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ON THE DIMENSION OF INFINITE COVERS

W. G. DWYER, S. STOLZ, AND L. R. TAYLOR

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ABSTRACT. We prove the following theorem and some generalizations.

Theorem A. *Let X be a connected CW complex which satisfies Poincaré duality of dimension $n \geq 4$. For any subgroup H of $\pi_1(X)$, let X_H denote the cover of X corresponding to H. If H has infinite index in $\pi_1(X)$, then X_H is homotopy equivalent to an $(n-1)$-dimensional CW complex.*

If X is also a $K(\pi, 1)$, Theorem A is a result of Strebel [St]. The lack of such a theorem in general is lamented in [H], and Theorem A allows the extension of Theorem 2 (p. 157) and its Corollary (p. 158) in [H] from 4-dimensional manifolds to 4-dimensional Poincaré spaces. We begin with some notation and propositions.

Let R be a commutative ring, G a discrete group, and RG the group ring. We say that a chain complex of left RG modules, E_*, satisfies condition $HD(n)$ iff $H^r(E_*; M) = 0$ for all left RG modules, M, and all $r > n$. Wall's techniques [W] can be used to show that E_* satisfies condition $HD(n)$ iff E_* is chain homotopy equivalent to a complex E'_* with $E'_r = 0$, $r > n$, by maps $f_r : E'_r \to E_r$ and $g_r : E_r \to E'_r$ such that $g_r f_r = 1_{E'_r}$ for all r, and $f_r g_r = 1_{E_r}$ for all $r < n$. If E_* is a complex of projective modules, Wall's condition $D(n)$ is equivalent to condition $HD(n)$. We say that a connected space satisfies $HD(n)$ iff the singular chain complex of its universal cover, $S_*(\widetilde{X})$, considered as a complex of $\mathbb{Z}\pi_1(X)$ modules, does. In particular, if $n \neq 2$, it follows from Wall [W] that a CW complex, X, satisfies $HD(n)$ iff X is homotopy equivalent to an n-dimensional complex.

Given a subgroup $H \subset G$ and a chain complex E_* on left RG-modules, we can consider E_* as a chain complex of RH modules by restriction: we write ${}_{RH}|E_*$ for this complex. Let $\mathcal{C}_H{}^R{}_G$ denote the collection of all right RG modules D such that $D \otimes_{RG} \operatorname{Hom}_{RH}(RG, L) = 0$ for all left RH modules L. Given any left RG module M and integer r, there exists a natural map $\mu_M[r] : H^r(E_*; RG) \otimes_{RG} M \to H^r(E_*; M)$ induced by the maps $\operatorname{Hom}_{RG}(E_r, RG) \otimes_{RG} M \to \operatorname{Hom}_{RG}(E_r, M)$ defined by $\gamma \otimes m \mapsto (e \mapsto \gamma(e) \cdot m)$.

Throughout this paper our conventions are the same as those in [CE], Chapter 2. The notation $\operatorname{Hom}_S(A, B)$ will only be used if A and B are left S modules and $C \otimes_S B$ will be used if C is a right S module and B is a left S module. Both Hom_S and $\otimes_S$ have additional module structure when A, B, or C has a bimodule structure. If H and K are subgroups of G, then RG is a RH-RK bimodule and

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this is the source of all module structure on tensor products, homology groups, cohomology groups, etc., occurring here.

Proposition 1. *Let E_* satisfy condition $HD(n)$. If $\mu_M[n]$ is onto for all left RG modules M and if $H^n(E_*; RG) \in \mathcal{C}_H{}^R{}_G$, then ${}_{RH}|E_*$ satisfies $HD(n-1)$. If $\mu_M[n]$ is injective for all left RG modules M and if ${}_{RH}|E_*$ satisfies $HD(n-1)$, then $H^n(E_*; RG) \in \mathcal{C}_H{}^R{}_G$.*

The next few results are useful for finding RG modules that are in $\mathcal{C}_H{}^R{}_G$.

