UNIQUENESS FOR NON-HARMONIC TRIGONOMETRIC SERIES

KAORA YONEDA

(Communicated by J. Marshall Ash)

Abstract. When $\lambda_n > 0$, $\lambda_n \uparrow \infty$ and

$$\frac{1}{2}|a_0| + \sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{\lambda_n^2} < \infty,$$

if

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x) = 0 \text{ everywhere } (-\infty, \infty),$$

then

$$a_0 = a_1 = b_1 = \cdots = a_n = b_n = \cdots = 0.$$

More generalized results are given.

1. Introduction

Let $\{\lambda_n\}_n$ be a strictly increasing sequence of positive numbers such that

$$\lim_{n \to \infty} \lambda_n = \infty.$$

For example, $\lambda_n = \log(n + 1)$ for $n = 1, 2, \ldots$. In this paper we shall discuss a uniqueness problem for non-harmonic trigonometric series:

$$(1) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x).$$

Many mathematicians have discussed some uniqueness problems for harmonic trigonometric series (see [1] and [3]).

Zygmund [2] discussed the same problem for the integral case

$$\int_{0}^{\infty} (c_s \cos sx + d_s \sin sx) ds,$$

where $c_s$ and $d_s$ are continuous.

Received by the editors March 30, 1994 and, in revised form, November 29, 1994.

1991 Mathematics Subject Classification. Primary 42A63.

Key words and phrases. Uniqueness, trigonometric series.

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It is easy to see that the series (1) is zero at $x$ and $-x$ if and only if

(1-a) \[ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x = 0; \]

(1-b) \[ \sum_{n=1}^{\infty} b_n \sin \lambda_n x = 0. \]

Using the same argument as in the proof of Theorem 2 in Section 68 of Chapter 1 of [1] (see p. 190), we can prove that if (1-a) and (1-b) hold, then

(2-a) \[ \lim_{h \to \infty} \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x \left( \sin \frac{\lambda_n h}{\lambda_n} \right)^2 = 0; \]

(2-b) \[ \lim_{h \to \infty} \sum_{n=1}^{\infty} b_n \sin \lambda_n x \left( \sin \frac{\lambda_n h}{\lambda_n} \right)^2 = 0. \]

When (1-a) and (1-b) hold, the convergences of the two series

\[ H_m^e(x) := \sum_{n=1}^{\infty} \frac{a_n \cos \lambda_n x}{\lambda^{2m}}, \]
\[ H_m^o(x) := \sum_{n=1}^{\infty} \frac{b_n \sin \lambda_n x}{\lambda^{2m}} \quad (m = 1, 2, \ldots) \]

are certified by the following lemma.

**Lemma 1.** When \( \{\theta_n\}_n \) is a strictly decreasing sequence of positive numbers such that \( \lim_{n \to \infty} \theta_n = 0 \), if a series \( \sum_{n=0}^{\infty} \alpha_n \) converges, then the series \( \sum_{n=0}^{\infty} \alpha_n \theta_n \) converges.

**Proof.** Put \( R_n := \sum_{k=n}^{\infty} \alpha_k \) for \( n = 1, 2, \ldots \). By the Abel transform, we have

\[ \sum_{k=n}^{\infty} \alpha_k \theta_k = R_n \theta_n - \sum_{k=n}^{N-1} R_{k+1}(\theta_k - \theta_{k+1}) - R_{N+1} \theta_N. \]

Thus,

\[ | \sum_{k=n}^{N} \alpha_k \theta_k | \leq |R_n \theta_n| + \sup_{n \leq k} |R_{k+1}| |(\theta_k + \theta_N) + |R_{N+1}| \theta_N \]

and each term in the right-hand side tends to zero when \( n \) and \( N \) tend to infinity. Hence the sequence \( \{\sum_{k=1}^{n} \alpha_k \theta_k\}_n \) is a Cauchy sequence. The lemma is proved. \( \square \)

In this paper, we shall give the following results:

**Theorem 2.** If

(3) \[ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x) = 0 \quad everywhere \quad (-\infty, \infty), \]
and if for \( m = 1, 2, \ldots \)

(4-a) \( H^e_m(x) \) and \( H^o_m(x) \) are continuous;
(4-b) \[ \lim_{x \to \infty} \frac{H^e_m(x)}{x^2} = \lim_{x \to \infty} \frac{H^o_m(x)}{x} = 0, \]

then

(5) \[ a_0 = a_1 = b_1 = \cdots = a_n = b_n = \cdots = 0. \]

**Corollary 3.** When

(6) \[ \sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{\lambda_n^2} < \infty, \]

if (3) holds, then (5) is valid.

