PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 6, June 1996

LIPSCHITZ DISTRIBUTIONS AND ANOSOV FLOWS

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(Communicated by Linda Keen)

Abstract. We show that if a distribution is locally spanned by Lipschitz vector fields and is involutive a.e., then it is uniquely integrable giving rise to a Lipschitz foliation with leaves of class $C^{1,\mathrm{Lip}}$. As a consequence, we show that every codimension-one Anosov flow on a compact manifold of dimension > 3 such that the sum of its strong distributions is Lipschitz, admits a global cross section.

The main purpose of this paper is to generalize the theorem of Frobenius on integrability of smooth vector distributions and to give an application of the theorem to the question of existence of global cross sections to Anosov flows. Accordingly, the paper is divided into two parts, A and B.

A. Integrability of Lipschitz distributions

Let M be a C^∞ n-dimensional Riemannian manifold equipped with a Lebesgue measure.

Definition 1. We will say that a distribution (or plane field) E on M is *Lipschitz* if it is locally spanned by Lipschitz continuous vector fields.

Recall that a map f between metric spaces (M_1, d_1) and (M_2, d_2) is called Lipschitz continuous (or simply Lipschitz) if there is a constant $C > 0$ such that $d_2(f(p), f(q)) \leq C d_1(p, q)$, for all $p, q \in M_1$. By saying that a vector field X on M is Lipschitz we mean that in some (and therefore in any) coordinate system, X can be written in the form

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},$$

where each a_i is a Lipschitz function. Recall that a Lipschitz vector field is uniquely integrable and each time t map of its flow is Lipschitz.

An important property of Lipschitz maps, discovered by H. Rademacher, is that they are almost everywhere (relative to Lebesgue measure) differentiable in the ordinary sense with locally essentialy bounded derivative. This property enables us to extend the notion of Lie bracket to Lipschitz vector fields. Namely, if X, Y are Lipschitz vector fields, and f is a C^∞ function, we define

$$[X, Y]f = X(Yf) - Y(Xf).$$

Received by the editors December 15, 1994.
1991 *Mathematics Subject Classification.* Primary 34C35, 58A30; Secondary 53C12.
Key words and phrases. Distribution, foliation, Anosov flow, cross section.
Part of this research was supported by the University of California Graduate Fellowship.

Note that this expression makes sense and is defined a.e. because Xf and Yf are Lipschitz functions. If in some local coordinates X and Y can be expressed as

$$X = \sum_i a_i \frac{\partial}{\partial x_i}, \quad Y = \sum_j b_j \frac{\partial}{\partial x_j},$$

then $[X, Y]$ in the same coordinate system looks like

$$[X, Y] = \sum_{i,j} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}.$$

Notice that the coefficients of $[X, Y]$ are locally bounded functions.

Definition 2. Let E be a Lipschitz distribution on M. We will say that E is *involutive a.e.* if for every two Lipschitz vector fields X, Y in E their bracket $[X, Y]_p$ belongs to E_p for a.e. $p \in M$.

So, for instance, if $X_1, \dots, X_k$ is a local basis for E on some open set consisting of Lipschitz vector fields, then E is involutive a.e. if and only if there exist locally bounded functions c_{ij}^l such that

$$[X_i, X_j] = \sum_{l=1}^{k} c_{ij}^l X_l.$$

We can now state the results of the first part of the paper.

Theorem A1. *Let E be a k-dimensional Lipschitz distribution on a compact smooth n-dimensional manifold M. If E is involutive almost everywhere, then every point of M has a coordinate neighborhood $(U; x_1, \dots, x_n)$ such that:*

(a) *Each map $x_i : U \to \boldsymbol{R}$ is Lipschitz.*

(b) *The slices $x_{k+1} = \text{constant}, \dots, x_n = \text{constant}$ are integral manifolds of E. Moreover, every connected integral manifold of E in U is of class $C^{1,\mathrm{Lip}}$ and lies in one of these slices.*

Theorem A2. *Let E be as above. Then through every point p of M passes a unique maximal connected integral manifold of E, and every connected integral manifold of E through p is contained in the maximal one.*

Corollary A3. *Let α be a 1-form on M which is everywhere nonsingular and Lipschitz, and let $E = \mathrm{Ker}(\alpha)$. Then E is uniquely integrable if and only if*

$$\alpha \wedge d\alpha = 0$$

almost everywhere on M.