Proposition 2. *The following are equivalent.*

(2.1) *The map $\lambda_H : D \to D \otimes_{RG} \operatorname{Hom}_{RH}(RG, RG)$ with $\lambda_H(d) = d \otimes 1_{RG}$ is the zero map.*

(2.2) $D \otimes_{RG} \operatorname{Hom}_{RH}(RG, RG) = 0$.

(2.3) $D \in \mathcal{C}_H{}^R{}_G$.

Here are some ways to construct elements in $\mathcal{C}_H{}^R{}_G$ from other elements.

Proposition 3. *The following results hold.*

(3.1) *If $H \subset G' \subset G$ and if $|G : G'| < \infty$, then for any RG module D, $D|_{RG'} \in \mathcal{C}_H{}^R{}_{G'}$ iff $D \in \mathcal{C}_H{}^R{}_G$.*

(3.2) *If $H' \subset H \subset G, \mathcal{C}_H{}^R{}_G \subset \mathcal{C}_{H'}{}^R{}_G$: if $|H : H'| < \infty$, then $\mathcal{C}_{H'}{}^R{}_G \subset \mathcal{C}_H{}^R{}_G$.*

(3.3) *If $D_1 \to D_2 \to 0$ is exact and if $D_1 \in \mathcal{C}_H{}^R{}_G$, then $D_2 \in \mathcal{C}_H{}^R{}_G$.*

(3.4) *If $D_0 \to D_1 \to D_2 \to 0$ is exact, and if $D_0, D_2 \in \mathcal{C}_H{}^R{}_G$, then $D_1 \in \mathcal{C}_H{}^R{}_G$.*

(3.5) *For any index set $\mathcal{S}$, $D_\alpha \in \mathcal{C}_H{}^R{}_G$ for each $\alpha \in \mathcal{S}$ iff $(\bigoplus_{\alpha \in \mathcal{S}} D_\alpha) \in \mathcal{C}_H{}^R{}_G$.*

(3.6) *If $p : F \to G$ is an epimorphism; if D is an RG module; and if D, considered as an RF module is in $\mathcal{C}_{p^{-1}(H)}{}^R{}_F$, then $D \in \mathcal{C}_H{}^R{}_G$.*

Let R^{tr} denote R considered as a trivial RG module. Here are two exercises, 4(b) and 4(c) on page 71, from Ken Brown's book [Brn] showing that $R^{\mathrm{tr}} \in \mathcal{C}_H{}^R{}_G$.

Proposition 4. *$R^{\mathrm{tr}} \in \mathcal{C}_H{}^R{}_G$ provided*

(4.1) *G is finitely generated and H has infinite index in G; or more generally*

(4.2) *there exists a finitely generated subgroup $G' \subset G$ such that the index of $G' \cap gHg^{-1}$ in G' is infinite for all $g \in G$.*

For any R module D, let $\operatorname{Aut}(D)$ denote the R automorphism group of D. Let $R^\times$ denote the group of units of R. Multiplication by $r \in R^\times$ gives an automorphism of D and hence a homomorphism $i : R^\times \to \operatorname{Aut}^{\mathrm{op}}(D)$: let $P\operatorname{Aut}^{\mathrm{op}}(D) = \operatorname{Aut}^{\mathrm{op}}(D)/i(R^\times)$. To give D a right RG module structure is equivalent to giving a homomorphism $s : G \to \operatorname{Aut}^{\mathrm{op}}(D)$: let $\hat{s} : G \to P\operatorname{Aut}^{\mathrm{op}}(D)$ denote the evident composition and let G_D denote the kernel of $\hat{s}$.

