DIFFEOMORPHISMS WITH PERSISTENCY

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Abstract. The $C^1$ interior of the set of all diffeomorphisms satisfying Lewowicz’s persistency is characterized as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition.

In [5], Lewowicz introduced a notion of persistency for a homeomorphism of a compact metric space $X$, and it is remarked that persistence is a weaker property than topological stability when $X$ is a manifold. It is also proved there that every pseudo-Anosov map (on a surface) is persistent. The purpose of this paper is to analyze the dynamics of diffeomorphisms having persistency. More precisely we shall prove the following theorem.

Let $M$ be a $C^\infty$ closed manifold and let $\text{Diff}(M)$ be the space of $C^1$ diffeomorphisms of $M$ endowed with $C^1$ topology. We denote by $\mathcal{P}(M)$ the set of all $f \in \text{Diff}(M)$ having persistency.

Theorem. The $C^1$ interior of $\mathcal{P}(M)$ in $\text{Diff}(M)$, $\text{int}\mathcal{P}(M)$, is characterized as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition.

It was proved in [7] and [9] respectively that the $C^1$ interior of the set of all $f \in \text{Diff}(M)$ having topological stability and the $C^1$ interior of the set of all $f \in \text{Diff}(M)$ having the pseudo-orbit tracing property were characterized as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition. Therefore, if the theorem is established, then these two open sets and $\text{int}\mathcal{P}(M)$ are equal.

Let $\mathcal{E}(M)$ be the set of all expansive diffeomorphisms of $M$. Gerber and Katok [3] proved that if $f$ is a pseudo-Anosov map on a surface $M$ and if $\mathcal{N}^0(f)$ is a $C^0$ neighborhood of $f$, then there exists a smooth diffeomorphism $g \in \mathcal{N}^0(f)$ conjugating to $f$. Thus it can be checked that $g \in \mathcal{E}(M) \cap \mathcal{P}(M)$, and more precisely, the following corollary implies that $g$ belongs to $\mathcal{E}(M) \cap \partial\mathcal{P}(M)$. Here $\partial\mathcal{P}(M)$ denotes the boundary of $\mathcal{P}(M)$ in $\text{Diff}(M)$.

Corollary. $\mathcal{E}(M) \cap \text{int}\mathcal{P}(M)$ is characterized as the set of all Anosov diffeomorphisms.

The corollary is an easy consequence of our theorem. Indeed, since every $f \in \text{Diff}(M)$ satisfying Axiom A and the strong transversality condition is structurally...
stable, if \( f \in \mathcal{E}(M) \), then it is Anosov (by [6]). Conversely, if \( f \in \text{Diff}(M) \) is Anosov, then \( f \) is persistent since \( f \) is topologically stable.

Let \( d \) be a metric on \( M \) which is induced from a Riemannian metric \( \| \cdot \| \) on \( TM \), and let us denote by \( \mathcal{H}(M) \) the set of all homeomorphisms of \( M \). We say that \( f \in \mathcal{H}(M) \) is persistent if for each \( \varepsilon > 0 \), there is \( \delta > 0 \) such that for every \( x \in M \) and \( g \in \mathcal{H}(M) \) with \( d(f,g) < \delta \), there is \( y \in M \) satisfying \( d(f^n(x),g^n(y)) < \varepsilon \) \((\forall n \in \mathbb{Z})\). The notion is independent of a metric for \( M \) and is conjugacy invariant.

Let \( \Lambda(f) \) be a hyperbolic set of \( f \in \text{Diff}(M) \). For any \( \varepsilon > 0 \) and \( x \in \Lambda(f) \), the local stable manifold and the local unstable manifold are denoted by \( W^s(x,f) \) and \( W^u(x,f) \) respectively. The stable manifold \( W^s(x,f) \) and the unstable manifold \( W^u(x,f) \) of \( x \in \Lambda(f) \) are defined by a usual way. Let \( f \in \text{Diff}(M) \) satisfy Axiom A. Then the non-wandering set of \( f \), \( \Omega(f) \), is a disjoint union of basic sets \( \Lambda_i(f) \cup \cdots \cup \Lambda_{\ell}(f) \). Recall that the periodic points of \( f \) are dense in \( \Lambda_i(f) \) \((1 \leq i \leq \ell)\) and that for every \( x \in M \) there are \( p \in \Lambda_i(f) \) and \( q \in \Lambda_j(f) \) \((1 \leq i \neq j \leq \ell)\) such that \( x \in W^s(p,f) \cap W^u(q,f) \). We say that \( f \) satisfies the strong transversality condition if for every \( x \in M \), \( T_xW^s(p,f) + T_xW^u(q,f) = T_xM \) for some \( p,q \in \Omega(f) \).

