

## PRODUCTS OF QUASI-MEASURES

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ABSTRACT. A quasi-state is a positive functional on  $C(X)$  that is only assumed to be linear on singly-generated subalgebras. We consider the “iterated integral” of two quasi-states and determine when this gives a quasi-state on the product space. We also provide explicit formulas for the corresponding quasi-measures in case it does. Finally, we show the general failure of Fubini’s Theorem for quasi-states.

If  $X$  is a compact, Hausdorff space, we let  $C(X)$  denote the collection of real-valued continuous functions on  $X$ . We let  $\text{sp } f$  denote the range of  $f$ . A quasi-state is a function  $\rho : C(X) \rightarrow \mathbb{R}$  such that:

- (i) If  $f \geq 0$ , then  $\rho(f) \geq 0$ .
- (ii)  $\rho(1) = 1$ .
- (iii) If  $r \in \mathbb{R}$ , then  $\rho(rf) = r\rho(f)$ .
- (iv) If  $\varphi, \psi \in C(\text{sp } f)$ , then  $\rho(\varphi \circ f + \psi \circ f) = \rho(\varphi \circ f) + \rho(\psi \circ f)$ .

In [1], Aarnes answered the question of whether every quasi-state must be linear in the negative. He did this by establishing a correspondence between quasi-states and certain set functions. In particular, a function  $\mu$  defined for the subsets of  $X$  that are either open or closed is a quasi-measure if:

- a)  $\mu(A) \geq 0$  for all  $A$ .
- b)  $A \subseteq B$  implies that  $\mu(A) \leq \mu(B)$ .
- c)  $A \cap B = \emptyset$  implies  $\mu(A \cup B) = \mu(A) + \mu(B)$ .
- d) If  $U$  is open,  $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ closed}\}$ .
- e)  $\mu(X) = 1$ .

The primary difference between a quasi-measure and a finitely additive measure is that quasi-measures do not have to be subadditive. The quasi-state  $\rho$  corresponds to the quasi-measure  $\mu$  if

$$\mu(U) = \sup\{\rho(f) : 0 \leq f \prec U\} \quad \text{for } U \text{ open}$$

and

$$\mu(K) = \inf\{\rho(f) : K \prec f\} \quad \text{for } K \text{ closed.}$$

Here,  $f \prec U$  means that  $0 \leq f \leq 1$  and the support of  $f$  is contained in  $U$ . Also,  $K \prec f$  means that  $f \geq 0$  and  $f \geq 1$  on  $K$ . This construction is detailed in [1], where a particular example of a quasi-measure that is not a measure is given. Other examples may be found in [2, 3] and [5].

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There are some basic properties of quasi-states that we will need for this paper. In particular, we will require the fact that if  $f$  and  $g$  are such that  $fg = 0$ , then  $\rho(f + g) = \rho(f) + \rho(g)$ . We also use the fact that  $\rho$  is continuous on  $C(X)$ . These facts may be found in [1]. For notational convenience, we will write  $\langle f, \rho \rangle = \rho(f)$ .

We will call a quasi-state simple if  $\langle \varphi \circ f, \rho \rangle = \varphi(\langle f, \rho \rangle)$  when  $\varphi \in C(\text{sp } f)$ . It is shown in [2] that the quasi-state  $\rho$  is simple if and only if the corresponding quasi-measure  $\mu$  satisfies  $\mu(A) \in \{0, 1\}$  for all  $A$ . We call such quasi-measures  $\{0, 1\}$ -quasi-measures.

From now on, we will let  $X$  and  $Y$  be compact, Hausdorff spaces,  $\rho$  a quasi-state on  $C(X)$  with corresponding quasi-measure  $\mu$  and  $\eta$  a quasi-state on  $C(Y)$  with corresponding quasi-measure  $\nu$ . We are interested in considering the functions on  $C(X \times Y)$  obtained from “repeated integration”. To do so, we need the following definitions.

**Definition 1.** For  $f \in C(X \times Y)$ , define  $f^y(x) = f(x, y) = f_x(y)$ . Then  $f^y \in C(X)$ ,  $f_x \in C(Y)$  and the functions  $y \mapsto f^y$  and  $x \mapsto f_x$  are continuous when  $C(X)$  and  $C(Y)$  are given the uniform topologies.