Proposition 5. *The following results hold.*

(5.1) *If $D = R$ as an R module, then $G_D = G$.*

If G_D has finite index in G, then $D \in \mathcal{C}_H{}^R{}_G$ provided either

(5.2) *$R^{\mathrm{tr}} \in \mathcal{C}_H{}^R{}_G$ and $i : R^\times \to \operatorname{Aut}(D)$ is injective; or*

(5.3) *$p : F \to G$ is an epimorphism from a free group and $R^{\mathrm{tr}} \in \mathcal{C}_{p^{-1}(H)}{}^R{}_F$.*

Proof of Theorem A. We prove that X_H satisfies $HD(n-1)$: the conclusion of the theorem follows from Wall [W], Theorem E, p. 63. It follows easily from Poincaré duality that $S_*(\widetilde{X})$ and $S_*(\widetilde{X}, \partial\widetilde{X})$ satisfy $HD(n)$. Moreover $G = \pi_1(X)$ is finitely generated [Brd] (Corollary 1′, p. 195), so (4.1) applies.

First consider the case $\partial X = \varnothing$ and let $\mathcal{D} = H^n(S_*(\widetilde{X}); \mathbb{Z}G)$. Since X is a connected Poincaré duality space, there is a fundamental class $[X] \in H_n(S_*(\widetilde{X}); \mathcal{D})$ such that the cap product with $[X]$ yields the usual isomorphisms. In particular, for any left $\mathbb{Z}G$ module M, the cap product by $[X]$ yields an isomorphism

$$\mathrm{ev}_M : H^n(S_*(\widetilde{X}); M) \to H_0(S_*(\widetilde{X}); \mathcal{D} \otimes_{\mathbb{Z}} M) = \mathcal{D} \otimes_{\mathbb{Z}G} M.$$

The diagram

$$\begin{array}{ccc} H^n(S_*(\widetilde{X}); \mathbb{Z}G) \otimes_{\mathbb{Z}G} M & \xrightarrow{\mathrm{ev}_{\mathbb{Z}G} \otimes 1_M} & (\mathcal{D} \otimes_{\mathbb{Z}G} \mathbb{Z}G) \otimes_{\mathbb{Z}G} M \\ \Big\downarrow \mu_M[n] & & \Big\downarrow A \\ H^n(S_*(\widetilde{X}); \mathbb{Z}G \otimes_{\mathbb{Z}G} M) & \xrightarrow{\mathrm{ev}_{\mathbb{Z}G \otimes M}} & \mathcal{D} \otimes_{\mathbb{Z}G} (\mathbb{Z}G \otimes_{\mathbb{Z}G} M) \end{array}$$

commutes, where A is the usual associativity isomorphism. It follows that the $\mu_M[n]$ are isomorphisms which is one of the hypotheses of Proposition 1. Since X is connected, $\mathcal{D} = \mathbb{Z}$ as $\mathbb{Z}$ modules, and (5.1) verifies the hypotheses of (5.2). Hence (5.2) verifies the remaining hypothesis of Proposition 1 and Theorem A follows.

If $\partial X \neq \varnothing$, then Theorem A has two possible interpretations. It follows easily from Poincaré duality that $S_*(\widetilde{X}|_H)$ satisfies $HD(n-1)$ for any subgroup H, so the more interesting result is that $S_*(\widetilde{X}|_H, \partial\widetilde{X}|_H)$ satisfies $HD(n-1)$ if H has infinite index in G. The proof is a straightforward generalization of the closed case.

Proof of Proposition 1. The Shapiro Lemma [Brn] (p. 73) says $H^r({}_{RH}|E_*; L) \cong H^r(E_*; \mathrm{Hom}_{RH}(RG, L))$. The first statement follows if these cohomology groups are 0 for $r \geq n$. For $r > n$ this follows from $HD(n)$. For $r = n$, $H^n(E_*; RG) \in \mathcal{C}_H{}^R{}_G$ implies $H^n(E_*; RG) \otimes_{RG} \mathrm{Hom}_{RH}(RG, L) = 0$. But $\mu[n] : H^n(E_*; RG) \otimes_{RG} \mathrm{Hom}_{RH}(RG, L) \to H^n(E_*; \mathrm{Hom}_{RH}(RG, L))$ is onto, so ${}_{RH}|E_*$ satisfies $HD(n-1)$. Turning to the second half of Proposition 1, if ${}_{RH}|E_*$ satisfies $HD(n-1)$, then $H^n(E_*; \mathrm{Hom}_{RH}(RG, L)) = 0$, so the injectivity of $\mu_K[n]$ implies $H^n(E_*; RG) \otimes_{RG} \mathrm{Hom}_{RH}(RG, L) = 0$.