Let \( P(f) \) denote the set of all periodic points of \( f \in \text{Diff}(M) \), and let \( \mathcal{F}(M) \) be the set of all \( f \in \text{Diff}(M) \) having a \( C^1 \)-neighborhood \( \mathcal{U}(f) \subset \text{Diff}(M) \) such that every \( p \in P(g) \) \((\forall g \in \mathcal{U}(f))\) is hyperbolic. Then such a set was characterized as the set of all diffeomorphisms satisfying Axiom A with no-cycles (see [1, 4]). It is well known that every \( f \in \text{Diff}(M) \) satisfying Axiom A and the strong transversality condition is persistent (because \( f \) is topologically stable (see [8])). Therefore our theorem follows from the following two propositions.

**Proposition A.** The \( C^1 \) interior of \( \mathcal{P}(M) \), \( \text{int}\mathcal{P}(M) \), is a subset of \( \mathcal{F}(M) \).

**Proposition B.** Let \( f \in \text{Diff}(M) \) satisfy Axiom A with no-cycles. If \( f \in \text{int}\mathcal{P}(M) \), then \( f \) satisfies the strong transversality condition.

1. **Proof of Proposition A**

Let \( f \in \text{int}\mathcal{P}(M) \). To get the conclusion, it is enough to show that every \( p \in P(f) \) is hyperbolic. Indeed, if this is established, then for every \( C^1 \) neighborhood \( \mathcal{V}(f) \subset \text{int}\mathcal{P}(M) \) of \( f \), every \( q \in P(g) \) \((\forall g \in \mathcal{V}(f))\) is hyperbolic because \( g \in \text{int}\mathcal{P}(M) \). Thus \( f \in \mathcal{F}(M) \) is obtained.

Fix a neighborhood \( \mathcal{U}(f) \subset \text{int}\mathcal{P}(M) \) of \( f \), and by assuming that there is a non-hyperbolic periodic point \( p = f^n(p) \), we shall derive a contradiction. Here \( n > 0 \) is the prime period of \( p \). The tangent space \( T_p M \) splits into the direct sum \( T_p M = E^n_p \oplus E^n_u \oplus E^n_c \) where \( E^n_p \), \( E^n_u \) and \( E^n_c \) are \( D_pf^n \)-invariant subspaces corresponding to the absolute values of the eigenvalues of \( D_pf^n \) greater than one, less than one and equal to one, and suppose \( E^n_c \neq 0 \). Then, for every \( \varepsilon > 0 \) there exists a linear automorphism \( \mathcal{O} : T_p M \to T_p M \) such that

\[
\begin{align*}
|\mathcal{O} - I| &\leq \varepsilon, \\
\mathcal{O}(E^n_\sigma) &\supseteq E^n_\sigma \text{ for } \sigma = s,u \text{ and } c, \\
\text{all eigenvalues of } \mathcal{O} \circ D_pf^n|_{E^n_\sigma} \text{ are of a root of unity,}
\end{align*}
\]

where \( I : T_p M \to T_p M \) is an identity map. By making use of Franks’s lemma (see [2, Lemma 1.1]), we can find \( \delta_0 > 0 \) and \( g \in \mathcal{U}(f) \) such that

\[
(i) \quad B_{4\delta_0}(f^i(p)) \cap B_{4\delta_0}(f^j(p)) = \emptyset \text{ for } 0 \leq i \neq j \leq n - 1,
\]
(ii) $g(x) = f(x)$ for $x \in \{p, f(p), \ldots, f^{n-1}(p)\} \cup \{M \setminus \cup_{i=0}^{n-1} B_{\delta_0}(f^i(p))\}$,

(iii) $g(x) = \exp_{f^{i+1}(p)} \circ D_{f^i(p)} f \circ \exp_{f^i(p)}^{-1}(x)$ for $x \in B_{\delta_0}(f^i(p)) (0 \leq i \leq n-2)$,

(iv) $g(x) = \exp_p \circ \mathcal{O} \circ D_{f^{n-1}(p)} f \circ \exp_{f^{n-1}(p)}^{-1}(x)$ for $x \in B_{\delta_0}(f^{n-1}(p))$,

where $B_{\varepsilon}(x) = \{y \in M | d(x, y) \leq \varepsilon\}$ for $\varepsilon > 0$.