Define  $T_\rho(f)(y) = \langle f^y, \rho \rangle$  and  $S_\eta(f)(x) = \langle f_x, \eta \rangle$ . By continuity of  $\rho$  and  $\eta$ , we see that  $T_\rho(f) \in C(Y)$  and  $S_\eta(f) \in C(X)$ .

Finally, define  $\rho \times_l \eta$  and  $\rho \times_r \eta$  on  $C(X \times Y)$  by  $\langle f, \rho \times_l \eta \rangle = \langle T_\rho(f), \eta \rangle$  and  $\langle f, \rho \times_r \eta \rangle = \langle S_\eta(f), \rho \rangle$ .

Thus,  $\rho \times_l \eta$  is obtained by first “integrating” with the use of  $\rho$  and then with  $\eta$ , while  $\rho \times_r \eta$  integrates with  $\eta$  first and then  $\rho$ . There is a real difference between  $\rho \times_l \eta$  and  $\rho \times_r \eta$  as will be seen later. It is easily seen that  $\rho \times_r \eta$  and  $\rho \times_l \eta$  satisfy (i)–(iii) in the definition of quasi-state.

**Proposition 1.** *If  $g \in C(X)$  and  $h \in C(Y)$ , let  $g \otimes h(x, y) = g(x)h(y)$ . Then*

$$\langle g \otimes h, \rho \times_l \eta \rangle = \langle g \otimes h, \rho \times_r \eta \rangle = \langle g, \rho \rangle \langle h, \eta \rangle.$$

*Proof.* In fact,  $(g \otimes h)^y = h(y) \cdot g$ , so  $T_\rho(g \otimes h)(y) = h(y)\langle g, \rho \rangle$ . Thus,  $\langle g \otimes h, \rho \times_l \eta \rangle = \langle T_\rho(g \otimes h), \eta \rangle = \langle h\langle g, \rho \rangle, \eta \rangle = \langle g, \rho \rangle \langle h, \eta \rangle$ . Similarly for the other equality.  $\square$

One fact that is perhaps surprising is that the functions  $\rho \times_l \eta$  and  $\rho \times_r \eta$  do not always give quasi-states. The exact situations when they do are given in the following theorem.

**Theorem 1.** *The function  $\rho \times_l \eta$  is a quasi-state if and only if  $\eta$  is linear or  $\rho$  is simple.*

*Proof.* The only consideration is whether property (iv) in the definition of a quasi-state is satisfied. Let  $f \in C(X \times Y)$  and  $\varphi, \psi \in C(\text{sp } f)$ . Then

$$\begin{aligned} T_\rho(\varphi \circ f + \psi \circ f)(y) &= \langle (\varphi \circ f + \psi \circ f)^y, \rho \rangle \\ &= \langle \varphi \circ f^y + \psi \circ f^y, \rho \rangle \\ &= \langle \varphi \circ f^y, \rho \rangle + \langle \psi \circ f^y, \rho \rangle \\ &= T_\rho(\varphi \circ f)(y) + T_\rho(\psi \circ f)(y). \end{aligned}$$

So  $T_\rho(\varphi \circ f + \psi \circ f) = T_\rho(\varphi \circ f) + T_\rho(\psi \circ f)$ .

If  $\eta$  is linear, we have

$$\begin{aligned} \langle \varphi \circ f + \psi \circ f, \rho \times_I \eta \rangle &= \langle T_\rho(\varphi \circ f + \psi \circ f), \eta \rangle \\ &= \langle T_\rho(\varphi \circ f) + T_\rho(\psi \circ f), \eta \rangle \\ &= \langle T_\rho(\varphi \circ f), \eta \rangle + \langle T_\rho(\psi \circ f), \eta \rangle \\ &= \langle \varphi \circ f, \rho \times_I \eta \rangle + \langle \psi \circ f, \rho \times_I \eta \rangle, \end{aligned}$$

so  $\rho \times_I \eta$  is a quasi-state in this case.