Proof of Proposition 2. Clearly (2.3) implies (2.2) implies (2.1). Composition, $c : \mathrm{Hom}_{RH}(RG, RG) \otimes_{RG} \mathrm{Hom}_{RH}(RG, L) \to \mathrm{Hom}_{RH}(RG, L)$, is a left RG module map. The identity from $D \otimes_{RG} \mathrm{Hom}_{RH}(RG, L)$ to itself factors as

$$\lambda_H \otimes 1 : D \otimes_{RG} \mathrm{Hom}_{RH}(RG, L) \to \mathcal{D} \otimes_{RG} \mathrm{Hom}_{RH}(RG, RG) \otimes_{RG} \mathrm{Hom}_{RH}(RG, L)$$

composed with

$$1 \otimes c : \mathcal{D} \otimes_{RG} \mathrm{Hom}_{RH}(RG, RG) \otimes_{RG} \mathrm{Hom}_{RH}(RG, L) \to \mathcal{D} \otimes_{RG} \mathrm{Hom}_{RH}(RG, L).$$

This shows (2.1) implies (2.3).

Proof of Proposition 3. Results (3.3) and (3.4) follow since the tensor product is right exact. Result (3.5) follows since tensor product preserves sums.

If $H \subset G' \subset G$, define $\Lambda : RG \otimes_{RG'} \mathrm{Hom}_{RH}(RG', L) \to \mathrm{Hom}_{RH}(RG, L)$ by $\Lambda(x \otimes f)(y) = f((yx)_{G'})$ where $(yx)_{G'} = \sum_{k \in G'} r_k k$ if $yx = \sum_{g \in G} r_g g$. Note Λ is a map of left RG modules. If $|G : G'| < \infty$, we construct an inverse for Λ as follows. Let $\mathcal{S} = \{x_i \in G\}$ be a set of coset representatives for $G' \backslash G$ and define $\Psi_{\mathcal{S}} : \mathrm{Hom}_{RH}(RG, L) \to RG \otimes_{RG'} \mathrm{Hom}_{RH}(RG', L)$ by $\Psi_{\mathcal{S}}(f) = \sum_i x_i^{-1} \otimes (x_i \cdot f)|_{G'}$, where here $(x_i \cdot f)|_{G'}$ denotes the homomorphism restricted to RG'. Check that $\Psi_{\mathcal{S}}$ is the inverse to Λ, so D is an RG module in $\mathcal{C}_H{}^R{}_G$ iff $D|_{RG'} \in \mathcal{C}_H{}^R{}_{G'}$. Result (3.1) follows.

Let $H' \subset H \subset G$. Since $\operatorname{Hom}_{RH}(RG, RG) \subset \operatorname{Hom}_{RH'}(RG, RG)$, (2.1) proves $\mathcal{C}_H{}^R{}_G \subset \mathcal{C}_{H'}{}^R{}_G$. If $|H : H'| < \infty$, choose coset representatives x_i for $H'\backslash H$, and define $\tau : \operatorname{Hom}_{RH'}(RG, RG) \to \operatorname{Hom}_{RH}(RG, RG)$ by $\tau(f)(y) = \sum_{x_i} x_i^{-1} f(x_i y)$. Check τ is a left RG module map. Choose coset representatives, g_α, for $H\backslash G$. Define

$$v(g) = \begin{cases} g & \text{if } g \in H' g_\alpha \text{ for some } \alpha, \\ 0 & \text{if } g \notin H' g_\alpha \text{ for any } \alpha. \end{cases}$$

Check $v \in \operatorname{Hom}_{RH'}(RG, RG)$ and $\tau(v) = 1_{RG}$. It follows that λ_H factors through $D \otimes_{RG} \operatorname{Hom}_{RH'}(RG, RG)$ so (2.2) and (2.1) prove $\mathcal{C}_{H'}{}^R{}_G \subset \mathcal{C}_H{}^R{}_G$. Result (3.2) follows.