Define $G = \mathcal{O} \circ D_p f^n$. Then there exists $m > 0$ such that $G^m_{|B_{\varepsilon}}$ is an identity map. For a sufficiently small $0 < \delta_1 < \delta_0$, we have

$$g^m_{|\exp_p T_p M(\delta_1)} = \exp_p \circ G^m \circ \exp_p^{-1}$$

where $T_p M(\delta_1) = \{v \in T_p M ||v|| \leq \delta_1\}$. Put $E^c_p(\delta_1) = E^c_p \cap T_p M(\delta_1)$. Then it is clear that

$$g^m_{|\exp_p E^c_p(\delta_1)} = id_{|\exp_p E^c_p(\delta_1)}.$$

Let $v = (v_1, v_2, \ldots, v_r)$ ($r = \dim E^c_p$) be the representation by components with respect to the fundamental vectors of $R^r = E^c_p$. Put $\varepsilon = \delta_1 / 8$ and fix any $0 < \delta < \varepsilon$. Let $\phi : R^r \to R^r$ be the time-one map given by the vector field

$$\psi_i = \delta \chi(v_1) \cdots \chi(v_r) v_i$$

for $1 \leq i \leq r$. Here $\chi : R \to R$ is a $C^\infty$ function $(0 \leq \chi(t) \leq 1)$ such that

$$\chi(t) = \begin{cases} 1 & \text{if } |t| \leq \delta_1 / 2, \\ 0 & \text{if } |t| \geq 2\delta_1 / 3, \end{cases}$$

and $\delta^\prime > 0$ is a number chosen so that $||\phi(v) - v|| < \delta$ for $v \in R^r$ and $||Dv\phi - id_{R^r}|| \leq |\delta^\prime - 1| < \delta$ for $v \in T_p M(\delta_1) \cap R^r$. We shall denote by $\hat{\phi} : T_p M(\delta_1) \to T_p M(\delta_1)$ the extension of $\phi$ such that $\hat{\phi}(v) = \phi(v)$ for $v \in E^c_p(\delta_1)$ and $||\hat{\phi}(v) - v|| < \delta$ for $v \in T_p M(\delta_1)$. Put

$$\psi(x) = \begin{cases} \exp_p \circ \hat{\phi} \circ \exp_p^{-1}(x) & \text{if } x \in \exp_p(T_p M(\delta_1)), \\ x & \text{otherwise}, \end{cases}$$

and define $\tilde{g} = \psi \circ g$. Let $\hat{\delta} = \delta(g, \varepsilon) > 0$ be a number as in the definition of persistency of $g$. Then $d(\tilde{g}, g) < \hat{\delta}$ ($\tilde{g} \in \mathcal{H}(M)$) for a sufficiently small $\hat{\delta}$. Take and fix $v \in E^c_p(\delta_1)$ such that $d(\exp_p(v), p) = 2\hat{\varepsilon}$. Clearly $g^m(\exp_p(v)) = \exp_p(v)$ for all $m \in Z$. On the other hand, it is easy to see that for every $y \in B_{\varepsilon}(\exp_p(v))$, there is $m(y) \in Z$ such that

$$d(\tilde{g}^m(y), g^m(\exp_p(v))) = d(\tilde{g}^m(y), \exp_p(v)) > \hat{\varepsilon}.$$

This is a contradiction.

2. Proof of Proposition B

Before starting the proof of this proposition, we need some preparation. Throughout this section let $f \in \text{Diff}(M)$ satisfy Axiom A with no-cycles. Take a basic set $\Lambda(f)$ of $f$ and fix $\varepsilon_0 > 0$ sufficiently small. Since $\dim W^s_{\varepsilon_0}(x, f) = \dim W^s_{\varepsilon_0}(y, f)$ for $x, y \in \Lambda(f)$, we denote by $\text{Ind}\Lambda(f)$ the dimension of $W^s_{\varepsilon_0}(x, f)$ for $x \in \Lambda(f)$. If $g \in \text{Diff}(M)$ is $C^1$ close to $f$, then the number of basic sets $\{\Lambda_i(g)\}$ of $g$ coincides with that of basic sets $\{\Lambda_i(f)\}$ because of $\Omega$-stability of $f$.

