

## ON CONTRAVARIANT FINITENESS OF SUBCATEGORIES OF MODULES OF PROJECTIVE DIMENSION $\leq I$

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**ABSTRACT.** Let  $\Lambda$  be an artin algebra. This paper presents a sufficient condition for the subcategory  $\mathcal{P}^i(\Lambda)$  of  $\text{mod } \Lambda$  to be contravariantly finite in  $\text{mod } \Lambda$ , where  $\mathcal{P}^i(\Lambda)$  is the subcategory of  $\text{mod } \Lambda$  consisting of  $\Lambda$ -modules of projective dimension less than or equal to  $i$ . As an application of this condition it is shown that  $\mathcal{P}^i(\Lambda)$  is contravariantly finite in  $\text{mod } \Lambda$  for each  $i$  when  $\Lambda$  is stably equivalent to a hereditary algebra.

### INTRODUCTION AND PRELIMINARIES

Throughout this paper, all algebras are artin algebras, all modules are finitely generated left modules, and all subcategories are full subcategories. For an artin algebra  $\Lambda$ , we denote by  $\text{mod } \Lambda$  the category of all finitely generated  $\Lambda$ -modules.

The notions of contravariantly and covariantly finite subcategories of  $\text{mod } \Lambda$  were first introduced and studied by Auslander and Smalø in connection with the study of the existence of almost split sequences in a subcategory of  $\text{mod } \Lambda$  (see [2] and [3]). We now recall these notions from [2].

**Definition.** A full subcategory  $\mathcal{A}$  of  $\mathcal{B}$  is said to be

- (i) contravariantly finite in  $\mathcal{B}$  if for each object  $X$  in  $\mathcal{B}$ , the representable functor  $\text{hom}_{\mathcal{B}}(\_, X)$  restricted to  $\mathcal{A}$  is finitely generated as a functor on  $\mathcal{A}$ ,
- (ii) covariantly finite in  $\mathcal{B}$  if for each object  $Y$  in  $\mathcal{B}$ , the representable functor  $\text{hom}_{\mathcal{B}}(Y, \_)$  restricted to  $\mathcal{A}$  is finitely generated, and
- (iii) functorially finite if  $\mathcal{A}$  is both contravariantly and covariantly finite in  $\mathcal{B}$ .

For an artin algebra  $\Lambda$ , an interesting class of subcategories of  $\text{mod } \Lambda$  is the subcategory  $\mathcal{P}^i(\Lambda)$  which consists of all  $\Lambda$ -modules of projective dimension  $\leq i$  for  $i \geq 0$ , as well as the subcategory  $\mathcal{P}^\infty(\Lambda)$  consisting of all  $\Lambda$ -modules of finite projective dimension. The contravariant and covariant finiteness of  $\mathcal{P}^i(\Lambda)$  and  $\mathcal{P}^\infty(\Lambda)$  is studied by many authors; see, for example, [1], [5], and [6].

The aim of this paper is to present a condition which is sufficient for the subcategory  $\mathcal{P}^i(\Lambda)$  to be contravariantly finite in  $\text{mod } \Lambda$ . As an application of this condition we show in section 2 that  $\mathcal{P}^i(\Lambda)$  is contravariantly finite in  $\text{mod } \Lambda$  for each  $i$  when  $\Lambda$  is stably equivalent to a hereditary algebra. The main idea of the

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proofs is from [1]. Note that in [1] Auslander and Reiten gave a sufficient condition for  $\mathcal{P}^\infty(\Lambda)$  to be contravariantly finite in  $\text{mod } \Lambda$  and proved that  $\mathcal{P}^\infty(\Lambda)$  is contravariantly finite in  $\text{mod } \Lambda$  when  $\Lambda$  is stably equivalent to a hereditary algebra.

The terminology used throughout this article is taken from [1].