Proof of Theorem A1. The proof is by induction on k. In constructing the desired coordinate system we closely follow Warner (see [Wa], Theorem 1.60), but when certain difficulties arise due to nonsmoothness of E, we use some standard approximation techniques to reach the correct conclusions.

If $k = 1$, the theorem follows directly from the flowbox theorem and the already mentioned fact that Lipschitz vector fields generate Lipschitz flows.

So assume $k \geq 2$ and that the theorem holds for $k - 1$. Given $p \in M$, let $(V; y_1, \dots, y_n)$ be a C^∞ coordinate neighborhood with $y_i(p) = 0 \quad (1 \leq i \leq n)$, on which E is spanned by Lipschitz vector fields $X_1, \dots, X_k$. Without loss of generality we may assume that $X_1(y_1) \geq 1$ on V. Set $Y_1 = X_1$ and

$$Y_i = X_i - \frac{X_i(y_1)}{X_1(y_1)} X_1,$$

for $2 \le i \le k$. Then $Y_1, \dots, Y_k$ are linearly independent *Lipschitz* vector fields on V spanning $E|_V$. Let H be the slice $y_1 = 0$ and let

$$Z_i = Y_i|_H,$$

for $2 \le i \le k$. Then, by construction, $Y_i(y_1) = 0$ for $2 \le i \le k$. So the vector fields $Z_2, \dots, Z_k$ are tangent to H and span a $(k-1)$-dimensional Lipschitz distribution F on H. We claim that F is involutive a.e. To see this, let $\iota : H \to M$ be the inclusion. Then $Y_i = \iota_*(Z_i)$, so

$$[Y_i, Y_j] = \iota_*([Z_i, Z_j]).$$

Since for $i, j \ge 2$,

$$[Y_i, Y_j](y_1) = Y_i(Y_j y_1) - Y_j(Y_i y_1) = 0,$$

there exist locally bounded functions c_{ij}^l, such that

$$[Y_i, Y_j] = \sum_{l=1}^{k} c_{ij}^l \cdot Y_l, \tag{1}$$

a.e. on U, where for $i, j \ge 2$, $c_{ij}^1 = 0$. Thus, since ι_* is 1-1, we have

$$[Z_i, Z_j] = \sum_{l=2}^{k} c_{ij}^l\big|_H \cdot Z_l$$

a.e. on H (with respect to the $(n-1)$-dimensional Lebesgue measure on H). So F is involutive a.e. By induction hypothesis, there exists a coordinate neighborhood $(U; z_2, \dots, z_n)$ of p with $U \subset V$, such that the slices

$$z_{k+1} = \text{constant}, \dots, z_n = \text{constant}$$

are precisely the integral manifolds of F.

Let ϕ_t be the local flow of $X_1 (= Y_1)$ on U. There exists a neighborhood of p which we (to simplify notation) also call U, such that the projection $\pi : U \to H \cap U$ along the orbits of ϕ_t is well defined and Lipschitz.

Now define maps from U to $\mathbf{R}$ by

$$x_1(q) = t,$$

$$x_j = z_j \circ \pi,$$

where $x_1(q) = t$ if and only if $\phi_{-t}(q) \in H \cap U$. It is clear that $(U; x_1, \dots, x_n)$ is a Lipschitz coordinate system. It remains to show that

$$Y_i(x_j) = 0$$

a.e. on U for $1 \le i \le k$ and $k+1 \le j \le n$. First let us approximate the functions x_j by smooth ones. Since the statements above are of local character, without loss of generality we may assume that we are in $\mathbf{R}^n$ where we have the standard mollifiers η_ϵ at our disposal (see [EG] or [Zi]). Let $x_j^\epsilon = x_j * \eta_\epsilon$ (the $*$ denotes convolution). Then:

(i) Each x_j^ϵ is of class C^∞.

(ii) As $\epsilon \to 0$, $x_j^\epsilon \to x_j$ uniformly on compact sets.

