned}$$

Therefore $\mathrm{Ass}_R\left(M/I^sN\right) \subseteq \mathrm{Ass}_R\left(M/I^tN\right)$. Clearly for any $u \in \mathbb{N}^g$ with $u \geq d$ there exists $e \in \mathbb{N}$ such that $ed \geq u$, and hence

$$\mathrm{Ass}_R\left(M/I^dN\right) = \mathrm{Ass}_R\left(M/I^{ed}N\right) \subseteq \mathrm{Ass}_R\left(M/I^uN\right) \subseteq \mathrm{Ass}_R\left(M/I^dN\right). \quad \square$$

1.6 **Theorem.** *Let R be a commutative ring (not necessarily Noetherian), let M be a Noetherian R-module and let N be a submodule of M.*

(i) *Let $(a_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{N}_0^g$ with $a_i \leq a_{i+1}$ for all $i \in \mathbb{N}$. Then $\mathrm{Ass}_R\left(M/I^{a_m}N\right)$ is ultimately constant for large m.*

(ii) *Furthermore, there exists $d \in \mathbb{N}^g$ such that $\mathrm{Ass}_R\left(M/I^nN\right)$ is independent of n for all $n \geq d$.*

Proof. Clearly M has a natural structure as an $R/(0 :_R M)$-module with the same submodule structure as the R-module, and so is still Noetherian. Let $b \in \mathbb{N}_0^g$. Since $(0 :_R M) \subseteq \left(0 :_R M/I^bN\right)$, M/I^bN is a Noetherian $R/(0 :_R M)$-module. Let $\phi : R \longrightarrow R/(0 :_R M)$ be the natural homomorphism, and $\phi^* : \mathrm{Spec}\left(R/(0 :_R M)\right) \longrightarrow \mathrm{Spec}\left(R\right)$ be the induced map. Then by [Ru, Theorem 5.2(i)], we see that

$$\mathrm{Ass}_R\left(M/I^bN\right) = \phi^*\left(\mathrm{Ass}_{R/(0:_RM)}\left(M/I^bN\right)\right).$$

Hence since $R/(0 :_R M)$ is a Noetherian ring the results follow from 1.5. $\square$

2. Attached prime ideals

2.1 **Matlis duality.** The classical duality originally developed by Matlis (see [Ma, Section 4]) dealt with a complete Noetherian local ring; it can, in fact, also be applied to a complete Noetherian semi-local ring (see [S-T, 3.5]). We shall be making substantial use of this theory, and so we shall include a summary for the reader. Let R be a complete semi-local Noetherian ring, and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the distinct maximal ideals of R. Set $S := \bigoplus_{j=1}^{n} R/\mathfrak{m}_j$ and $E := E(S)$, the injective envelope of S. We shall use D to denote the additive, exact, R-linear [N, p.35] functor $\mathrm{Hom}_R(-, E)$ from the category of all R-modules and R-homomorphisms to itself. For each R-module T, let

$$\mu_T : T \longrightarrow DD(T) = \mathrm{Hom}_R(\mathrm{Hom}_R(T, E), E)$$

be the natural R-homomorphism for which $(\mu_T(x))(f) = f(x)$ for all $x \in T$ and $f \in \mathrm{Hom}_R(T, E)$.

Then Matlis duality states (among other things) that, whenever M is a finitely generated R-module, $\mu_M : M \longrightarrow DD(M)$ is an isomorphism and $D(M)$ is an Artinian R-module, and whenever A is an Artinian R-module, $\mu_A : A \longrightarrow DD(A)$ is an isomorphism and $D(A)$ is a Noetherian R-module.

2.2 **Lemma.** *Let the situation be as in* 2.1. *Let A be an Artinian R-module, let B be a submodule of A and let $M := D(A)$. Then there exists a submodule N of M such that the sets* $\mathrm{Ass}_R(M/JN)$ *and* $\mathrm{Att}_R(B :_A J)$ *are equal for all ideals J of R.*

Proof. The argument of [S2, Corollary 2.7(ii)] can be extended to this case to show that, whenever M' is a Noetherian R-module, then $\mathrm{Att}_R(D(M')) = \mathrm{Ass}_R(M')$.