If, instead,  $\rho$  is simple, we have that  $T_\rho(\varphi \circ f)(y) = \langle \varphi \circ f^y, \rho \rangle = \varphi(\langle f^y, \rho \rangle) = \varphi \circ T_\rho(f)(y)$ . So we have for this case,  $T_\rho(\varphi \circ f + \psi \circ f) = \varphi \circ T_\rho(f) + \psi \circ T_\rho(f)$ , which gives

$$\begin{aligned} \langle \varphi \circ f + \psi \circ f, \rho \times_I \eta \rangle &= \langle T_\rho(\varphi \circ f + \psi \circ f), \eta \rangle \\ &= \langle \varphi \circ T_\rho(f) + \psi \circ T_\rho(f), \eta \rangle \\ &= \langle \varphi \circ T_\rho(f), \eta \rangle + \langle \psi \circ T_\rho(f), \eta \rangle \\ &= \langle T_\rho(\varphi \circ f), \eta \rangle + \langle T_\rho(\psi \circ f), \eta \rangle \\ &= \langle \varphi \circ f, \rho \times_I \eta \rangle + \langle \psi \circ f, \rho \times_I \eta \rangle, \end{aligned}$$

so again,  $\rho \times_I \eta$  is a quasi-state.

Conversely, assume that  $\rho$  is not simple, but that  $\rho \times_I \eta$  is a quasi-state. We will show that  $\eta$  is linear.

Since  $\rho$  is not simple, the corresponding quasi-measure,  $\mu$ , is not a  $\{0, 1\}$ -quasi-measure. Let  $A \subseteq X$  be closed with  $0 < \mu(A) < 1$ . Use inner regularity of  $\mu$  on  $X \setminus A$  to find a closed set  $B$  with  $0 < \mu(B) < 1$  and  $B$  disjoint from  $A$ . Now pick two positive functions  $k_1$  and  $k_2$  such that  $k_1 k_2 = 0$ ,  $A \prec k_1$  and  $B \prec k_2$ . Then  $a = \langle k_1, \rho \rangle \neq 0$  and  $b = \langle k_2, \rho \rangle \neq 0$ . Define  $f_1 = k_1/a$  and  $f_2 = k_2/b$ . Notice that  $\langle f_1, \rho \rangle = \langle f_2, \rho \rangle = 1$ .

Now, for  $g, h \in C(Y)$ , we obtain

$$\begin{aligned} \langle g, \eta \rangle + \langle h, \eta \rangle &= \langle f_1, \rho \rangle \langle g, \eta \rangle + \langle f_2, \rho \rangle \langle h, \eta \rangle \\ &= \langle f_1 \otimes g, \rho \times_I \eta \rangle + \langle f_2 \otimes h, \rho \times_I \eta \rangle \\ &= \langle f_1 \otimes g + f_2 \otimes h, \rho \times_I \eta \rangle \\ &= \langle T_\rho(f_1 \otimes g + f_2 \otimes h), \eta \rangle, \end{aligned}$$

where we have used  $(f_1 \otimes g)(f_2 \otimes h) = 0$  and the fact that  $\rho \times_I \eta$  is a quasi-state. But we check that  $(f_1 \otimes g + f_2 \otimes h)^y = g(y)f_1 + h(y)f_2$ . Since  $f_1 f_2 = 0$ , we have

$$\begin{aligned} T_\rho(f_1 \otimes g + f_2 \otimes h)(y) &= \langle (f_1 \otimes g + f_2 \otimes h)^y, \rho \rangle \\ &= \langle g(y)f_1 + h(y)f_2, \rho \rangle \\ &= g(y)\langle f_1, \rho \rangle + h(y)\langle f_2, \rho \rangle \\ &= g(y) + h(y), \end{aligned}$$

so the previous calculation yields

$$\langle g + h, \eta \rangle = \langle g, \eta \rangle + \langle h, \eta \rangle.$$

This states the linearity of  $\eta$ . □

**Corollary 1.** *If  $\rho$  and  $\eta$  are both simple quasi-states, then  $\rho \times_I \eta$  is also a simple quasi-state.*

*Proof.* If  $f \in C(X \times Y)$  and  $\varphi \in C(\text{sp } f)$ , we have that  $T_\rho(\varphi \circ f) = \varphi \circ T_\rho(f)$ , so that  $\langle \varphi \circ f, \rho \times_I \eta \rangle = \langle \varphi \circ T_\rho(f), \eta \rangle = \varphi(\langle T_\rho(f), \eta \rangle) = \varphi(\langle f, \rho \times_I \eta \rangle)$ . □

Thus, if  $\mu$  is a quasi-measure on  $X$  and  $\nu$  is a quasi-measure on  $Y$ , we may define a product quasi-measure  $\mu \times_l \nu$  on  $X \times Y$  if either  $\nu$  is a measure or  $\mu$  is a  $\{0, 1\}$ -quasi-measure. If both  $\mu$  and  $\nu$  are  $\{0, 1\}$ -quasi-measures, so is  $\mu \times_l \nu$ . It should be pointed out that there are analogous results to those above for the function  $\rho \times_r \eta$ , which will give a quasi-measure  $\mu \times_r \nu$  if  $\nu$  is a  $\{0, 1\}$ -quasi-measure or if  $\mu$  is a measure.

