ON THE SIZE OF LEMNISCATES OF POLYNOMIALS IN ONE AND SEVERAL VARIABLES

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Abstract. In the convergence theory of rational interpolation and Padé approximation, it is essential to estimate the size of the lemniscatic set $E := \{ z : |z| \leq r \text{ and } |P(z)| \leq \epsilon^n \}$, for a polynomial $P$ of degree $\leq n$. Usually, $P$ is taken to be monic, and either Cartan’s Lemma or potential theory is used to estimate the size of $E$, in terms of Hausdorff contents, planar Lebesgue measure $m_2$, or logarithmic capacity cap. Here we normalize $\|P\|_{L_\infty(\{z\leq r\})} = 1$ and show that $\text{cap}(E) \leq 2r\epsilon$ and $m_2(E) \leq \pi(2r\epsilon)^2$ are the sharp estimates for the size of $E$. Our main result, however, involves generalizations of this to polynomials in several variables, as measured by Lebesgue measure on $\mathbb{C}^n$ or product capacity and Favorov’s capacity. Several of our estimates are sharp with respect to order in $r$ and $\epsilon$.

§1. Introduction

In the convergence theory of Padé approximation, and more generally rational interpolation, an essential ingredient is an estimate on the size of the lemniscate

\begin{equation}
E(P, \epsilon) := \{ z : |P(z)| \leq \epsilon^n \},
\end{equation}

where $P$ is a polynomial of degree $\leq n$. There are several ways to provide this estimate. Cartan’s Lemma shows that if $P$ is normalized to be monic of degree $n$, then we can cover this set by a union of $\ell \leq n$ balls $B_j$, $1 \leq j \leq \ell$, whose diameters $d(B_j)$ satisfy, for a given $\alpha > 0$,

\begin{equation}
\sum_{j=1}^{\ell} (d(B_j))^\alpha \leq e^{4\alpha} \epsilon^\alpha.
\end{equation}

The remarkable thing about the estimate is its independence of the degree of $P$. See [1, p. 194], [7], [9], [12], [14] for further details and extensions. As far as we know, the sharp constant (that should replace $e^{4\alpha}$) in Cartan’s Lemma is still an unsolved problem. The authors thank Peter Borwein for informing them that the conjectured sharp constant for $\alpha = 1$ is 4.
An even more appropriate set function to measure $E(P; \epsilon)$ for monic $P$ is logarithmic capacity. Amongst the many equivalent definitions, we mention the one involving the Chebyshev constant: For compact $F \subset \mathbb{C}$,

$$\text{cap}(F) := \lim_{n \to \infty} \left[ \min \left\{ \|P\|_{L_\infty(F)} : P \text{ monic of degree } n \right\} \right]^{1/n}.$$  

See [7], [9], [12]. Here we have the identity

$$\text{cap}(E(P; \epsilon)) = \epsilon.$$  

(1.3)

In applications of these to Padé approximation, one usually has to estimate

$$\|P\|_{L_\infty(|t|=r)}/|P(z)|,$$  

where $|z| < r$ lies outside some exceptional set. Normalizing $P$ to be monic helps us to estimate the denominator in (1.4), but then zeros of $P$ of large modulus are troublesome in estimating the numerator. To circumvent this, researchers in Padé approximation such as Nuttall, Pommerenke, Goncar, and others [8], [13], [15] split the zeros of $P$ into sets \{u_j : |u_j| \leq 2r\} and \{v_j : |v_j| > 2r\} and normalized $P$ as

$$P(z) = \prod_j (z - u_j) \prod_j (1 - z/v_j).$$

Since for $|z| \leq r$,

$$\frac{1}{2} < |1 - z/v_j| < \frac{3}{2}; \quad |z - u_j| \leq 3r$$

we easily see that

$$\|P\|_{L_\infty(|t|=r)}/|P(z)| \leq \left( 3 \max\{1, r\} \right)^n\left|\prod_j (z - u_j)\right|$$

and now the size of the exceptional set can be estimated by (1.2) or (1.3).

In studying convergence theory of Padé approximants of several variables [5], [8], [11], one can try to extend this approach to several variable polynomials $P(z_1, z_2, \ldots, z_\ell)$. One can fix $z_2, z_3, \ldots, z_\ell$ and then factorize as above in terms of $z_1$. However the $u_j$ and $v_j$ depend in a complicated way (implicit function theorem, etc.) on the other variables $z_j$, $2 \leq j \leq \ell$, and normalization becomes a real problem.