Proof of Proposition 5. Result (5.1) holds since the R module automorphisms of R are $R^\times$. To prove the remaining results, let $\omega : G \to R^\times$ be a homomorphism, and let $\omega : RG \to RG$ be the ring homomorphism defined by $\omega(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g \omega(g) g$. Given any right RG module D, let D^ω denote D with a new RG module structure given by $d \cdot_\omega x = d \cdot \omega(x)$. Given $f \in \operatorname{Hom}_{RH}(RG, RG)$ define $f^\omega(x) = \omega(f(\omega^{-1}(x)))$ and check that $f^\omega \in \operatorname{Hom}_{RH}(RG, RG)$. Define $\Lambda : D^\omega \otimes_{RG} \operatorname{Hom}_{RH}(RG, RG) \to D \otimes_{RG} \operatorname{Hom}_{RH}(RG, RG)$ by $\Lambda(d \otimes f) = d \otimes f^\omega$. Note Λ is an isomorphism. Since $1_{RG}^\omega = 1_{RG}$, it follows from (2.1) that

$$D \in \mathcal{C}_H{}^R{}_G \quad \text{iff} \quad D^\omega \in \mathcal{C}_H{}^R{}_G. \tag{5.4}$$

Finally, we show (5.2) and (5.3). Let $H_D = H \cap G_D$ and note H_D has finite index in H. It follows from (3.1) and (3.2) that we may assume $G = G_D$ without loss of generality. By (3.6) in case (5.3) we may assume further that G is free. Under either of our two assumptions, $s : G \to \operatorname{Aut}^{\mathrm{op}}(D)$ factors through a homomorphism $\omega : G \to R^\times$. If D^{tr} denotes D as an R module but with trivial G action, $D = (D^{\mathrm{tr}})^\omega$. From (5.4), we need only prove $D^{\mathrm{tr}} \in \mathcal{C}_H{}^R{}_G$. This follows from (3.5), from (3.3) and from the assumption that $R^{\mathrm{tr}} \in \mathcal{C}_H{}^R{}_G$.

Remarks 1. 1. Strebel [St], p. 324, shows that $\mathbb{Z}^{\mathrm{tr}} \notin \mathcal{C}_e{}^{\mathbb{Z}}{}_G$ whenever G is infinite but locally finite. His technique generalizes to show a partial converse for (4.2): if for every finitely generated subgroup $G' \subset G$ there exists a $g \in G$ such that the index of $G' \cap gHg^{-1}$ in G' is finite, then $\mathbb{Z}^{\mathrm{tr}} \notin \mathcal{C}_H{}^{\mathbb{Z}}{}_G$.

2. For duality groups in the sense of Bieri and Eckmann [Brn], pp. 219–225, $\mu_M[n]$ is still an isomorphism, so Proposition 1 implies that, for any subgroup H of a duality group G,

$$K(H,1) \text{ satisfies } HD(n-1) \quad \text{iff} \quad H^n(G; \mathbb{Z}G) \in \mathcal{C}_H{}^{\mathbb{Z}}{}_G. \tag{6}$$

As Strebel remarks, there are examples of duality groups and subgroups for which $|G : H| = \infty$ but $K(H,1)$ does not satisfy $HD(n-1)$. This shows that more than infinite index is required in general for the dimension to drop. The only if part of (6) also gives restrictions on the dualizing module: e.g., since $K(\{e\},1)$ satisfies $HD(0)$, $H^n(G; \mathbb{Z}G) \otimes_{\mathbb{Z}G} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, \mathbb{Z}G) = 0$.

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Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556
E-mail address: `dwyer.1@nd.edu`
E-mail address: `stolz.1@nd.edu`
E-mail address: `taylor.2@nd.edu`