It is of interest to see how the quasi-measure  $\mu \times_l \nu$  acts on sets when it is defined. This is the content of the next two results. The first is a generalization of a construction first considered in [4]. There, the relevant quasi-measure was obtained from a weak-\* limit procedure. However, no description of how this quasi-measure acts on sets was given.

**Theorem 2.** *Let  $\mu$  be a quasi-measure on  $X$  and  $\nu$  a measure on  $Y$ . Then for  $A$  either open or closed in  $X \times Y$ , we have*

$$\mu \times_l \nu(A) = \int_Y \mu(A^y) d\nu(y)$$

where  $A^y = \{x : (x, y) \in A\}$ .

*Proof.* *Claim 1.* If  $U \subseteq X \times Y$  is open, then the function  $y \rightarrow \mu(U^y)$  is lower semi-continuous and so is  $\nu$ -measurable.

Suppose that  $\mu(U^{y_0}) > \alpha$ . Pick  $K \subseteq U^{y_0}$  compact such that  $\mu(K) > \alpha$ . Then  $K \times \{y_0\} \subseteq U$ , so there is a neighborhood  $V$  of  $y_0$  such that  $K \times V \subseteq U$ . Then for  $y \in V$ , we have  $K \subseteq U^y$ , so  $\alpha < \mu(U^y)$ .

*Claim 2.* If  $U \subseteq X \times Y$  is open, then  $\mu \times_l \nu(U) \leq \int_Y \mu(U^y) d\nu(y)$ .

If  $f \prec U$ , then  $f^y \prec U^y$  for all  $y \in Y$ . Thus  $\rho(f^y) \leq \mu(U^y)$ , which gives  $\rho \times_l \eta(f) = \int_Y \rho(f^y) d\nu(y) \leq \int_Y \mu(U^y) d\nu(y)$ . Now use the fact that  $\mu \times_l \nu(U)$  is the supremum of such  $\rho \times_l \eta(f)$ .

*Claim 3.* If  $U \subseteq X \times Y$  is open, then  $\mu \times_l \nu(U) = \int_Y \mu(U^y) d\nu(y)$ .

Since the function  $y \rightarrow \mu(U^y)$  is lower semi-continuous, we have that

$$\int_Y \mu(U^y) d\nu(y) = \sup \left\{ \int_Y g d\nu : 0 \leq g(y) \leq \mu(U^y), g \in C(Y) \right\}.$$

Let  $g \in C(Y)$  such that  $0 \leq g(y) \leq \mu(U^y)$  for  $y \in Y$ . Let  $\varepsilon > 0$ . For each  $y \in Y$ , let  $K_y \subseteq U^y$  be compact with  $\mu(K_y) > g(y) - \varepsilon/2$ . Then  $K_y \times \{y\} \subseteq U$ , so there is a neighborhood  $V_y$  of  $y$  such that  $K \times \overline{V}_y \subseteq U$  and for  $z \in V_y$  we have  $|g(z) - g(y)| < \varepsilon/2$ . Choose  $V_{y_1}, \dots, V_{y_n}$  that cover  $Y$ . Define

$$E = \bigcup_{i=1}^n K_{y_i} \times \overline{V}_{y_i} \subseteq U.$$

Then  $E$  is compact. Choose  $f$  so that  $E \prec f \prec U$ . Then, if  $y \in Y$ , say  $y \in V_{y_i}$ , we have  $K_{y_i} \subseteq E^y \prec f^y$ , so

$$g(y) - \varepsilon < g(y_i) - \varepsilon/2 \leq \mu(K_{y_i}) \leq \rho(f^y).$$

Thus

$$\int_Y g d\nu - \varepsilon \leq \int_Y \rho(f^y) d\nu(y) = \rho \times_l \eta(f) \leq \mu \times_l \nu(U).$$

Now let  $\varepsilon \rightarrow 0$  to get  $\int_Y g d\nu \leq \mu \times_l \nu(U)$ . Since this happens with every  $g$  as above, we are done.