(iii) As $\epsilon \to 0$, $D^\alpha x_j^\epsilon \to D^\alpha x_j$ in L^1_{loc} and also pointwise almost everywhere, for every "multiindex" α, $|\alpha| = 1$.

Furthermore, by the properties of convolution,

$$\begin{aligned} Y_1(x_j^\epsilon) &= (Y_1 x_j) * \eta_\epsilon \\ &= 0 \end{aligned}$$

for all $2 \le j \le n$ and small $\epsilon > 0$. Therefore $Y_1(Y_i x_j^\epsilon) = [Y_1, Y_i](x_j^\epsilon) \quad (j \ge 2)$. By (1), we have

$$Y_1(Y_i x_j^\epsilon) = \sum_{l=2}^{k} c_{1i}^l \cdot Y_l x_j^\epsilon \tag{2}$$

almost everywhere. Since Y_1 is Lipschitz, the foliation of U by orbits of ϕ_t is absolutely continuous which implies that for a.e. $q \in H$ (with respect to the $(n-1)$-dimensional Lebesgue measure on H) the ϕ_t-orbit of q intersects any set of set of n-dimensional Lebesgue measure zero along a set of 1-dimensional measure zero. So integration of (2) along $\phi_s(q)$, $0 \le s \le t$, yields

$$(Y_i x_j^\epsilon)(\phi_t(q)) - (Y_i x_j^\epsilon)(q) = \int_0^t \sum_{l=2}^{k} c_{1i}^l(\phi_s(q))\, (Y_l x_j^\epsilon)(\phi_s(q))\, ds, \tag{3}$$

for a.e. $q \in H$ and a.e. $t \in J(q)$, where $J(q)$ is some open interval in $\mathbf{R}$ depending on q.

Now let $\epsilon \to 0$. By (i)–(iii), we obtain

$$(Y_i x_j)(\phi_t(q)) - (Y_i x_j)(q) = \int_0^t \sum_{l=2}^{k} c_{1i}^l(\phi_s(q))\, (Y_l x_j)(\phi_s(q))\, ds, \tag{4}$$

for a.e. $q \in H$ and a.e. $t \in J(q)$. Since for $2 \le i \le k$ the vector fields $Y_i|_H (= Z_i)$ belong to the distribution F, for $k+1 \le j \le n$ we have

$$\begin{aligned} (Y_i x_j)(q) &= (Z_i z_j)(q) \\ &= 0 \end{aligned}$$

a.e., because the slices $z_j = \text{constant} \quad (k+1 \le j \le n)$ are integral manifolds of F.

Fix $q \in H$ so that (4) holds for a.e. $t \in J(q)$. (There will be a set of full measure of such q.) The right-hand side of (4) is a continuous function of t. Therefore the functions $t \mapsto (Y_i x_j)(\phi_t q)$ (j fixed, $k+1 \le j \le n$) are a.e. continuous and satisfy the following $(k-1) \times (k-1)$ homogeneous system of linear integral equations (along the orbit of q) with L^∞ coefficients

$$Y_i x_j = \int_0^t \sum_{l=2}^{n} c_{1i}^l\, Y_l x_j\, ds. \tag{5}$$

Let $C(t)$ be the matrix with entries $c_{1i}^l(\phi_t q)$ and let

$$f(t) = (Y_2 x_j(\phi_t(q)), \dots, Y_k x_j(\phi_t(q)))\,.$$

Then (5) becomes

$$f(t) = \int_0^t C(s) f(s) ds.$$

So

$$\|f(t)\| \le \int_0^t \|C(s)\| \cdot \|f(s)\| ds.$$

As remarked above, f is continuous a.e. and $\|C\| \in L^\infty$. Gronwall's inequality (see [Hi], Theorem 1.5.7) implies $f(t) = 0$ a.e.

This proves that $Y_i x_j = 0$ a.e. It remains to show that the slices

$$x_{k+1} = \text{constant}, \dots, x_n = \text{constant}$$

are integral manifolds of E.