**Theorem 4.** When \( E \) is an enumerable set (without loss of generality, we can assume that \( E \) satisfies \( -x \in E \) if \( x \in E \)), if

(7) \[ \frac{1}{2} a_n + \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x) = 0 \quad \text{everywhere } (-\infty, \infty) \text{ except } E, \]

and if

(8) \( H^e_1(x) \) and \( H^o_1 \) are smooth in \( E \)

and (4-a) and (4-b) hold for \( m = 1, 2, \ldots \), then (5) is valid.

**Corollary 5.** When

(9) \[ \sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{\lambda_n} < \infty, \]

if (7) holds, then (5) is valid.

2. Proof of Theorem 2

Put

\[ F_1(x) := \frac{1}{4} a_0 x^2 - H^e_1(x). \]

Thus we have

\[ \frac{1}{4h^2} \{ F_1(x + 2h) - 2F_1(x) + F_1(x - 2h) \} = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x \left( \frac{\sin \lambda_n h}{\lambda_n h} \right)^2 \]

and

\[ \frac{1}{4h^2} \{ H^e_1(x + 2h) - 2H^e_1(x) + H^e_1(x - 2h) \} = \sum_{n=1}^{\infty} b_n \sin \lambda_n x \left( \frac{\sin \lambda_n h}{\lambda_n h} \right)^2. \]
From (3), the two second symmetric derivatives satisfy
\begin{equation}
D^2 F_1(x) = D^2 H^o_1(x) = 0 \quad \text{everywhere } (-\infty, \infty).
\end{equation}
From (4-a) and by Lemma (3.4) in Section 3 of Chapter IX of [3] (see p. 327), $F_1(x)$ and $H^o_1$ are linear, that is,
\begin{align}
F_1(x) &= A_1 x + \frac{1}{2} B_1; \\
H^o_1(x) &= C_1 x + D_1 \quad \text{everywhere } (-\infty, \infty).
\end{align}
From (11-a),
\begin{align*}
A_1 x &= \frac{1}{4} a_0 x^2 - \frac{1}{2} B_1 - H^o_1(x),
\end{align*}
where the left-hand side is an odd function and the right-hand side is an even function. Thus, $A_1 = 0$. And from (4-b), $a_0 = 0$. Thus
\begin{align}
\frac{1}{2} B_1 + H^o_1(x) &= 0 \quad \text{everywhere } (-\infty, \infty).
\end{align}
Arguing analogously, for $H^q_1(x)$, we can prove that $C_1 = D_1 = 0$ and
\begin{align}
H^q_1(x) &= 0 \quad \text{everywhere } (-\infty, \infty).
\end{align}
Let us discuss similarly to the above for non-harmonic trigonometric series, (12-a) and (12-b). And we can prove that $B_1 = 0$ and for some $B_2$
\begin{align*}
\frac{1}{2} B_2 + H^e_2(x) &= 0; \quad H^e_2(x) = 0 \quad \text{everywhere } (-\infty, \infty).
\end{align*}
Continuing this process, we have $B_{m-1} = 0$ and for some $B_m$
\begin{align}
\frac{1}{2} B_m + H^e_m(x) &= 0; \quad H^e_m(x) = 0 \quad \text{everywhere } (-\infty, \infty).
\end{align}
Obviously $B_m = 0$ for all $m$; then
\begin{align}
H^o_m(x) &= 0 \quad \text{everywhere } (-\infty, \infty).
\end{align}
The conclusion follows if the following lemma is proved.

**Lemma 6.** Let $\{\theta_n\}_n$ be a sequence satisfying the condition of Lemma 1 and $\theta_1 < 1$. If $\sum_{n=0}^{\infty} \alpha_n$ converges and
\begin{align*}
\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \theta_n^m = 0 \quad \text{for all } m = 1, 2, \ldots,
\end{align*}
then $\alpha_0 = 0$. 