The following lemma is induced by Franks’s lemma (see [9, Lemma 3] for details).
Lemma 1. Let $\Lambda_1(f)$ and $\Lambda_2(f)$ be basic sets for $f$. Suppose that there are $p = f^n(p) \in \Lambda_1(f)$ $(n > 0), q \in \Lambda_2(f)$ and $x \in M \setminus \Omega(f)$ satisfying $x \in W^s(p, f) \cap W^u(q, f)$. Then, for a neighborhood $\mathcal{U}(f) \subset \text{Diff}(M)$ of $f$, there are $0 < \varepsilon_1 < \varepsilon_0/2, g \in \mathcal{U}(f)$ and two distinct basic sets $\Lambda_1(g)$ and $\Lambda_2(g)$ such that

(i) $B_{4\varepsilon_1}(f^i(p)) \cap B_{4\varepsilon_1}(f^j(p)) = \emptyset$ for $0 \leq i \neq j \leq n - 1$,

(ii) $g(x) = \begin{cases} \exp_{f^{i+1}(p)} \circ Df^{i+1}(p)f \circ \exp_{f^i(p)}^{-1}(x) & \text{if } x \in B_{\varepsilon_1}(f^i(p)) \text{ for } 0 \leq i \leq n - 1, \\ f(x) & \text{if } x \notin \bigcup_{i=0}^{n-1} B_{4\varepsilon_1}(f^i(p)), \end{cases}$

(iii) $\begin{cases} g^n(p) = p \in \Lambda_1(g) \text{ and } q \in \Lambda_2(g), \\ x \in W^s(p, g) \cap W^u(q, g), \\ T_xW^s(p, g) = T_xW^s(p, f) \text{ and } T_xW^u(q, g) = T_xW^u(q, f). \end{cases}$

Since $f$ satisfies Axiom A, there exist a $Df$-invariant continuous splitting $T_{\Omega(f)}M = E^s \oplus E^u$ and a constant $0 < \lambda < 1$ such that

$\|Df^n_x\| < \lambda^n, \|Df^{-n}_x\| < \lambda^n$

for all $m \geq 0$. We denote by $E^s_v$ a fiber of $E^s$ at $x \in \Omega(f)$ and put $E^s_{x, \varepsilon} = \{v \in E^s_x \|v\| \leq \varepsilon\}$ for $\varepsilon > 0$ $(\sigma = s, u)$. Let $g \in \text{Diff}(M), p = g^n(p) \in \Lambda_1(g)$ and $\varepsilon_1 > 0$ be given as in Lemma 1. Then it is easily checked that for $0 < \varepsilon < \varepsilon_1$, we have

$\exp_p(E^s_{x, \varepsilon}) \subset W^s_{x, \varepsilon_0}(p, g) \text{ and } \dim \exp_p(E^s_{x, \varepsilon}) = \dim W^s_{x, \varepsilon_0}(p, g)$

for $\sigma = s, u$.

Now we shall prove Proposition B. Fix $x \in M \setminus \Omega(f)$ and let $\Lambda_i(f)$ and $\Lambda_j(f)$ be basic sets of $f$ such that $x \in W^s(\Lambda_i(f), f) \cap W^u(\Lambda_j(f), f)$. To simplify the proof we assume $i = 1$ and $j = 2$. If $\text{Ind} \Lambda_1(f) = \dim M$ or $\text{Ind} \Lambda_2(f) = 0$, then the conclusion of this proposition is clear. Thus we shall prove $T_xM = T_xW^s(x, f) + T_xW^u(x, f)$ when $\text{Ind} \Lambda_1(f) \leq \dim M - 1$ and $\text{Ind} \Lambda_2(f) \geq 1$.

Since $f \in \text{intP}(M)$ and $\Omega(f) = \overline{\Omega(f)}$, there is $f' \in \text{intP}(M)$ arbitrarily near to $f$ in the $C^1$ topology such that

(a) $f(y) = f'(y)$ for all $y$ outside of a small neighborhood of $x$,

(b) there are $p = f^n(p) \in \Lambda_1(f)$ for some $n > 0$ and $q \in \Lambda_2(f)$ such that $x \in W^s(p, f') \cap W^u(q, f'), T_xW^s(p, f') = T_xW^s(x, f)$ and $W^u(q, f') = W^u(x, f)$.