So we found it desirable to instead normalize

$$\|P\|_{L_\infty(|z|=r)} = 1$$

(1.5)

and study directly the size of

$$\begin{align*}
E(P; r; \epsilon) &:= \left\{ z : |z| \leq r \text{ and } |P(z)| \leq \epsilon^n \right\},
\end{align*}$$

(1.6)
in the hope of producing an approach that will more easily extend to polynomials in several variables. Of course, this normalization avoids having to separate zeros of $P$ into large and small modulus when we estimate the ratio (1.4).

Let $m_2$ denote planar Lebesgue measure and, for $\alpha > 0$, let $h_\alpha$ denote $\alpha$-dimensional Hausdorff content, so that

$$(1.7) \quad h_\alpha(E) := \inf \left\{ \sum_{j=1}^{\infty} (d(B_j))^\alpha : \{B_j\} \text{ are balls with } E \subset \bigcup_{j=1}^{\infty} B_j \right\}.$$  

Here $d(B_j)$ denotes the diameter of $B_j$. Of course, for measurable $E$,

$$m_2(E) = \frac{\pi}{4} h_2(E).$$  

The sharp form of (a) of the following simple one-variable result is apparently new:

**Theorem 1.1.** (a) For polynomials $P$ of degree $\leq n$, normalized by (1.5), and $\epsilon > 0$, we have

$$\begin{align*}
(1.8) & \quad \text{cap}(E(P; r; \epsilon)) \leq 2r\epsilon; \\
(1.9) & \quad m_2(E(P; r; \epsilon)) \leq \pi(2r\epsilon)^2.
\end{align*}$$

If $L$ is any line in the plane, then

$$\begin{align*}
(1.10) & \quad h_1(L \cap E(P; r; \epsilon)) \leq 8r\epsilon.
\end{align*}$$

Given $n \geq 1$ and $r > 0$, (1.8) and (1.9) are sharp in the sense that

$$\begin{align*}
(1.11) & \quad \sup_{P, \epsilon} \text{cap}(E(P; r; \epsilon))/\epsilon = 2r; \\
(1.12) & \quad \sup_{P, \epsilon} m_2(E(P; r; \epsilon))/\epsilon^2 = \pi(2r)^2.
\end{align*}$$

In each case the sup is taken over $\epsilon > 0$ and polynomials $P$ of degree $n$ satisfying (1.5). Moreover, (1.10) is almost sharp in the sense that given $n \geq 1$ and $r > 0$,

$$\begin{align*}
(1.13) & \quad \sup_{L, P, \epsilon} h_1(L \cap E(P; r; \epsilon))/\epsilon \geq 8r2^{-1/n}.
\end{align*}$$

In the last sup, $L$ refers to all lines in $\mathbb{C}$.

(b) Given $\alpha > 0$ and $P$ of degree $\leq n$, normalized by (1.5), we have

$$h_\alpha(E(P; r; \epsilon)) \leq 18(4r\epsilon)^\alpha.$$ 

Of course, (1.10) shows that the diameter of $E(P; r; \epsilon)$ is at most $8r\epsilon$, and our examples that prove (1.13) show this is sharp as $n \to \infty$. We remark that using Nuttall’s method, Pommerenke [15] established the weaker estimate

$$\text{cap}(E(P; r; \epsilon)) \leq 3\epsilon.$$
Our proof of (1.8) involves the Walsh–Bernstein lemma and simple estimates on Green’s functions. Then standard inequalities relating \(h_\alpha\) and \(m_2\) to cap give (1.9), (1.10), (1.14).

As we have mentioned, our main goal is estimation of the lemniscates of polynomials of several variables. Some intuition is provided by the polynomial

\[ P(z, w) := (zw)^n. \]

We see that given \(r \geq \epsilon > 0\),

\[
E(P; r; \epsilon) := \left\{(z, w): |z|, |w| \leq r \quad \text{and} \quad |P(z, w)| \leq \epsilon^n \right\}
\]

\[
= \left\{(z, w): |z|, |w| \leq r \quad \text{and} \quad |zw| \leq \epsilon \right\}
\]

\[
= \bigcup_{|w| \leq r} \left\{(z, w): |z| \leq \min \left\{r, \frac{\epsilon}{|w|}\right\}\right\}.
\]