Let S be one such slice. Then its tangent bundle can be expressed as

$$TS = \bigcap_{j=k+1}^{n} \operatorname{Ker}(dx_j|_S),$$

which clearly contains the vector fields $Y_1|_S, \dots, Y_k|_S$ a.e. Thus $T_q S = E_q$ a.e. on S. Since E is a continuous distribution defined on a compact space, we can extend this relation over all of S. Thus S is an integral manifold of E. Note also that since the tangent bundle of S is Lipschitz, it follows that S is a manifold of class $C^{1,\mathrm{Lip}}$.

This completes the first part of the proof.

It remains to show uniqueness of integral manifolds.

Let N be a connected integral manifold of E in U. Denote the projection $\mathbf{R}^n \to \mathbf{R}^{n-k}$ to the last $n-k$ coordinates by pr. If $\varphi = (x_1, \dots, x_n)$ is the coordinate map defined above, then

$$\begin{aligned} T(\mathrm{pr} \circ \varphi)(TN) &= T(\mathrm{pr} \circ \varphi)(E\,|_N) \\ &= 0 \end{aligned}$$

a.e. on N. Since N is connected, it follows (see e.g. [Zi]) that $\mathrm{pr} \circ \varphi = \text{const}$ a.e. on N. By continuity, this equality holds everywhere on N, which implies that N is contained in a slice $x_{k+1} = \text{const}, \dots, x_n = \text{const}$.

This completes the proof. □

Proof of Theorem A2 and Corollary A3. The proof of existence and uniqueness of maximal integral manifolds in the classical case given in [Wa](Theorem 1.64) is valid in our setting.

To prove Corollary A3, choose a continuous vector field X such that $\alpha(X) = 1$. Let Y and Z be arbitrary Lipschitz vector fields in E. Then

$$\begin{aligned} (\alpha \wedge d\alpha)(X, Y, Z) &= \alpha(X)\, d\alpha(Y, Z) \\ &= d\alpha(Y, Z) \\ &= Y(\alpha(Z)) - Z(\alpha(Y)) - \alpha([Y, Z]) \\ &= -\alpha([Y, Z]). \end{aligned}$$

(A remark is in place here. The third equality above is well known for C^1 forms. However, since Lipschitz forms are nicely approximable by smooth forms, as demonstrated above, this equality continues to hold almost everywhere for Lipschitz forms.) So if $\alpha \wedge d\alpha = 0$ a.e., then $[Y, Z]_p \in E_p$ for a.e. $p \in M$, and the Corollary follows from Theorems A1 and A2. □

Remark. Note that Corollary A3 as well as Theorems A1 and A2 do not follow from Hartman's version of Frobenius theorem (see [Ha], Theorem 3.1), because Hartman requires that $d\alpha$ be continuous. In fact, Hartman deals with the Stokes exterior differential of α, but it is possible to show that for Lipschitz forms the Stokes differential coincides with the ordinary differential. (See, for instance, Whitney's book [Wh] and note that Lipschitz forms are "flat".)

B. Cross sections to Anosov flows

A nonsingular flow f_t of class C^r $(r \geq 1)$ on a compact Riemannian manifold M is called *Anosov* if there exist a continuous invariant splitting of the tangent bundle of M, $TM = E^{ss} \oplus E^{uu} \oplus \mathbf{R}X$ (here X is the vector field generating the flow), such that in the forward direction the tangent Tf_t to the flow exponentially contracts E^{ss} and exponentially expands E^{uu}. (Note that, since M is compact, the Anosovness of f_t is independent of the choice of Riemannian metric.) Subbundles E^{ss} and E^{uu} are called strong stable and strong unstable respectively. An Anosov flow is called *of codimension one* if E^{ss} or E^{uu} is one dimensional. For basic results about Anosov flows (such as unique integrability of $E^{ss}, E^{uu}, E^{ss} \oplus \mathbf{R}X$ and $E^{uu} \oplus \mathbf{R}X$, etc.) the reader should consult [An] and [Pl]; for analogous results about Anosov diffeomorphisms, see [Fr].