By (a), there are basic sets $\Lambda_i(f')$ $(i = 1, 2)$ for $f'$ such that $\Lambda_i(f') = \Lambda_i(f)$ since $f$ is $\Omega$-stable. Let us prove $T_xM = T_xW^s(p, f') + T_xW^u(q, f')$. We identify $f'$ with $f$ for simplicity, and let $\mathcal{U}(f)$ be a small neighborhood of $f$ such that $\mathcal{U}(f) \subset \text{P}(M)$.

Then, by Lemma 1 there are $g \in \mathcal{U}(f)$ and basic sets $\Lambda_i(f)$ $(i = 1, 2)$ satisfying Lemma 1 (i), (ii) and (iii). Thus $T_xW^s(p, g) = T_xW^s(x, f)$ and $W^u(q, g) = W^u(x, f)$. Pick $\ell > 0$ such that $g(\ell x) \in W^s_{x, \varepsilon_1/2}(p, g)$ and $g^{-\ell}(x) \in W^u_{x, \varepsilon_0/2}(g^{-\ell}(q), g)$, and define

$C^u(g^\ell(x)) = \text{the connected component of } g^\ell(x) \text{ in } W^u(g^\ell(q), g) \cap B_{\varepsilon_1}(p)$. 

Clearly, $\exp_p^{-\ell}(C^u(g^\ell(x))) \subset T_pM$. 

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For a linear subspace $E$ of $T_pM$ and $\nu > 0$, we write
\[ E_p(g^\ell(x)) = \{v + \exp_p^{-1}(g^\ell(x))| v \in E \text{ with } \|v\| \leq \nu\}. \]
Then there are a linear subspace $E' \subset T_pM$ and a number $0 < \nu_0 \leq \varepsilon_1$ such that
\[ T_{g^\ell(x)} \exp_p(E'_\nu(g^\ell(x))) = T_{g^\ell(x)} C^u(g^\ell(x)) \]
and $\exp_p(E'_\nu(g^\ell(x))) \subset B_{\varepsilon_1}(p)$ for $0 < \nu \leq \nu_0$. Since $g^\ell(x) \notin \Omega(g)$, there is $0 < \nu_1 \leq \nu_0$ such that $B_{\varepsilon_1}(g^\ell(x)) \cap g^\ell(B_{\nu_1}(g^\ell(x))) = \emptyset$ for all $i \in \mathbb{Z} \setminus \{0\}$. If $U(g) \subset U(f)$ is a neighborhood of $g$, then there are $0 < \nu_2 < \nu_1/4$ and $\varphi \in \text{Diff}(M)$ such that
\[
\begin{align*}
\phi((B_{\varepsilon_1}(g^\ell(x))) &= \text{id}, \\
\phi(g^\ell(x)) &= g^\ell(x), \\
\varphi(\exp_p(E'_\nu(g^\ell(x)))) &\subset C^u(g^\ell(x)), \\
\dim \varphi(\exp_p(E'_\nu(g^\ell(x)))) &= \dim C^u(g^\ell(x)), \\
g' \in U(g) \text{ where } g' = \varphi^{-1} \circ g.
\end{align*}
\]
We denote $\exp_p(E'_\nu(g^\ell(x)))$ by $\exp_p(E'_\nu(g^\ell(x)))$ because of $g^\ell(x) = g^\ell(x)$. It is clear that there are two distinct basic sets $\Lambda_i(g')$ ($i = 1, 2$) such that $\Lambda_i(g') = \Lambda_i(g)$ since $g$ is $\Omega$-stable, and such that
\[
\begin{align*}
T_xW^\sigma(x, g') &= T_xW^\sigma(x, g) \ (\sigma = s, u), \\
\exp_p(E'_\nu(g^\ell(x))) &\subset W^u(g^\ell(q), g') \cap B_{\varepsilon_1}(p), \\
\dim \exp_p(E'_\nu(g^\ell(x))) &= \dim W^u(q, g') = \dim C^u(g^\ell(x)).
\end{align*}
\]
Lemma 2. Under the above notation, $\exp_p(E'_\nu(g^\ell(x)))$ meets transversely $W_{\varepsilon_1}^s(p, g')$ at $g^\ell(x)$.