Then if \(m_4\) denotes Lebesgue measure on \(\mathbb{C}^2\), Fubini’s theorem gives

\[
m_4(E(P; r; \epsilon)) = m_2 \times m_2(E(P; r; \epsilon)) = \int_{|w| \leq r} \pi \min \left\{r, \frac{\epsilon}{|w|}\right\}^2 dm_2(w)
\]

\[
= \pi^2 \epsilon^2 \left[1 + 2 \log \frac{r^2}{\epsilon}\right],
\]

provided \(r^2 \geq \epsilon\). If \(r^2 < \epsilon\), we obtain instead \((\pi r^2)^2\). (We used polar coordinates to compute the integral.) As \(r \to \infty\), the measure of \(E(P; r; \epsilon)\) \(\to \infty\), which is surprising when one thinks of one variable, for which the measure/content/cap is bounded independent of \(r\). If we consider the normalized polynomial

\[
P_1(z, w) := (zw/r^2)^n,
\]

which has

\[
\max_{|z|, |w| \leq r} |P_1(z, w)| = 1,
\]

then we see that

\[
E(P_1; r; \epsilon) := \left\{(z, w): |z|, |w| \leq r \quad \text{and} \quad |P_1(z, w)| \leq \epsilon^n \right\}
\]

\[
= \left\{(z, w): |z|, |w| \leq r \quad \text{and} \quad |zw| \leq (\epsilon r^2) \right\}
\]

so we can apply (1.15) if we replace \(\epsilon\) there by \(\epsilon r^2\). Thus if \(\epsilon \leq 1\),

\[
m_4(E(P_1; r; \epsilon)) = (\pi r^2 \epsilon)^2 \left[1 + 2 \log \frac{1}{\epsilon}\right].
\]

(If \(\epsilon > 1\), it is instead \((\pi r^2)^2\).) This simple example shows that our next result has estimates of the correct order in \(r\) and \(\epsilon\) for 2 dimensions, and for more general
k dimensions, one can perform analogous calculations with \( P(z_1, z_2, \ldots, z_k) := (z_1 z_2 \ldots z_k / r^k)^n \).

Our two main theorems treat polynomials \( P(z_1, z_2, \ldots, z_k) \) that are of degree \( \leq n \) in each variable \( z_j \) (so that no higher power than \( z_j^n \) appears in \( P \)), \( 1 \leq j \leq k \), normalized by

\[
\text{(1.20)} \quad \max \left\{ |P(z_1, z_2, \ldots, z_k)| : |z_j| \leq r, \ 1 \leq j \leq k \right\} = 1.
\]

We denote its lemniscate by

\[
\text{(1.21)} \quad E(P; r) := \left\{ (z_1, z_2, \ldots, z_k) : |z_j| \leq r, \ 1 \leq j \leq k \right\}.
\]

Let \( m_{2k} \) denote Lebesgue measure on \( \mathbb{C}^k \) and let \( \log_2 \) denote the log to the base 2.

**Theorem 1.2.** For polynomials \( P \) that are of degree \( \leq n \) in each of their \( k \) variables \( z_1, z_2, \ldots, z_k \), normalized by (1.20), and for \( \epsilon > 0 \), we have

\[
\text{(1.22)} \quad m_{2k}(E(P; r; \epsilon)) \leq (16\pi r^2)^k \epsilon^2 \max \left\{ 1, \ \log_2 \frac{2k-1}{\epsilon} \right\}^{k-1}.
\]

We note that the estimate (1.22) remains valid if we replace \( = 1 \) in (1.20) by \( \geq 1 \). There is a well-developed theory of capacities in \( \mathbb{C}^n \) [3], [6], [17], [18], [20], but for our purposes these are difficult to estimate, especially as there is no longer such a simple relationship between potentials and logs of polynomials. We prefer to use product capacity and Favarov’s capacity (a close cousin of Ronkin’s \( \gamma \)-capacity), as discussed by Cegrell [6, p.86, p.81].

For compact \( E \subset \mathbb{C}^2 \), we define its product capacity \( \text{cap}^{(2)}(E) \) by

\[
\text{(1.23)} \quad \text{cap}^{(2)}(E) := \int_0^\infty \text{cap}\left\{ z_1 : \text{cap}\left\{ z_2 : (z_1, z_2) \in E \right\} > s \right\} ds.
\]

More generally, for \( E \subset \mathbb{C}^k \), we define \( \text{cap}^{(k)}(E) \) inductively by

\[
\text{(1.24)} \quad \text{cap}^{(k)}(E) := \int_0^\infty \text{cap}\left\{ z_1 : \text{cap}^{(k-1)}\left\{ (z_2, \ldots, z_k) : (z_1, z_2, \ldots, z_k) \in E \right\} > s \right\} ds.
\]

