\( \delta_2^1 \) WITHOUT SHARPS

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Abstract. We show that the supremum of the lengths of \( \Delta^1_2 \) prewellorderings of the reals can be \( \omega_2 \), with \( \omega_1 \) inaccessible to reals, assuming only the consistency of an inaccessible.

\( \delta^1_\sim_2 \) denotes the supremum of the lengths of \( \Delta^1_2 \) prewellorderings of the reals. A result of Kunen and Martin (see Martin [77]) states that \( \delta^1_\sim_2 \) is at most \( \omega_2 \), and it is known that in the presence of sharps the assumption \( \delta^1_\sim_2 = \omega_2 \) is strong: it implies the consistency of a strong cardinal (see Steel-Welch [?]).

In this paper we show how to obtain the consistency of \( \delta^1_\sim_2 = \omega_2 \) in the absence of sharps, without strong assumptions.

Theorem. Assume the consistency of an inaccessible. Then it is consistent that \( \delta^1_\sim_2 = \omega_2 \) and \( \omega_1 \) is inaccessible to reals (i.e., \( \omega_1^{L[x]} \) is countable for each real \( x \)).

The proof is obtained by combining the \( \Delta^1_1 \)-coding technique of Friedman-Velickovic [96] with the use of a product of Jensen codings of Friedman [94].

We begin with a description of the \( \Delta^1_1 \)-coding technique.

Definitions. Suppose \( x \) is a set, \( \langle x, \epsilon \rangle \) satisfies the axiom of extensionality and \( A \subseteq \text{ORD} \). \( x \) preserves \( A \) if \( \langle x, A \cap x \rangle \cong \langle x, A \cap \bar{x} \rangle \) where \( \bar{x} = \text{transitive collapse of } x \). For any ordinal \( \delta \), \( x[\delta] = \{ f(\gamma) | \gamma < \delta, f \in x, f \text{ a function}, \gamma \in \text{Dom}(f) \} \). \( x \) strongly preserves \( A \) if \( x[\delta] \) preserves \( A \) for every cardinal \( \delta \). A sequence \( x_0 \subseteq x_1 \subseteq \ldots \) is tight if it is continuous and for each \( i \), \( \langle \bar{x}_j | j < i \rangle \) belongs to the least \( ZF^- \)-model which contains \( x_i \) as an element and correctly computes \( \text{card}(\bar{x}_i) \).

Condensation Condition for \( A \). Suppose \( t \) is transitive, \( \delta \) is regular, \( \delta \in t \) and \( x \in t \). Then:

(a) There exists a continuous, tight \( \delta \)-sequence \( x_0 \prec x_1 \prec \cdots \prec t \) such that \( \text{card}(x_i) = \delta, x \in x_0 \) and \( x_1 \) strongly preserves \( A \), for each \( i \).

(b) If \( \delta \) is inaccessible, then there exist \( x_i \)'s as above but where \( \text{card}(x_i) = \aleph_i \).

The following is proved in Friedman-Velickovic [96].
\(\Delta_1\)-Coding. Suppose \(V = L\) and the Condensation Condition holds for \(A\). Then \(A\) is \(\Delta_1\) in a class-generic real \(R\), preserving cardinals.

Now we are ready to begin the proof of the Theorem. Suppose \(\kappa\) is the least inaccessible and \(V = L\). Let \(\langle \alpha_i | i < \kappa^+ \rangle\) be the increasing list of all \(\alpha \in (\kappa, \kappa^+)\) such that \(L_\alpha = \text{Skolem hull}(\kappa)\) in \(L_\alpha\). For each \(i < \kappa^+\) define \(f_i : \kappa \rightarrow \kappa\) by \(f_i(\gamma) = \text{ordertype}(\text{ORD} \cap \text{Skolem hull}(\gamma)\text{ in }L_{\alpha_i})\). By identifying \(f_i\) with its graph and using a pairing function we can think of \(f_i\) as a subset of \(\kappa\). The following is straightforward.