The case where  $A$  is closed in  $X \times Y$  follows by taking complements.  $\square$

The proof of the next result is very similar in conception to that of the previous theorem. The differences arise from the lack, as yet, of a suitable integration theory for lower semi-continuous functions with respect to a quasi-measure.

**Theorem 3.** *Let  $\mu$  be a  $\{0, 1\}$ -quasi-measure on  $X$  and  $\nu$  a quasi-measure on  $Y$ . Construct  $\mu \times_I \nu$  as above. Then for  $A$  either open or closed in  $X \times Y$ , we have*

$$\mu \times_I \nu(A) = \nu(\{y : \mu(A^y) = 1\}).$$

*Proof.* For  $A \subseteq X \times Y$  either open or closed, we define  $B(A) = \{y : \mu(A^y) = 1\}$ . We use the notation of  $A^c$  for the complement of  $A$ . Notice that  $B(A)^c = \{y : \mu(A^y) = 0\} = \{y : \mu((A^y)^c) = 1\} = \{y : \mu((A^c)^y) = 1\} = B(A^c)$ .

*Claim 1.* If  $A$  is open, then  $B(A)$  is open. If  $A$  is closed,  $B(A)$  is closed.

It is enough to show this for  $A$  open. Suppose this is so and assume that  $y_0 \in B(A)$ , i.e.  $\mu(A^{y_0}) = 1$ . Since  $A^{y_0}$  is open, and  $\mu$  is a  $\{0, 1\}$ -quasi-measure, there is a compact set  $K \subseteq A^{y_0}$  such that  $\mu(K) = 1$ . Then  $K \times \{y_0\} \subseteq A$ , so there are open sets  $K \subseteq U \subseteq X$  and  $y_0 \in V \subseteq Y$  such that  $U \times V \subseteq A$ . Now, if  $y$  is in the neighborhood  $V$  of  $y_0$ , we have  $K \subseteq U \subseteq A^y$ , so  $\mu(A^y) = 1$ , i.e.  $V \subseteq B(A)$ .

*Claim 2.* If  $A$  is closed, then  $\nu(B(A)) \leq \mu \times_I \nu(A)$ .

We have that  $\mu \times_I \nu(A) = \inf\{\langle f, \rho \times_I \eta \rangle : A \prec f\}$ . Suppose that  $A \prec f$ . Then  $A^y \prec f^y$  for all  $y \in Y$ , so  $\mu(A^y) \leq \langle f^y, \rho \rangle = T_\rho(f)(y)$ . Thus  $B(A) \prec T_\rho(f)$ . This shows that  $\nu(B(A)) \leq \langle T_\rho(f), \eta \rangle = \langle f, \rho \times_I \eta \rangle$ . This gives the claim.

*Claim 3.* If  $A$  is open, then  $\nu(B(A)) \geq \mu \times_I \nu(A)$ .

In fact,  $\nu(B(A)) = 1 - \nu(B(A)^c) = 1 - \nu(B(A^c)) \geq 1 - \mu \times_I \nu(A^c) = \mu \times_I \nu(A)$ .

*Claim 4.* If  $A$  is open,  $\nu(B(A)) = \mu \times_I \nu(A)$ .

We use the fact that  $\nu(B(A)) = \sup\{\nu(K) : K \subseteq B(A), K \text{ closed}\}$ . Suppose that  $K \subseteq B(A)$  is closed. For each  $y \in K$ ,  $\mu(A^y) = 1$ , so there is a compact set  $C_y \subseteq A^y$  such that  $\mu(C_y) = 1$ . Then  $C_y \times \{y\} \subseteq A$ , so there are open sets  $C_y \subseteq U_y \subseteq X$  and  $y \in W_y \subseteq Y$  such that  $U_y \times \overline{W_y} \subseteq A$ . Finitely many  $W_{y_1}, W_{y_2}, \dots, W_{y_n}$  cover  $K$ .