If this lemma is established, then we have $T_xM = T_xW^s(x, f) + T_xW^u(x, f)$ since $T_xW^\sigma(x, g') = T_xW^\sigma(x, g) = T_xW^\sigma(x, f)$ for $\sigma = s, u$.

Proof of Lemma 2. Put $C^u_x(g^\ell(x)) = B_x(g^\ell(x)) \cap g^{2\ell}(W^u_\varepsilon(g^{-\ell}(q), g'))$ for $\varepsilon > 0$. Take $0 < \varepsilon < \nu_2$ such that $C^u_x(g^\ell(x))$ is the connected component of $g^\ell(x)$ in $B_{\varepsilon}(g^\ell(x)) \cap g^{2\ell}(W^u_\varepsilon(g^{-\ell}(q), g'))$ for $0 < \varepsilon \leq \varepsilon$, and such that $B_{\varepsilon}(g^\ell(x)) \cap g^{2\ell}(W^u_\varepsilon(g^{-\ell}(q), g')) \subset \exp_p(E'_\nu(g^\ell(x)))$.

Claim. For every $0 < \varepsilon \leq \varepsilon$, if $d(g^{-\ell}(g^\ell(x)), g^{-\ell}(w)) < \varepsilon$ for all $i \geq 0$, then $w \in C^u_x(g^\ell(x))$.

Indeed, it is clear that $d(g^{-\ell}(g^\ell(x)), g^{-2\ell}(w)) < \varepsilon \leq \varepsilon_0/2$ for all $i \geq 0$. On the other hand, since $d(g^{-\ell}(g^\ell(x)), g^{-i}(q)) < \varepsilon_0/2$ for all $i \geq 0$,
\[
d(g^{-2\ell}(w), g^{-\ell}(q)) \leq d(g^{-2\ell}(w), g^{-i}(q)) + d(g^{-\ell}(g^\ell(x)), g^{-i}(q)) < \varepsilon_0,
\]
for all $i \geq 0$ and so $g^{-2\ell}(w) \in W^u_\varepsilon(g^{-\ell}(q), g')$. Thus $w \in C^u_x(g^\ell(x)) = B_x(g^\ell(x)) \cap g^{2\ell}(W^u_\varepsilon(g^{-\ell}(q), g'))$ since $d(g^\ell(x), w) < \varepsilon$. The claim is proved.

Suppose that $\exp_p(E'_\nu(g^\ell(x)))$ does not meet transversely $W^s_{\varepsilon_1}(p, g')$ at $g^\ell(x)$. Then there exist $0 < \nu_3 < \min\{\varepsilon, \nu_2/2\}$ such that for every $\delta > 0$ ($\delta \ll \nu_3$) there is $\psi_\delta \in \text{Diff}(M)$ satisfying
\[
\begin{align*}
\psi_\delta((B_{\varepsilon_1}(g^\ell(x))) &= \text{id}, \\
d(\psi_\delta, \text{id}) &< \delta, \\
\psi_\delta(\exp_p(E'_\nu(g^\ell(x)))) &\cap W^s_{\varepsilon_1}(p, g') = \emptyset.
\end{align*}
\]
Fix $0 < \varepsilon' < \nu_3/2$ and let $\delta' = \varepsilon'(g', \varepsilon') > 0$ be a number as in the definition of persistency of $g'$. Take $\delta > 0$ such that $\tilde{g} = g' \circ \psi_\delta \in \mathcal{H}(M)$ and $d(\tilde{g}, g') < \delta'$. Then, for $g'$-orbit $\{g'^i(x)\}_{i \in \mathbb{Z}}$ of $x$, there is $y \in B_{\varepsilon'}(g'^i(x))$ such that $d(\tilde{g}^i(y), g'^i(g'^i(x))) < \varepsilon'$ for all $i \in \mathbb{Z}$. By the claim $y \in \exp_p(E'_{\nu_3/2}(g'^i(x)))$, from which $\tilde{g}(y) = g \circ \psi_\delta(y) \notin W^s_\varepsilon(p, g')$. Thus, by the hyperbolicity, $d(\tilde{g}^i(\tilde{g}(y)), g'^{i+1}(g'^i(x))) > \varepsilon'$ for some $i \geq 0$. This is a contradiction.

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