1. A SUFFICIENT CONDITION FOR  $\mathcal{P}^i(\Lambda)$  TO BE CONTRAVARIANTLY FINITE

The proofs in this section are analogous to those in sect.4 in [1], but for the completeness of the article, we give here the proofs. Before presenting the sufficient condition, we first give the following more general result.

**Proposition 1.1.** *Suppose that  $\underline{a}$  is an ideal in  $\Lambda$  with  $\text{pd}_\Lambda \Lambda / \underline{a} \leq i$  such that if  $M$  is a  $\Lambda$ -module with  $\text{pd}_\Lambda M \leq i$ , then  $M / \underline{a}M$  is a projective  $\Lambda / \underline{a}$ -module. Let  $C$  be a  $\Lambda / \underline{a}$ -module. Then we have the following.*

- (a) *A map  $B \rightarrow C$  in  $\text{mod } \Lambda / \underline{a}$  is a right  $\mathcal{P}^0(\Lambda / \underline{a})$ -approximation of  $C$  if and only if it is a right  $\mathcal{P}^i(\Lambda)$ -approximation of  $C$ .*
- (b) *If  $A \rightarrow C$  is a right  $\mathcal{P}^i(\Lambda)$ -approximation of  $C$ , then  $A / \underline{a}A \rightarrow C$  is a right  $\mathcal{P}^0(\Lambda / \underline{a})$ -approximation of  $C$ .*

*Proof.* (a) Suppose  $f : B \rightarrow C$  is a right  $\mathcal{P}^0(\Lambda / \underline{a})$ -approximation of  $C$ . Since  $B$  is a projective  $\Lambda / \underline{a}$ -module and  $\text{pd}_\Lambda \Lambda / \underline{a} \leq i$ , as a  $\Lambda$ -module  $B$  is in  $\mathcal{P}^i(\Lambda)$ . Let  $g : X \rightarrow C$  be a morphism in  $\text{mod } \Lambda$  with  $X$  in  $\mathcal{P}^i(\Lambda)$ . Then  $g$  is the composition of the canonical projection  $\pi : X \rightarrow X / \underline{a}X$  and the induced map  $g_1 : X / \underline{a}X \rightarrow C$ . Since  $X / \underline{a}X$  is a projective  $\Lambda / \underline{a}$ -module, the morphism  $g_1$  can be lifted to  $B$ , so  $g$  can be lifted to  $B$ , that is,  $f : B \rightarrow C$  is a right  $\mathcal{P}^i(\Lambda)$ -approximation of  $C$ .

Conversely, suppose that a morphism  $f : B \rightarrow C$  in  $\text{mod } \Lambda / \underline{a}$  is a right  $\mathcal{P}^i(\Lambda)$ -approximation of  $C$ . Since  $\text{pd}_\Lambda B \leq i$ , as a  $\Lambda / \underline{a}$ -module  $B = B / \underline{a}B$  is projective. Hence  $f : B \rightarrow C$  is a right  $\mathcal{P}^0(\Lambda / \underline{a})$ -approximation of  $C$ .

(b) This is trivial.

**Corollary 1.2.** *Let  $\underline{a}$  be an ideal in  $\Lambda$  satisfying the hypothesis of Proposition 1.1. If  $\underline{a}S = 0$  for each simple  $\Lambda$ -module  $S$  with  $\text{pd}_\Lambda S > i$ , then  $\mathcal{P}^i(\Lambda)$  is contravariantly finite in  $\text{mod } \Lambda$ .*