Let E^{su} denote the direct sum $E^{ss} \oplus E^{uu}$. In general E^{su} is only a Hölder continuous distribution (see, for instance, [HP]). In [Gh] E. Ghys showed that if for a codimension one Anosov flow, E^{su} is of class C^1 and $\dim M > 3$, then the flow admits a global cross section. Recall that a compact codimension-one submanifold Σ of M is called a *(global) cross section* for a flow on M if it intersects every orbit of the flow transversely and the orbit of every point in Σ intersects Σ after some positive time. If an Anosov flow admits a global cross section Σ, then the corresponding first return map f on Σ is an Anosov diffeomorphism. If the Anosov flow is also of codimension one, then so is f and by the well-known result of Newhouse (see [Nh]), Σ is homeomorphic to a torus and f is topologically conjugate to a linear toral automorphism.

Verjovsky conjectured that as soon as $\dim M > 3$, every codimension one Anosov flow on M admits a global cross section. We prove a special case of this conjecture, generalizing a result of Ghys (see [Gh]).

Recall that if A is a linear isomorphism of normed vector spaces, the conorm (or minimum norm) of A is defined to be

$$m(A) = \inf\{\|Av\| : \|v\| = 1\}.$$

Now we can state the results of the second part of the paper.

Theorem B1. *Let $\{f_t\}$ be an Anosov flow on a compact manifold M such that E^{su} is Lipschitz and*

$$\mu := \inf_{x \in M} [m(T_x f_\tau|_{E^{ss}}) \cdot m(T_x f_\tau|_{E^{uu}})] > 1,$$

for some $\tau > 0$. Then $\{f_t\}$ admits a global cross section.

Theorem B2. *Let $\{f_t\}$ be a codimension one Anosov flow on a compact manifold M of dimension $n > 3$. If E^{su} is Lipschitz, then $\{f_t\}$ is topologically conjugate to the suspension of a linear toral automorphism.*

Proof of Theorem B1. To show the existence of a global cross section to an Anosov flow it suffices to prove that E^{su} is an integrable distribution. For a proof of this fact, see [Pl], Theorem 3.1.

We will need the following generalization of Lemma 1.2 from [Gh].

Lemma. *Let E_i $(i = 1, 2)$ be Euclidean spaces and assume the splitting $E_i = S_i \oplus U_i$ is orthogonal $(i = 1, 2)$. Let $f : E_1 \to E_2$ be a linear isomorphism such that $f(S_1) = S_2$ and $f(U_1) = U_2$.*

(a) *If* $\mu := m(f|_{S_1})\ m(f|_{U_1})$, *then for all* $w_s \in S_2$ *and* $w_u \in U_2$

$$\|f^{-1}(w_s \wedge w_u)\| \leq \frac{1}{\mu}\|w_s \wedge w_u\|.$$

(b) *If* $\dim E_i = n-1$, $\dim U_1 = 1$ *and* $\|f(w)\| \leq \nu\|w\|$, *for some* $\nu > 0$ *and all* $w \in S_1$, *then*

$$(\det f)\ \|f^{-1}(w_s \wedge w_u)\| \leq \nu^{n-3}\|w_s \wedge w_u\|,$$

for all $w_s \in S_2$, $w_u \in U_2$

Proof. (a) Let $v_s \in S_1$, $v_u \in U_1$ be arbitrary. Then

$$\begin{aligned}\|f(v_s \wedge v_u)\| &= \|f(v_s)\|\ \|f(v_u)\| \\ &\geq m(f|_{S_1})\ m(f|_{U_1})\ \|v_s\|\ \|v_u\| \\ &= \mu\|v_s \wedge v_u\|.\end{aligned}$$

Part (a) now follows if we take $w_s = f(v_s)$ and $w_u = f(v_u)$.

(b) Let v_s, v_u be as above. Choose unit vectors $e_3, \ldots, e_{n-1}$ in S_1 such that $e_1 = v_u, e_2 = v_s, e_3, \ldots, e_{n-1}$ is an orthogonal basis of E_1. Then we have

$$\begin{aligned}\|v_u \wedge v_s\| \cdot \det f &= \|f(e_1 \wedge \ldots \wedge e_{n-1})\| \\ &= \|f(v_u \wedge v_s)\| \prod_{i=3}^{n-1} \|f(e_i)\| \\ &\leq \nu^{n-3}\|f(v_u \wedge v_s)\|.\end{aligned}$$