We know from Matlis duality that there exists a Noetherian R-module L and an isomorphism $\psi : D(L) \xrightarrow{\cong} A/B$. It is easily shown that

$$0 \longrightarrow (0 :_{A/B} J) \xrightarrow{\beta} A/B \xrightarrow{D(i)\circ\psi^{-1}} D(JL) \longrightarrow 0$$

is an exact sequence, where β is the inclusion map and i is the inclusion map of JL into L. It follows from this that

$$0 \longrightarrow J(D(A/B)) \xrightarrow{\pi} D(A/B) \xrightarrow{D(\beta)} D(0 :_{A/B} J) \longrightarrow 0$$

is an exact sequence, where

$$\pi = D(\psi)^{-1} \circ DD(i) \circ \mu_{JL} \circ \mu_L^{-1} |_{J(DD(L))} \circ D(\psi) |_{J(D(A/B))} .$$

Hence from the commutative diagrams

$$\begin{array}{ccc} JL & \xrightarrow{i} & L \\ \cong\Big\downarrow \mu_{JL} & & \cong\Big\downarrow \mu_L \\ DD(JL) & \xrightarrow{DD(i)} & DD(L) \end{array} \qquad \begin{array}{ccc} J(D(A/B)) & \xrightarrow[\cong]{D(\psi)|} & J(DD(L)) \\ \subseteq\Big\downarrow & & \subseteq\Big\downarrow \\ D(A/B) & \xrightarrow[\cong]{D(\psi)} & DD(L) \end{array}$$

we see that π is just inclusion, and so $\operatorname{Ker}(D(\beta)) = J(D(A/B))$. Therefore, applying the Snake Lemma to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D(A/B) & \xrightarrow{D(\sigma)} & D(A) & \longrightarrow & D(B) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle D(\beta)} & & \downarrow{\scriptstyle D(\alpha)} & & \| & & \\
0 & \longrightarrow & D(0 :_{A/B} J) & \longrightarrow & D(B :_A J) & \longrightarrow & D(B) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where α is the inclusion map of $(B :_A J)$ into A and $\sigma : A \longrightarrow A/B$ is the canonical epimorphism, we obtain $\operatorname{Ker}(D(\alpha)) = J(D(\sigma)(D(A/B)))$.

Hence $D(B :_A J) \cong M/JN$ where $N := (D(\sigma)(D(A/B)))$ and therefore, since the choice of N is not dependent upon J, the proof is finished. $\square$

2.3 **Theorem.** *Let R be a complete semi-local Noetherian ring, let A be an Artinian R-module and let B be a submodule of A.*

(i) *Let $(a_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{N}_0^g$ with $a_i \leq a_{i+1}$ for all $i \in \mathbb{N}$. Then $\operatorname{Att}_R(B :_A I^{a_m})$ is ultimately constant for large m.*

(ii) *Furthermore, there exists $d \in \mathbb{N}^g$ such that $\operatorname{Att}_R(B :_A I^n)$ is independent of n for all $n \geq d$.*

Proof. This follows from 2.1, 2.2 and 1.6. $\square$

2.4 *Note.* (i) Here we shall outline some results concerning Artinian modules; for further details see [S3, Theorem 3.2]. Let A be a non-zero Artinian module over the commutative ring R. There exist only finitely many maximal ideals $\mathfrak{m}$ of R for which $\operatorname{Soc}(A)$ has a submodule isomorphic to $R/\mathfrak{m}$: let the distinct such maximal ideals be $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$. Set $J := \bigcap_{i=1}^s \mathfrak{m}_i$ and

$$\hat{R}^{(J)} := \varprojlim_n R/J^n,$$

the J-adic completion of R. Then A has a natural structure as a module over $\hat{R}^{(J)}$, and the ring $R' := \hat{R}^{(J)}/(0 :_{\hat{R}^{(J)}} A)$ is a complete semi-local commutative Noetherian ring. Also the module A is a faithful Artinian module over R'; moreover, a subset of A is an R-submodule if and only if it is an R'-submodule.