**Lemma 1.** Each \(f_i\) obeys the Condensation Condition. Indeed \(\langle f_i | i < \kappa^+ \rangle\) jointly obeys the Condensation Condition in the following sense: Suppose \(t\) is transitive, \(\delta\) is regular, \(\delta \in t\), \(x \in t\). Then there exists a tight \(\delta\)-sequence \(x_0 < x_1 < \cdots < t\) such that \(\text{card}(x_i) = \delta\), \(x \in x_0\) and each \(x_i\) strongly preserves all \(f_j\) for \(j < i\) (and if \(\delta = \kappa\), then we can alternatively require \(\text{card}(x_i) = \aleph_i\)).

Now, following Friedman [94] we use a “diagonally-supported” product of Jensen-style codings. For each \(i < \kappa^+\) let \(\mathcal{P}(i)\) be the forcing from Friedman-Velickovic [96] to make \(f_i\) \(\Delta_1\)-definable in a class-generic real. Then \(\mathcal{P}\) consists of all \(p \in \prod_{i < \kappa^+} \mathcal{P}(i)\) such that for infinite cardinals \(\gamma\), \(\{i | p(i)(\gamma) \neq (\phi, \phi)\}\) has cardinality at most \(\gamma\) and in addition \(\{i | p(i)(0) \neq (\phi, \phi)\}\) is finite.

Now note that for successor cardinals \(\gamma < \kappa\) the forcing \(\mathcal{P}\) factors as \(\mathcal{P}_\gamma \ast \mathcal{P}_{\gamma}^{G_{\gamma}}\) where \(\mathcal{P}_\gamma\) forces that \(\mathcal{P}_{\gamma}^{G_{\gamma}}\) has the \(\gamma^+-\text{CC}\). Also the joint Condensation Condition of Lemma 1 implies that the argument of Theorem 3 of Friedman-Velickovic [96] can be applied here to show that \(\mathcal{P}_\gamma\) is \(\leq \gamma\)-distributive, and also that \(\mathcal{P}\) is \(\Delta\)-distributive (if \(\langle D_i | i < \kappa \rangle\) is a sequence of predense sets, then it is dense to reduce each \(D_i\) below \(\aleph_{i+1}\)). So \(\mathcal{P}\) preserves cofinalities.

Thus in a cardinal-preserving forcing extension of \(L\) we have produced \(\kappa^+\) reals \(\langle R_i | i < \kappa^+ \rangle\) where \(R_i\) \(\Delta_1\)-codes \(f_i\) and hence there are wellorderings of \(\kappa\) of any length \(< \kappa^+\) which are \(\Delta_1\) in a real. Finally Lévy collapse to make \(\kappa = \omega_1\) and we have \(\delta_2 = \omega_2\), \(\omega_1\) inaccessible to reals.

The above proof also shows the following, which may be of independent interest.

**Theorem 2.** Let \(\delta_1(\kappa)\) be the sup of the lengths of wellorderings of \(\kappa\) which are \(\Delta_1\) over \(L_\kappa[x]\) for some \(x\), a bounded subset of \(\kappa\). Then (relative to the consistency of an inaccessible) it is consistent that \(\kappa\) be weakly inaccessible and \(\delta_1(\kappa) = \kappa^+\).

**Remark.** The conclusion of Theorem 2 cannot hold in the context of sharps: if \(\kappa\) is weakly inaccessible and every bounded subset of \(\kappa\) has a sharp, then \(\delta_1(\kappa) < \kappa^+\). This is because \(\delta_1(\kappa)\) is then the second uniform indiscernible for bounded subsets of \(\kappa\), which can be written as the direct limit of the second uniform indiscernible for subsets of \(\delta_1\), as \(\delta_1\) ranges over cardinals less than \(\kappa\); so \(\delta_1(\kappa)\) has cardinality \(\kappa\).

Using the least inner model closed under sharp, we can also obtain the following.

**Theorem 3.** Assuming it is consistent for every set to have a sharp, then this is also consistent with \(\delta_3(\kappa) = \omega_2\).
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