*Proof.* First note that  $\mathcal{P}^i(\Lambda)$  is a resolving subcategory (i.e.  $\mathcal{P}^i(\Lambda)$  satisfying the following three conditions: (a) closed under extension, (b) closed under kernels of surjections, and (c) contains all projective  $\Lambda$ -modules). Then by [1, Proposition 3.7],  $\mathcal{P}^i(\Lambda)$  is contravariantly finite in  $\text{mod } \Lambda$  if and only if each simple  $\Lambda$ -module has a right  $\mathcal{P}^i(\Lambda)$ -approximation. Suppose  $S$  is a simple  $\Lambda$ -module. If  $\text{pd}_\Lambda S \leq i$ , we are done. Suppose now  $\text{pd}_\Lambda S > i$ ; then  $S$  is a  $\Lambda / \underline{a}$ -module. Hence there is a right  $\mathcal{P}^0(\Lambda / \underline{a})$ -approximation  $A \rightarrow S$  of  $S$  since  $\mathcal{P}^0(\Lambda / \underline{a})$  is contravariantly finite in  $\text{mod } \Lambda / \underline{a}$ . By (a) of Proposition 1.1,  $A \rightarrow S$  is also a right  $\mathcal{P}^i(\Lambda)$ -approximation of  $S$ . Therefore,  $\mathcal{P}^i(\Lambda)$  is contravariantly finite.

For an artin algebra  $\Lambda$ , we denote by  $\Omega(\text{mod } \Lambda)$  the subcategory consisting of the syzygy modules  $\Omega(C)$  of all  $C$  in  $\text{mod } \Lambda$ . Further, we denote by  $\tau_{\mathcal{A}}(M)$  the trace of a category  $\mathcal{A}$  in the module  $M$ , that is, the submodule of  $M$  generated by the images of all maps  $A \rightarrow M$  with  $A$  in  $\mathcal{A}$ . Finally, for each  $i \geq 0$ , we denote by  $\underline{a}_i$  the trace of  $\Omega(\text{mod } \Lambda) \cap \mathcal{P}^i(\Lambda)$  in  $\underline{r}$ , i.e.  $\underline{a}_i = \tau_{\Omega(\text{mod } \Lambda) \cap \mathcal{P}^i(\Lambda)}(\underline{r})$ , where  $\underline{r}$  denotes the radical of  $\Lambda$ . It is obvious that  $\underline{a}_i \subset \underline{r}$  is an ideal in  $\Lambda$ .

**Proposition 1.3.** *If  $\text{pd}_{\wedge \underline{a}_i} \leq i$ , then  $\mathcal{P}^{i+1}(\wedge)$  is contravariantly finite in  $\text{mod } \wedge$ .*

*Proof.* We first observe that  $\underline{a}_i P = \tau_{\Omega(\text{mod } \wedge) \cap \mathcal{P}^i(\wedge)}(\underline{r}P)$  if  $P$  is a projective module, where  $\underline{r}P$  is the radical of  $P$ .

Let  $M$  be a  $\wedge$ -module with projective dimension  $\leq i + 1$ . We then consider the following exact sequence

$$0 \longrightarrow \Omega(M) \longrightarrow P \xrightarrow{f} M \longrightarrow 0$$

with  $f$  a projective cover of  $M$ . Since  $\text{pd}_{\wedge} \Omega(M) \leq i$ , one gets that  $\Omega(M)$  is in  $\Omega(\text{mod } \wedge) \cap \mathcal{P}^i(\wedge)$ . Therefore, it holds that  $\Omega(M) \subset \tau_{\Omega(\text{mod } \wedge) \cap \mathcal{P}^i(\wedge)}(\underline{r}P) = \underline{a}_i P$  and that  $P/\underline{a}_i P \cong M/\underline{a}_i M$ , that is,  $M/\underline{a}_i M$  is a projective  $\wedge/\underline{a}_i$ -module. The condition  $\text{pd}_{\wedge} \underline{a}_i \leq i$  implies that  $\text{pd}_{\wedge} \wedge/\underline{a}_i \leq i + 1$ . Because each simple  $\wedge$ -module is annihilated by  $\underline{a}_i$ , one has by Corollary 1.2 that  $\mathcal{P}^{i+1}(\wedge)$  is contravariantly finite.