This apparently strange definition really does yield a product capacity: If \( E = E_1 \times E_2 \times \cdots \times E_k \),

where each \( E_j \subset \mathbb{C} \), then

\[
\text{cap}^{(k)}(E) = \prod_{j=1}^k \text{cap} E_j.
\]

Recall that a unitary transformation \( A \) is a \( k \times k \) matrix with complex entries such that \( A^* A = I \). Favarov’s capacity \( \Gamma_k^f(E) \) of \( E \subset \mathbb{C}^k \) is defined by [6, p. 93]

\[
\text{(1.25)} \quad \Gamma_k^f(E) = \sup \{ \text{cap}^{(k)}(A(E)) : A \ a \ \text{unitary transformation} \}.
\]

We say that a polynomial \( P(z_1, z_2, \ldots, z_k) \) is of total degree \( \leq n \), if each term \( cz_1^{j_1}z_2^{j_2} \cdots z_k^{j_k} \) in its Maclaurin series has \( j_1 + j_2 + \cdots + j_k \leq n \).
Theorem 1.3. For polynomials $P$ that are of degree $\leq n$ in each of their $k$ variables $z_1, z_2, \ldots, z_k$, normalized by (1.20), and for $\epsilon > 0$, we have

\begin{equation}
\text{cap}^{(k)}(E(P; r; \epsilon)) \leq C_1 r^k \epsilon \max \left\{ 1, \log_2 \frac{1}{\epsilon} \right\}^{k-1}
\end{equation}

and

\begin{equation}
\Gamma^F_k(E(P; r; \epsilon)) \leq C_1 r^k \epsilon^{1/k} \max \left\{ 1, \log_2 \frac{1}{\epsilon} \right\}^{k-1}.
\end{equation}

Here $C_1$ is independent of $r, P, \epsilon, n$. If in addition $P$ is of total degree $\leq n$, then

\begin{equation}
\Gamma^F_k(E(P; r; \epsilon)) \leq C_1 r^k \epsilon \max \left\{ 1, \log_2 \frac{1}{\epsilon} \right\}^{k-1}.
\end{equation}

The estimate (1.26) is sharp with respect to order in $\epsilon$ and $r$. For simplicity, consider $k = 2$ and $P_1$ of (1.16), and recall (1.17), (1.18). Now for fixed $z$,

\[
\text{cap}\left\{ w: |w| \leq r \text{ and } |w| \leq \epsilon r^2/|z| \right\} = r \min\{1, \epsilon r/|z|\},
\]

and hence, if $\epsilon \leq 1$,

\[
\text{cap}^{(2)}(E(P_1; r; \epsilon)) = \int_0^\infty \text{cap}\left\{ z: |z| \leq r \text{ and } r \min\{1, \epsilon r/|z|\} > s \right\} ds
\]

\[= r \int_0^r \min\{1, \epsilon r/s\} ds = r^2 \epsilon \left[ 1 + \log \frac{1}{\epsilon} \right].
\]

We prove Theorem 1.1 in Section 2, and Theorems 1.2 and 1.3 in Section 3.

§2. Proof of Theorem 1.1

We recall that if $E$ is a compact set with $\text{cap} E > 0$ and connected complement, then its Green function with pole at $\infty$ is

\[g_E(z) := \log \frac{1}{\text{cap} E} + \int_E \log |z - t| d\mu(t),\]

where $\mu$ is the so-called equilibrium measure of $E$. This $\mu$ is a probability measure supported on the outer boundary $\partial E$ of $E$. If $E$ is a set regular with respect to the Dirichlet problem (as our lemniscates certainly are), then $g_E(z) = 0$, $z \in \partial E$, and $g_E$ is harmonic in $\mathbb{C}\setminus E$, with

\[g_E(z) - \log |z| \to \log \frac{1}{\text{cap} E}, |z| \to \infty.
\]

All this may be found in [9], [10], [12].
Proof of (1.8) – (1.10) of Theorem 1.1. Let $P(z)$ be a polynomial of degree $\leq n$, normalized by (1.5). Let $E := E(P; r; \epsilon)$. As the ball $\{z: |z| \leq r\}$ has cap $r$, we need prove (1.8) only for $\epsilon \leq \frac{r}{2}$. The well-known Walsh–Bernstein Lemma states that

$$
|P(z)| \leq \|P\|_{L_\infty(E)} \left( e^{g_E(z)} \right)^n, \quad z \in \mathbb{C}\setminus E.
$$