(ii) Let $\phi : R \longrightarrow R'$ be the natural map and let $r \in R$. Then the multiplication by r on A has the same effect as multiplication by $\phi(r)$ on A (see [S3, 2.2]).

2.5 **Theorem.** *Let R be a commutative ring (not necessarily Noetherian), let A be an Artinian R-module and let B be a submodule of A.*

(i) *Let $(a_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{N}_0^g$ with $a_i \leq a_{i+1}$ for all $i \in \mathbb{N}$. Then $\operatorname{Att}_R(B :_A I^{a_m})$ is ultimately constant for large m.*

(ii) *Furthermore, there exists $d \in \mathbb{N}^g$ such that $\operatorname{Att}_R(B :_A I^n)$ is independent of n for all $n \geq d$.*

Proof. If $A = 0$ the result is clear, and so we may assume that $A \neq 0$. Let the notation be as in 2.4. Let $\phi : R \longrightarrow R'$ be the natural map and, for an ideal K of

R, let K^e denote the extension of K to R' under ϕ. Let $\phi^* : \operatorname{Spec}(R') \longrightarrow \operatorname{Spec}(R)$ be the induced map. Then, by [Ru, Theorem 5.2(ii)], we see that

$$\operatorname{Att}_R (B :_A I^n) = \phi^* (\operatorname{Att}_{R'} (B :_A I^n)) \qquad \text{for all } n \in \mathbb{N}_0^g,$$

and so it suffices to show that (i) $\operatorname{Att}_{R'} ((B :_A I^{a_m}))$ is ultimately constant for large m, and (ii) there exists $d \in \mathbb{N}^g$ such that $\operatorname{Att}_{R'} (B :_A I^n)$ is independent of n for all $n \geq d$. Let $(I^e)^n$ denote $(I_1^e)^{n(1)} \dots (I_g^e)^{n(g)}$ for all $n \in \mathbb{N}_0^g$. Then using the natural structure of A as an R'-module we see from 2.4(ii) that

$$(B :_A I^n) = (B :_A (I^n)^e) = (B :_A (I^e)^n)$$

for all $n \in \mathbb{N}_0^g$. Hence the results follow from 2.4(i) and 2.3. □

References

[B] M. Brodmann, *Asymptotic stability of* Ass (M/I^nM), Proc. Amer. Math. Soc. **74** (1979), 16–18. MR **80c:**13012

[K-M-R] D. Katz, S. McAdam and L. J. Ratliff, Jr., *Prime divisors and divisorial ideals*, J. Pure Appl. Algebra **59** (1989), 179–186. MR **90g:**13030

[Ma] E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528. MR **20:**5800

[Mc] S. McAdam, *Primes associated to an ideal*, Contemporary Mathematics, vol. 102, Amer. Math. Soc., Providence, 1989. MR **90m:**13004

[N] D. G. Northcott, *An introduction to homological algebra*, Cambridge Univ. Press, Cambridge, 1960. MR **22:**9523

[Ra] L. J. Ratliff, Jr., *On prime divisors of I^n, n large*, Michigan Math. J. **23** (1976), 337–352. MR **56:**15626

[Ru] D. E. Rush, *Asymptotic primes and integral closure in modules*, Quart. J. Math. Oxford Ser. (2) **43** (1992), 477–499. MR **93m:**13009

[S1] R. Y. Sharp, *Asymptotic behaviour of certain sets of attached prime ideals*, J. London Math. Soc. (2) **34** (1986), 212–218. MR **87i:**13007

[S2] ———, *A method for the study of Artinian modules, with an application to asymptotic behavior*, Commutative algebra (Proceedings of a Microprogram held June 15-July 2, 1987), Mathematical Sciences Research Institute Publications, vol. 15, Springer-Verlag, New York, 1989, pp. 443–465. MR **91a:**13011

[S3] ———, *Artinian modules over commutative rings*, Math. Proc. Cambridge Phil. Soc. **111** (1992), 25–33. MR **93a:**13009

[S-T] R. Y. Sharp and Y. Tiraş, *Asymptotic behaviour of integral closures of ideals relative to Artinian modules*, J. Algebra **153** (1992), 262–269. MR **94b:**13007

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