**Corollary 1.4.** *Suppose  $\text{pd}_{\wedge} \underline{a}_i \leq i$ . Then the  $\wedge$ -modules of projective dimension  $\leq i + 1$  are the summands of modules  $M$  which have filtrations  $M = M_0 \supset M_1 \supset \dots \supset M_n = 0$  such that each subquotient  $M_i/M_{i+1}$  is an indecomposable projective  $\wedge/\underline{a}_i$ -module.*

*Proof.* Note that the minimal right  $\mathcal{P}^0(\wedge/\underline{a})$ -approximation of a  $\wedge/\underline{a}$ -module is its projective cover. It then follows that the  $\wedge/\underline{a}$ -projective covers of the simple  $\wedge$ -modules are just the minimal right  $\mathcal{P}^i(\wedge)$ -approximations of the simple  $\wedge$ -modules. The corollary then follows directly from [1, Proposition 3.7].

Combining the above results with the results in [1] and [3], we get the following corollary.

**Corollary 1.5.** *Suppose  $\underline{a}_0$  is projective. Then  $\mathcal{P}^1(\wedge)$  has almost split sequences.*

*Proof.* From [1] one knows that  $\mathcal{P}^1(\wedge)$  is covariantly finite in  $\text{mod } \wedge$  for all artin algebras. By Proposition 1.3, one has that  $\mathcal{P}^1(\wedge)$  is also contravariantly finite in  $\text{mod } \wedge$ , so  $\mathcal{P}^1(\wedge)$  is functorially finite. Then from [3, Theorem 2.4] it follows that  $\mathcal{P}^1(\wedge)$  has almost split sequences.

As seen the example in [5], one knows that  $\mathcal{P}^i(\wedge)$  is not always contravariantly finite. In the following we construct an algebra with finite global dimension  $n \geq 2$  which satisfies that  $\mathcal{P}^i(\wedge)$  are not contravariantly finite for  $1 \leq i \leq n - 1$ .

**Example.** Let  $k$  be an algebraically closed field. For each  $n \geq 2$ , let  $\wedge_n$  be given as the path algebra of the following quiver

$$1 \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2 \xrightarrow{\gamma_1} \dots \xrightarrow{\gamma_{n-2}} n \xrightarrow{\gamma_{n-1}} n+1$$

modulo the ideal generated by the paths  $\gamma_1 \alpha$  and  $\gamma_{i+1} \gamma_i$ ,  $1 \leq i \leq n - 2$ . It is easy to calculate that  $\wedge_n$  has global dimension  $n$  and that the simple module  $S_1$  corresponding to the vertex 1 has projective dimension  $n$ . On one hand, the module  $S_1$  is a preinjective module (see [2]) and each indecomposable preinjective module has projective dimension  $n$ . On the other hand, there is a family of indecomposable  $\wedge$ -modules  $\{Y_\lambda \mid 0 \neq \lambda \in k\}$  in  $\mathcal{P}^1(\wedge)$  of dimension 2 with support containing vertices 1 and 2 only (i.e. the  $Y_\lambda$  can be considered as modules over double arrows; see [4, sect. 8]). Moreover, it holds that  $\text{hom}_{\wedge}(Y_\lambda, Y_\mu) = 0$  for  $\lambda \neq \mu$  and there is a

nonzero morphism from each  $Y_\lambda$  to  $S_1$ . Suppose  $f : X \rightarrow S_1$  is a right  $\mathcal{P}^1(\Lambda)$ -approximation. One can easily show that each  $Y_\lambda$  should appear as a summand of  $X$ , but this is impossible. Hence  $S_1$  has no right  $\mathcal{P}^1(\Lambda)$ -approximation, that is,  $\mathcal{P}^1(\Lambda)$  is not contravariantly finite in  $\text{mod } \Lambda$ . Similarly, one can prove that  $\mathcal{P}^i(\Lambda)$  is not contravariantly finite in  $\text{mod } \Lambda$  for  $2 \leq i \leq n-1$ .

An easy observation shows that  $\text{pd}_\Lambda \underline{a}_i = n-1$  for  $0 \leq i \leq n-2$ .