Using our normalization, we obtain

$$
1 = \|P\|_{L_\infty(|z| \leq r)} \leq \epsilon^n \exp \left( n \sup \{g_E(z): |z| \leq r, \quad z \notin E \} \right).
$$

But $\mu$ is a probability measure on $E \subset \{t: |t| \leq r\}$ so, for $|z| \leq r$, $z \notin E$,

$$
g_E(z) \leq \log \frac{1}{\text{cap } E} + \int_E \log(2r) d\mu(t) = \log \left( \frac{2r}{\text{cap } E} \right).
$$

Thus

$$
1 \leq \left( \frac{e2r}{\text{cap } E} \right)^n,
$$

from which (1.8) follows. The well-known inequalities [7, pp. 300–302]

$$
m_2(E) \leq \pi(\text{cap } E)^2;
$$
$$
h_1(L \cap E) \leq 4\text{cap } E
$$

then give (1.9) and (1.10). □

Proof of (1.11) – (1.13). Fix $0 < a < r$, and let

$$
P_1(z) := \left( \frac{z + a}{r + a} \right)^n.
$$

Then $P_1$ satisfies (1.5), and

$$
|P_1(z)| \leq \epsilon^n \iff |z + a| \leq \epsilon(r + a).
$$

We see that for

$$
0 < \epsilon \leq \frac{r - a}{r + a},
$$

the whole of the ball centre $-a$, radius $\epsilon(r + a)$, is contained in $\{z: |z| \leq r\}$. Thus for such $\epsilon$,

$$
E(P_1; r; \epsilon) = \left\{ z: |z + a| \leq \epsilon(r + a) \right\},
$$

so

$$
cap(E(P_1; r; \epsilon)) = \epsilon(r + a);
$$
$$
m_2(E(P_1; r; \epsilon)) = \pi(\epsilon(r + a))^2.
$$
Hence
\[
\sup_{P, \epsilon} \frac{\text{cap}(E(P; r))}{\epsilon} \geq r + a;
\]
\[
\sup_{P, \epsilon} \frac{m_2(E(P; r; \epsilon))}{\epsilon^2} \geq \pi (r + a)^2.
\]

Since we may make \(a\) arbitrarily close to \(r\), we obtain (1.11) – (1.12). The proof of (1.13) is a little more complicated: Let \(0 < a < r\), and \(T_n(x) = \cos(n \arccos x)\) denote the usual Chebyshev polynomial for \([-1, 1]\), and for small \(\delta > 0\) (actually \(\delta < r - a\) will do), let
\[
P_1(z) := T_n \left( \frac{z + a}{\delta} \right) / \left\| T_n \left( \frac{u + a}{\delta} \right) \right\|_{L_\infty \{ |u| \leq r \}}.
\]

Then \(P_1\) satisfies (1.5). Moreover, with
\[
\epsilon := \left\| T_n \left( \frac{u + a}{\delta} \right) \right\|_{L_\infty \{ |u| \leq r \}}^{-1/n},
\]
we see that
\[
E(P_1; r; \epsilon) = \left\{ z : |z| \leq r \text{ and } \left| T_n \left( \frac{z + a}{\delta} \right) \right| \leq 1 \right\} = [-a - \delta, -a + \delta],
\]
so
\[
h_1(E(P_1; r; \epsilon)) / \epsilon = 2\delta T_n \left( \frac{r + a}{\delta} \right)^{1/n}.
\]

Now \(T_n\) has leading coefficient \(2^{n-1}\), so behaves for large \(x\) like \(2^{n-1} x^n\). Then given \(\eta \in (0, 1)\), we have if \(\delta\) is small enough,
\[
h_1(E(P_1; r; \epsilon)) / \epsilon \geq 2\delta \eta 2^{-1/n} \left( \frac{r + a}{\delta} \right)^{1/n} = 4(r + a)2^{-1/n} \eta.
\]

Since \(a\) may be chosen arbitrarily close to \(r\), and \(\eta\) may be chosen arbitrarily close to 1, we obtain (1.13).

\[\square\]

Proof of (1.14) of Theorem 1.1. This follows from (1.8) and the estimate [12, p.203]
\[
h_n(E) \leq 18(2 \text{cap} E)^n.
\]

\[\square\]

§3. PROOF OF THEOREMS 1.2 AND 1.3

We begin with a lemma on the maximum of a polynomial along a slice:

Lemma 3.1. Let \(P(z_1, z_2, \ldots, z_k)\) be a polynomial of degree \(\leq n\) in each variable that satisfies (1.20). For fixed \(z_1\), let
\[
M(z_1) := \max \left\{ |P(z_1, z_2, \ldots, z_k)| : |z_j| \leq r, \ 2 \leq j \leq k \right\},
\]
\[\hfill (3.1)\]
and let
\[ E := \left\{ z_1 : |z_1| \leq r \text{ and } M(z_1) \leq \epsilon^n \right\}. \]

Then
\[ \text{cap}(E) \leq 2r\epsilon; \quad m_2(E) \leq \pi(2r\epsilon)^2. \]

**Proof.** Choose \( z_j, 2 \leq j \leq k \), such that each \( |z_j| \leq r \) and
\[ \max\left\{ \left| P(u, z_1, z_2, \ldots, z_k) \right| : |u| \leq r \right\} = 1. \]

This is possible by our normalization (1.20). With these variables chosen, \( Q(z_1) := P(z_1, z_2, \ldots, z_k) \) is a polynomial of degree \( \leq n \) in \( z_1 \) with
\[ \left| Q(z_1) \right| := \left| P(z_1, z_2, \ldots, z_k) \right| \leq M(z_1) \leq \epsilon^n, \quad z_1 \in E, \]
and
\[ \|Q\|_{L_\infty(|z_1| \leq r)} = 1. \]

Then
\[ E \subset E(Q; r; \epsilon), \]
so
\[ \text{cap}(E) \leq \text{cap}(E(Q; r; \epsilon)) \leq 2r\epsilon, \]
by Theorem 1.1. Then (2.2) gives the estimate for \( m_2(E) \).

**Proof of Theorem 1.2.** We do this by induction on \( k \). We can assume that \( \epsilon < 1 \), since if \( \epsilon \geq 1 \), then \( E(P; r; \epsilon) \) is all of the polydisc \( P := \left\{ |z_j| \leq r, 1 \leq j \leq k \right\} \), so has \( m_2k \) measure \((\pi r^2)^k\) and (1.22) is immediate.

(1.22) is true for \( k = 1 \). For \( k = 1 \), the result follows from Theorem 1.1.

**Assume (1.22) is true for \( k - 1 \), and prove true for \( k \).** Let us write
\[ z' = (z_2, z_3, \ldots, z_k); \quad z := (z_1, z') = (z_1, z_2, \ldots, z_k). \]

We let \( P \) be as above and we let \( P' \) denote the polydisc \( \zeta' : |z_j| \leq r, 2 \leq j \leq k \).

For \( z_1 \) fixed, let \( \hat{M}(z_1) \) denote the maximum modulus of \( P(\hat{z}) \) along a slice, as in (3.1). Note that for fixed \( z_1 \),
\[ Q(z') := P(\hat{z})/\hat{M}(z_1) \]
has
\[ \max\left\{ \left| Q(z') \right| : \hat{z} \in P' \right\} = 1. \]

By our induction step (recall \( z_1 \) is fixed),
\[ m_{2(k-1)} \left\{ z' \in P' : |P(\hat{z})| \leq \epsilon^n \right\} \]
\[ = m_{2(k-1)} \left\{ z' \in P' : |Q(z')| \leq \epsilon^n/M(z_1) \right\} \]
\[ \leq (16\pi r^2)^{k-1} \frac{\epsilon^2}{M(z_1)^2/n} \max\left\{ 1, \log_2 \frac{2^{k-2}M(z_1)^{1/n}}{\epsilon} \right\}^{k-2}. \]
Let us set
\[ E^{-1} := \left\{ z_1 : |z_1| \leq r \text{ and } M(z_1) \leq \epsilon^n \right\}; \]
\[ E_j := \left\{ z_1 : |z_1| \leq r \text{ and } (2^j \epsilon)^n < M(z_1) \leq (2^{j+1} \epsilon)^n \right\}, \ j \geq 0. \]

Since \( M(z_1) \leq 1 \), \( E_j \) is empty if
\[ 2^j \epsilon \geq 1 \iff j \geq \log_2 \frac{1}{\epsilon}. \]

By Lemma 3.1,
\[ m_2(E^{-1}) \leq \pi(2\epsilon)^2; \]
\[ m_2(E_j) \leq \pi(2\epsilon^{2^{j+1}})^2. \]