Let  $D = \bigcup_{i=1}^n C_{y_i} \times \overline{W_{y_i}}$ . Then  $D$  is compact and  $D \subseteq A$ . Choose  $f$  such that  $D \prec f \prec A$ . Then for  $y \in K$ , say  $y \in W_{y_i}$ , we have  $C_{y_i} \subseteq D^y \prec f^y$ , so  $1 = \mu(C_{y_i}) \leq \langle f^y, \rho \rangle = T_\rho(f)(y)$ . Thus  $K \prec T_\rho(f)$ . Finally, this shows that

$$\nu(K) \leq \langle T_\rho(f), \eta \rangle = \langle f, \rho \times_I \eta \rangle \leq \mu \times_I \nu(A).$$

If we take the supremum over  $K \subseteq B(A)$  and use the previous claim, we are finished.

*Claim 5.* For  $A$  closed,  $\nu(B(A)) = \mu \times_I \nu(A)$ .

This is now easy. □

**Corollary 2.** *Let  $\mu$  be a quasi-measure on  $X$  and  $\nu$  a quasi-measure on  $Y$  such that  $\mu \times_I \nu$  is a quasi-measure. If  $A \subseteq X$  and  $B \subseteq Y$  are either both open or both closed, then  $\mu \times_I \nu(A \times B) = \mu(A)\nu(B)$ .*

We now turn to the question of when a version of Fubini's Theorem holds for quasi-measures and quasi-states. In other words, in what circumstances does  $\rho \times_I \eta = \rho \times_r \eta$ ? If  $\rho \times_I \eta$  is not a quasi-state, in other words, if  $\rho$  is not simple and  $\eta$  is not linear, we choose  $f_1, f_2 \in C(X)$  such that  $f_1 f_2 = 0$  and  $1 = \langle f_1, \rho \rangle = \langle f_2, \rho \rangle$  as

in the proof of Theorem 1. Also pick  $g, h \in C(Y)$  such that  $\langle g+h, \eta \rangle \neq \langle g, \eta \rangle + \langle h, \eta \rangle$ . If we let  $k = f_1 \otimes g + f_2 \otimes h$ , we see that  $\langle k, \rho \times_l \eta \rangle = \langle g+h, \eta \rangle$ , while  $\langle k, \rho \times_r \eta \rangle = \langle g, \eta \rangle + \langle h, \eta \rangle$ . Thus  $\rho \times_l \eta \neq \rho \times_r \eta$  in this case.

If, however,  $\rho \times_l \eta$  is a quasi-state, we would need for  $\rho \times_r \eta$  to be one also. An enumeration of cases shows that this situation occurs only if one of  $\mu$  or  $\nu$  is a point-mass measure, when both of  $\rho$  and  $\eta$  are simple, or when both  $\mu$  and  $\nu$  are measures. In the last case,  $\rho \times_l \eta = \rho \times_r \eta$  by Fubini's Theorem. If either  $\mu$  or  $\nu$  is a point-mass, an easy calculation shows that  $\rho \times_l \eta = \rho \times_r \eta$ . In the final case when  $\rho$  and  $\eta$  are both simple, but non-linear, we have the following.

**Corollary 3.** *If  $\mu$  and  $\nu$  are both  $\{0, 1\}$ -quasi-measures that are not measures, then  $\mu \times_l \nu \neq \mu \times_r \nu$ .*

*Proof.* Since neither  $\mu$  nor  $\nu$  are measures, they must violate subadditivity. Thus we may find  $A, C \subseteq X$  and  $B, D \subseteq Y$  open such that  $\mu(A) = \mu(C) = \nu(B \cap D) = 0$  and  $\mu(A \cup C) = \nu(B) = \nu(D) = 1$ . If we set  $E = (A \times B) \cup (C \times D)$ , we see that  $\mu \times_l \nu(E) = 0$ , but  $\mu \times_r \nu(E) = 1$ .  $\square$

An interesting consequence of this is that both  $\mu \times_l \nu$  and  $\mu \times_r \nu$  are  $\{0, 1\}$ -quasi-measures on  $X \times Y$  that agree on rectangles, but are distinct. This occurs even if  $X = Y$  and  $\mu = \nu$ . If we consider quasi-measures of the form  $\alpha \cdot \mu \times_l \nu + (1-\alpha) \cdot \mu \times_r \nu$ , we get an uncountable family of quasi-measures that agree on rectangles, but are distinct. In contrast, a measure on the product space is determined by its action on rectangles.

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