To complete the proof, take $w_u = f(v_u)$, $w_s = f(v_s)$. □

Let us now prove Theorem B1. Define a 1-form α by requiring that

$$\mathrm{Ker}(\alpha) = E^{su}, \quad \alpha(X) = 1,$$

where X is the vector field which generates the flow. Since E^{su} is a Lipschitz distribution, α is a Lipschitz form, so $d\alpha$ exists on an f_t-invariant set of full measure in M. Clearly, $f_t^*\alpha = \alpha$ for all $t \in \mathbf{R}$. Since $d\alpha$ is the ordinary exterior differential, it commutes with pullbacks by diffeomorphisms and it follows that $f_t^*(d\alpha) = d\alpha$ for all $t \in \mathbf{R}$.

Let $x \in M$ be a point where $d\alpha$ is defined, and let $w_s \in E_x^{ss}$ and $w_u \in E_x^{uu}$ be arbitrary. Part (a) of the Lemma implies that $\|(f_{-\tau})_*(w_s \wedge w_u)\| \leq \mu^{-1}\|w_s \wedge w_u\|$. Therefore,

$$\begin{aligned}|d\alpha(w_s, w_u)| &= |f_{-k\tau}^*(d\alpha)(w_s, w_u)| \\ &= |d\alpha\left((f_{-k\tau})_*\,(w_s \wedge w_u)\right)| \\ &\leq \|d\alpha\|_\infty\ \mu^{-k}\ \|w_s \wedge w_u\| \\ &\to 0\end{aligned}$$

as $k \to \infty$. (Here $\|d\alpha\|_\infty$ denotes the L^∞ norm of $d\alpha$. Since α is Lipschitz and M is compact, this norm is finite.) Similarly we can show that $d\alpha_x(X, v) = 0$ for almost every $x \in M$, where $v \in E^{su}$. (In fact, we don't need the lemma to prove this. For details see [Gh].) Therefore $d\alpha = 0$ a.e. so by Theorems A1 and A2 it follows that E^{su} is an integrable distribution. By the already mentioned result of Plante, the flow admits a global cross section. □

Proof of Theorem B2. First note that if f_t preserves a volume form on M, then Theorem B2 follows directly from Theorem B1. If f_t is not volume preserving, denote by $\Delta(x,t)$ the determinant of $T_x f_t$. Define the 1-form α as above and let $\nu = \sup_{x\in M} \|T_x f_1|_{E^{ss}}\|$. Clearly, $\nu < 1$.

Assume that E^{uu} is 1-dimensional, let Y be a unit continuous vector field in E^{uu} (it is no loss of generality to assume that all the bundles are orientable) and let Z be a continuous, not necessarily nonvanishing vector field in E^{ss}. Define a function $h : M \to \mathbf{R}$ by

$$h(x) = d\alpha_x(Y_x, Z_x).$$

Then $h \in L^\infty(M)$ and

$$\begin{aligned}
\int_M |h(x)|\,dx &= \int_M \Delta(x,t)\,|h(f_t x)|\,dx \\
&= \int_M \Delta(x,t)\,|d\alpha_{f_t x}(Y_{f_t x}, Z_{f_t x})|\,dx \\
&= \int_M \Delta(x,t)\,|(f_{-t}^* d\alpha)_{f_t x}(Y_{f_t x}, Z_{f_t x})|\,dx \\
&= \int_M \Delta(x,t)\,|(d\alpha)_x((f_{-t})_*(Y\wedge Z))|\,dx \qquad (6)\\
&\le \|d\alpha\|_\infty \int_M \Delta(x,t)\,\|(f_{-t})_*(Y\wedge Z)\|\,dx \\
&\le C\int_M \nu^{(n-3)t}\,dx \\
&= C\,\mathrm{volume}(M)\,\nu^{(n-3)t} \\
&\to 0
\end{aligned}$$

as $t \to \infty$. (Inequality (6) follows from part (b) of the Lemma.) Therefore $h(x) = 0$ a.e. Since $d\alpha(X,v) = 0$ a.e. for $v \in E^{su}$, and since Z was arbitrary, it follows that $d\alpha = 0$ a.e., which completes the proof. □

Acknowledgement

I would like to thank Professor Charles Pugh for his help and encouragement.

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