Then by (3.4), if \( \ell = \text{greatest integer} \leq \log_2 \frac{1}{\epsilon} - 1 \),
\[
m_{2k}(E(P; r; \epsilon)) = \int_{|z_1| \leq r} m_{2(k-1)}(\left\{ z' \in P' : |P(z)| \leq \epsilon^n \right\}) dm_2(z_1) \\
\leq \int_{|z_1| \leq r} \min \left\{ \left(\pi r^2\right)^{k-1}, \left(16\pi r^2\right)^{k-1} \frac{\epsilon^2}{M(z_1)^{2/n}} \right\} dm_2(z_1) \\
\times \max \left\{ 1, \log_2 \frac{2k-2M(z_1)^{1/n}}{\epsilon} \right\}^{k-2} dm_2(z_1) \\
\leq \left(\pi r^2\right)^{k-1} \int_{E^{-1}} dm_2(z_1) \\
+ \sum_{j=0}^{\ell} \int_{E_j} \frac{16^{k-1} \epsilon^2}{(2\epsilon)^2} \left(\log_2 \left[2^{k-2} \epsilon^{2^{j+1}}\right]\right)^{k-2} dm_2(z_1) \\
\leq \left(\pi r^2\right)^k \left[ 4\epsilon^2 + 16^{k-1}16\epsilon^2 \sum_{j=0}^{\ell} (\log_2 \left[2^{k-2} \epsilon^{2}\right])^{k-2} \right] \\
\leq (16\pi r^2)^k \epsilon^2 \left[ 1 + (\log_2 \left[2^{k-2} \epsilon^{2}\right])^{k-1} \right],
\]
where we have used our choice of \( \ell \), and also \( \epsilon \leq 1 \). Finally,
\[
\left[ 1 + (\log_2 \left[2^{k-2} \epsilon^{2}\right])^{k-1} \right] \leq \left[ 1 + \log_2 \left[2^{k-2} \epsilon^{2}\right]^{k-1} \right] = \left[ \log_2 \left[2^{k-1} \epsilon^{2}\right]^{k-1} \right].
\]

So we have completed the proof for \( k \). \( \square \)

Proof of (1.26) of Theorem 1.3. We keep the notation \( z, z', P, P' \) from the previous proof. We can assume \( \epsilon \leq 1 \), for if \( \epsilon > 1 \), then \( E(P; r; \epsilon) = P \), and as \( \text{cap}^{(k)}(P) = r^k \) (this is easily proved by induction on \( k \)), (1.26) is immediate. So we assume \( \epsilon < 1 \), and proceed by induction on \( k \):
(1.26) is true for $k = 1$. This follows directly from Theorem 1.1, with $C_1 = 2$.

Assume (1.26) true for $k - 1$, some $k \geq 2$. Let $P(z_1, z_2, \ldots, z_k)$ be of degree $\leq n$ in each variable, normalized by (1.20). Let $M(z_1)$ be the maximum modulus along a slice, as in (3.1). By definition,

$$\text{cap}^{(k)}(E(P; r; \epsilon)) = \int_0^\infty \text{cap}\{z_1: |z_1| \leq r\} \text{ and } \text{cap}^{(k-1)}\{z': z \in E(P; r; \epsilon)\} > s\} ds.$$

By our induction hypothesis, namely (1.26) for $k - 1$,

$$\text{cap}^{(k-1)}\{z': z \in E(P; r; \epsilon)\} = \text{cap}^{(k-1)}\{z': z \in P': |P(z)|/M(z_1) \leq \epsilon^n/M(z_1)\} \leq C_1 r^{k-1} \epsilon \frac{1}{M(z_1)^{1/n}} \max \left\{1, \log_2 \frac{M(z_1)^{1/n}}{\epsilon}\right\}^{k-2}.$$

Moreover, this set is contained in $P'$, so has $\text{cap}^{(k-1)} \leq r^{k-1}$. Thus

$$\text{cap}^{(k-1)}\{z': z \in E(P; r; \epsilon)\} \leq r^{k-1} F(\epsilon/M(z_1)^{1/n}),$$

where

$$F(u) := \min \left\{1, C_1 \max \left\{1, \log_2 \frac{1}{u}\right\}^{k-2}\right\}.$$

So,

$$\text{cap}^{(k)}(E(P; r; \epsilon)) \leq \int_0^{r^{k-1}} \text{cap}\{z_1: |z_1| \leq r\} \text{ and } r^{k-1} F(\epsilon/M(z_1)^{1/n}) > s\} ds \leq \int_0^{r^{k-1}} \text{cap}\{z_1: |z_1| \leq r\} \text{ and } F(\epsilon/M(z_1)^{1/n}) > t\} dt.$$

(3.5)