## 2. AN APPLICATION

As an application of the sufficient condition given in section 1, in this section we prove that  $\mathcal{P}^i(\Lambda)$  is contravariantly finite in  $\text{mod } \Lambda$  when  $\Lambda$  is stably equivalent to a hereditary algebra. The proof is based on the following two lemmas.

**Lemma 2.1** [1, Proposition 4.12]. *For an artin algebra, the following are equivalent.*

- (a)  $\Lambda$  satisfies the conditions:
  - (i) A simple module  $S$  is a torsion module (i.e.  $S^* = \text{hom}_\Lambda(S, \Lambda) = 0$ ) if it is a composition factor of  $\underline{r}P/\text{soc } P$  for some indecomposable projective module  $P$ .
  - (ii) Every indecomposable torsionless module (i.e. a submodule of a free  $\Lambda$ -module) is simple or projective.
- (b)  $\Lambda$  is stably equivalent to a hereditary algebra.

**Lemma 2.2.** *Suppose that  $\Lambda$  is stably equivalent to a hereditary algebra. Then it holds that  $\text{pd}_\Lambda \underline{a}_i \leq i$  for each  $i \geq 0$ , where  $\underline{a}_i = \tau_{\Omega(\text{mod } \Lambda) \cap \mathcal{P}^i(\Lambda)}(\underline{r})$ .*

*Proof.* By Lemma 2.1 one gets that

$$\underline{a}_i = P \oplus S_1 \oplus \cdots \oplus S_t,$$

where  $P$  is projective and the  $S_j$  are simple modules.

Suppose now that  $\text{pd}_\Lambda \underline{a}_i > i$ . Then there is a simple module  $S := S_j$  satisfying that  $\text{pd}_\Lambda S > i$ . By the construction of  $\underline{a}_i$ , there is an epimorphism  $f : L \rightarrow S$  with  $L$  indecomposable in  $\Omega(\text{mod } \Lambda) \cap \mathcal{P}^i(\Lambda)$ . We claim that  $f(\text{soc}(L)) = 0$ . Otherwise,  $f$  would be a splittable epimorphism, so  $\text{pd}_\Lambda S \leq i$ , but this is impossible. Thus  $S$  is a composition factor of  $L/\text{soc } L$ . Since  $L$  is in  $\Omega(\text{mod } \Lambda)$ , it follows that  $L \subset \underline{r}Q$  for some projective module  $Q$ . Then  $L/\text{soc } L \subset \underline{r}Q/\text{soc } Q$ ; that is,  $S$  is a composition factor of  $\underline{r}Q/\text{soc } Q$ . This implies by Lemma 2.1 (ii) that  $S$  is a torsion module. This contradicts that  $S \subset \underline{a}_i$ . Therefore, it holds that  $\text{pd}_\Lambda \underline{a}_i \leq i$ .

**Proposition 2.3.** *Suppose that  $\Lambda$  is stably equivalent to a hereditary algebra. Then  $\mathcal{P}^i(\Lambda)$  is contravariantly finite in  $\text{mod } \Lambda$  for each  $i \geq 0$ .*

*Proof.* By Lemma 2.2 one knows that  $\text{pd}_\Lambda \underline{a}_i \leq i$  for each  $i$ . Then by applying Proposition 1.3 one gets that  $\mathcal{P}^{i+1}(\Lambda)$  is contravariantly finite in  $\text{mod } \Lambda$ . The proposition then follows from the fact that the subcategory  $\mathcal{P}^0(\Lambda)$  is always contravariantly finite in  $\text{mod } \Lambda$ .

*Remark.* In [1] it is proved that  $\mathcal{P}^\infty(\Lambda)$  is contravariantly finite in  $\text{mod } \Lambda$  when  $\Lambda$  is stably equivalent to a hereditary algebra. Combining this result with Proposition 2.3, one can see that if  $\Lambda$  is stably equivalent to a hereditary algebra, then  $\mathcal{P}^i(\Lambda)$  is contravariantly finite in  $\text{mod } \Lambda$  for all  $i \in \mathbb{N} \cup \{\infty\}$ .

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