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Proof of Lemma 6. Put \( R_n = \sum_{k=n}^{\infty} \alpha_k \) for \( n = 1, 2, \ldots \). For each \( \varepsilon > 0 \), there exists \( N \) such that \( |R_n| < \varepsilon \) for \( n > N \). Since

\[
\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \theta_n^m = \alpha_0 + \alpha_1 \theta_1^m + \cdots + \alpha_N \theta_N^m - R_{N+1}^{m} \theta_{N+1}^m - \sum_{k=N+1}^{\infty} R_{k+1}(\theta_k^m - \theta_{k+1}^m),
\]

\[
|R_{N+1}^{m} \theta_{N+1}^m| < \varepsilon,
\]

we have

\[
|\sum_{k=N+1}^{\infty} R_{k+1}(\theta_k^m - \theta_{k+1}^m)| \leq \sum_{k=N+1}^{\infty} |R_{k+1}| |\theta_k^m - \theta_{k+1}^m| \leq \varepsilon \sum_{k=N+1}^{\infty} \theta_k^m - \theta_{k+1}^m = \varepsilon \theta_{N+1}^m < \varepsilon
\]

and

\[
\lim_{m \to \infty} \sum_{k=1}^{N} \alpha_k \theta_k^m = 0,
\]

we have

\[
|\alpha_0| = |\sum_{k=1}^{\infty} \alpha_k \theta_k^m| \leq |\sum_{k=1}^{N} \alpha_k \theta_k^m| + |\sum_{k=N+1}^{\infty} \alpha_k \theta_k^m| \leq |\sum_{k=1}^{N} \alpha_k \theta_k^m| + 2\varepsilon.
\]

Thus

\[
|\alpha_0| \leq \lim_{m \to \infty} |\sum_{k=1}^{N} \alpha_k \theta_k^m| + 2\varepsilon = 2\varepsilon.
\]

Consequently \( \alpha_0 = 0 \). Lemma 6 is proved. \( \square \)

Now put \( \theta_n = \frac{\lambda_{n+1}}{\lambda_1} \) for \( n = 1, 2, \ldots \). Thus \( \{\theta_n\}_n \) satisfies the condition of Lemma 6. And put \( \alpha_n = a_{n+1} \cos \lambda_{n+1}x \). Then from (13-a')

\[
a_1 \cos \lambda_1 x = 0 \quad \text{everywhere } (-\infty, \infty).
\]

And analogously from (13-b),

\[
b_1 \sin \lambda_1 x = 0 \quad \text{everywhere } (-\infty, \infty).
\]

Continuing this process we can easily prove

\[
a_n \cos \lambda_n x = b_n \sin \lambda_n x = 0 \quad \text{everywhere for all } n.
\]

Consequently

\[
a_n = b_n = 0 \quad \text{for } n = 1, 2, \ldots.
\]

We have proved Theorem 2.
3. Proofs of Theorem 4 and corollaries

By Lemma (3.20) in Section 3 of Chapter IX of [3] (see p.328) and from (7), (8) and (4-a), \( F_1(x) \) and \( H_0^o(x) \) are linear. Then using the same argument as in the proof of Theorem 2, we can easily prove Theorem 4.

Obviously condition (6) in Corollary 3 is stronger than (4-a) and (4-b) in Theorem 2, and (9) in Corollary 5 is stronger than (4-a), (4-b) and (8) in Theorem 4.

Remark 1. Under the conditions (4-a) and (4-b), \( H_m^c(x) \) and \( H_m^o(x) \) are continuous if and only if

\[
\lim_{h \to 0} \sum_{n=1}^{\infty} \frac{a_n \sin \lambda_n x}{\lambda_n^{2m}} \sin \lambda_n h = \lim_{h \to 0} \sum_{n=1}^{\infty} \frac{b_n \cos \lambda_n x}{\lambda_n^{2m}} \sin \lambda_n h = 0
\]

everywhere \((-\infty, \infty)\).

Remark 2. \( H_1^c(x) \) and \( H_1^o(x) \) are smooth at \( x \) if and only if

\[
\lim_{h \to 0} (h \left( \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x \left( \frac{\sin \lambda_n h}{\lambda_n h} \right)^2 \right) = \lim_{h \to 0} (\sum_{n=1}^{\infty} b_n \sin \lambda_n x \left( \frac{\sin \lambda_n h}{\lambda_n h} \right)^2) = 0.
\]

(See p. 43 (3.1) in Chapter II and p. 328 (3.21) in Chapter IX of [3].)

References


Department of Mathematics and Information Sciences, Osaka Prefecture University, 1-1 Gakuen-cho, Sakai, Osaka 593, Japan

E-mail address: yoneda@mathsun.cias.osakafu-u.ac.jp