We see that there exists $C_2 > 0$ such that for $t \in (0, 1]$,

$$F(u) > t \Rightarrow u > C_2 t \max \left\{1, \log_2 \frac{1}{t}\right\}^{-(k-2)}.$$

Hence

$$F(\epsilon/M(z_1)^{1/n}) > t \Rightarrow M(z_1) < \left(\epsilon \max \left\{1, \log_2 \frac{1}{t}\right\}^{k-2}\right)^n / C_2 t.$$
By Lemma 3.1, the set of $|z_1| \leq r$ with $M(z_1)$ satisfying this inequality has cap at most
\[ 2r \epsilon \max \left\{ 1, \log_2 \frac{1}{\epsilon} \right\}^{k-2} \frac{1}{C_2}, \]
and also has cap $\leq r$. So (3.5) gives
\[
cap^{(k)} \left( E(P; r; \epsilon) \right) \leq r^k \int_0^1 \min \left\{ 1, \frac{2 \epsilon \max \left\{ 1, \log_2 \frac{1}{\epsilon} \right\}^{k-2}}{C_2} \right\} \frac{1}{t} dt \leq r^k \epsilon \max \left\{ 1, \log_2 \frac{1}{\epsilon} \right\}^{k-1},
\]
where $C_3$ depends only on $k$. \hfill $\square$

**Proof of (1.27) and (1.28) of Theorem 1.3.** We let $\bar{z} = (z_1, z_2, \ldots, z_k)$ and $\|\bar{z}\| := \left\{ \sum_{j=1}^k |z_j|^2 \right\}^{1/2}$. We shall use the following properties of a unitary matrix $A$: The inverse $A^{-1}$ is also unitary, and [19, p.74]
\[ \|A\bar{z}\| = \|\bar{z}\|. \]
Now if $P(\bar{z})$ is of degree $\leq n$ in each variable, and $Q(\bar{z}) := P(A^{-1}\bar{z})$, then $Q(\bar{z})$ is of degree $\leq kn$ in each variable. If in addition $P$ is of total degree $\leq n$, then we see that $Q(\bar{z})$ is of degree $\leq n$ in each variable. Moreover, setting $\bar{w} = A\bar{z}$, we see that
\[
A(E(P; r; \epsilon)) = \left\{ A\bar{z}: \text{each } |z_j| \leq r \text{ and } |P(\bar{z})| \leq \epsilon^n \right\} = \left\{ \bar{w}: \text{each } |(A^{-1}\bar{w})_j| \leq r \text{ and } |Q(\bar{w})| \leq \epsilon^n \right\}.
\]
Here, of course, $(A^{-1}\bar{w})_j$ denotes the $j$th component of the $k$-vector $A^{-1}\bar{w}$. Then $\forall j$
\[ |w_j| \leq \|\bar{w}\| = \|A^{-1}\bar{w}\| \leq \sqrt{k} \max_j |(A^{-1}\bar{w})_j| \leq \sqrt{kr}. \]
Thus, regarding $Q$ as a polynomial of degree $\leq kn$ in each variable,
\[
A(E(P; r; \epsilon)) \subseteq E(Q; \sqrt{kr}; \epsilon^{1/k}).
\]
(If $P$ is of total degree $\leq n$, we can regard $Q$ as a polynomial of degree $\leq n$ in each variable, and replace $\epsilon^{1/k}$ by $\epsilon$.) Next, if $\bar{w} = A\bar{z}$, and each $|z_j| \leq r$, we have shown each $|w_j| \leq \sqrt{kr}$, so
\[
\max \left\{ |Q(\bar{w})|: \text{each } |w_j| \leq \sqrt{kr} \right\} \geq \max \left\{ |P(\bar{z})|: \text{each } |z_j| \leq r \right\} = 1.
\]
Thus our (1.26) applied to $Q$ gives
\[
cap^{(k)} \left[ A(E(P;r;\epsilon)) \right] \leq \cap^{(k)} \left[ E(Q;\sqrt{kr};\epsilon^{1/k}) \right] \\
\leq C_1 \sqrt{kr}^{k} \epsilon^{1/k} \max \left\{ 1, \frac{1}{k} \log_2 \frac{1}{\epsilon} \right\}^{k-1}.
\]
So we have (1.27). When $P$ has total degree $\leq n$, we can replace $\epsilon^{1/k}$ by $\epsilon$ and hence obtain (1.28). □

**Note added in proof**

After this paper was accepted, Prof. Tom Bloom of the University of Toronto provided the authors with related references for the classical capacities in $\mathbb{C}^k$:


**References**


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