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TAYLOR SPECTRAL INVARIANCE FOR CRISSCROSS COMMUTING PAIRS ON BANACH SPACES

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ABSTRACT. A theorem on the commuting property of Taylor's spectrum for crisscross commuting pairs is proved in this paper.

It is well known that for two operators A and B on a Banach space X equality $\sigma(BA)\backslash\{0\} = \sigma(AB)\backslash\{0\}$ holds; that is, $I - BA$ is invertible if and only if $I - AB$ is invertible and

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A.$$

In the paper [3], while Crimus and Ecker study the set of matrices which can be simultaneously diagonalized by two unitary matrices, the property of crisscross commuting is introduced and it guarantees that the tuples of multiplication operators are commuting, i.e., if A and B are crisscross commuting pairs, then AB and BA are commuting tuples of operators. This may have some applications to mathematical physics.

In the paper [2] we proved that for crisscross commuting pairs $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ it holds that

$$\operatorname{Sp}(AB) \cup \{(0, \dots, 0)\} = \operatorname{Sp}(BA) \cup \{(0, \dots, 0)\}, \tag{1}$$

where $AB = (A_1B_1, \dots, A_nB_n)$ and $\operatorname{Sp}(\cdot)$ denotes Taylor's spectrum. In this paper we will discuss the problem for Taylor's essential spectrum.

First, we introduce some notation.

Let $\bigwedge_n(e)$ be the exterior algebra generated by $e_1, \dots, e_n$ and $O_n = \{I = \{i_1 < i_2 < \cdots < i_k\} \colon I \subset \{1, \dots, n\}\}$. For $I \in O_n$ we denote $e_I = e_{i1} \cdots e_{ik}$. Note that when $I = \Phi$, $e_I = e_\Phi = 1$. In this case, $\bigwedge_n(e)$ becomes a Hilbert space with orthonormal basis $\{e_I \colon I \in O_n\}$. If we denote $\bigwedge_p(e) = \operatorname{Span}(e_I \colon I \in O_n, |I| = p)$, then $\bigwedge_n(e) = \bigoplus_{p=0}^{n} \bigwedge^p(e)$. In the Hilbert space $\bigwedge_n(e)$ we define operators E_i, $i = 1, 2, \dots, n$, by $E_i x = e_i x$. It is obvious that

$$E_iE_j + E_jE_i = 0, \qquad E_i^*E_j + E_jE_i^* = \delta_i I. \tag{2}$$

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Let $T = (T_1, \dots, T_1)$ be a commuting tuple of the operators on Banach space X. Corresponding to T, there is a complex sequence

$$D_T\colon 0 \xrightarrow{D_{-1}} X \otimes \bigwedge{}^0(e) \xrightarrow{D_0} \cdots \xrightarrow{D_{n-1}} X \otimes \bigwedge{}^n(e) \xrightarrow{D_n} 0, \tag{3}$$

where $D_T = \sum_{i=1}^n T_i \otimes E_i$. If $\ker(D_k) = R(D_{k-1})$ for $k = 0, 1, \dots, n$, then T is non-singular. If $R(D_{k-1})$ is closed and $\dim \ker(D_k)/R(D_{k-1}) < \infty$ for $k = 0, 1, \dots, n$, then T is Fredholm with its index defined by $\operatorname{ind}(T) = \sum_{K=0}^n (-1)^k \dim H^k(T)$, where $H^k = \operatorname{Ker}(D_k)/R(D_{k-1})$. Of course, we can consider D_T as an operator on $X \otimes \bigwedge_n(e)$. It is obvious that T is Fredholm if and only if $R(D_T)$ is closed and $\dim \ker(D_T)/R(D_T)$ is finite.

The following is the main result of this note:

Theorem 1. *Let $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ be crisscross commuting pairs on Banach space X, i.e., they satisfy the conditions:*

$$A_i B_j A_k = A_k B_j A_i, \qquad B_i A_j B_k = B_k A_j B_i. \tag{4}$$

Let $T = (\varepsilon_1 - A_1 B_1, \dots, \varepsilon_n - A_n B_n)$ and $S = (\varepsilon_1 - B_1 A_1, \dots, \varepsilon_n - B_n A_n)$, where there exists at least one k such that $\varepsilon_k \neq 0$. Then T is Fredholm if and only if S is Fredholm and in this case

$$\operatorname{ind}(T) = \operatorname{ind}(S), \quad \dim H^k(T) = \dim H^k(S), \quad k = 0, 1, \dots, n. \tag{5}$$

Proof. Without loss of generality, we may suppose $\varepsilon_1 = 1$ and denote

$$D_1 = \sum_{K=1}^n (\varepsilon_k - A_k B_k) \otimes E_k, \qquad D_2 = \sum_{K=1}^n (\varepsilon_k - B_k A_k) \otimes E_k.$$

We can regard A_i andB_i as operators on $X \otimes \bigwedge_n(e)$ by $A_i(x \otimes e_I) = A_i x \otimes e_I$ and $B_i(x \otimes e_I) = B_i x \otimes e_I$. It is easy to know $A_i D_2 = D_1 A_i$ and $B_i D_1 = D_2 B_i$ from the conditions.

If T is Fredholm, then there exist a finite-dimensional subspace M such that $\ker(D_1) = R(D_1) \dotplus M$, i.e., M is equal to $H(T)$. Suppose $x \in \ker(D_2)$. From $A_1 D_2 = D_1 A_1$ we know that $D_1 A_1 x = A_1 D_2 x = 0$, i.e., $A_1 x \in \ker(D_1)$. Therefore there exist a vector $y \in X \otimes \bigwedge_n(e)$ and $u \in M$ such that

$$A_1 x = D_1 y + u.$$

Let $v = B_1 y + E_1^* x$; then we have

$$\begin{aligned}
D_2 v &= D_2 B_1 y + D_2 E_1^* x \\
&= B_1 D_1 y + \sum_{k=1}^n S_k \otimes E_1^* x \\
&= B_1 A_1 x - B_1 u - \sum_{k=1}^n S_k \otimes E_1^* E_k x + S_1 x \\
&= B_1 A_1 x - B_1 u - E_1^* D_2 x + (I - B_1 A_1) x \\
&= x - B_1 u.
\end{aligned}$$

Thus we get

$$x = D_2 v + B_1 u.$$

Therefore $R(D)$ has finite codimension in $\ker(D_2)$ and $\dim \ker(D_2)/R(D_2) \leqslant \dim B_1 M \leqslant \dim H(T)$. Thus S is Fredholm and $\dim H(S) \leqslant \dim H(T)$. From symmetry, we know that $\dim H(S) = \dim H(T)$, and B_1 is bijective from $H(T)$ onto $H(S)$. It is obvious that B_1 is also one-to-one and maps $H^k(T)$ onto $H^k(S)$ for $k = 0, 1, \ldots, n$. Thus the relations (5) are proved. □

Corollary 2. *Under the conditions of Theorem 1 we have*

$$\mathrm{Sp}_e(AB) \cup \{(0, \ldots, 0)\} = \mathrm{Sp}_e(BA) \cup \{(0, \ldots, 0)\}$$

where $\mathrm{Sp}_e(A)$ *is Taylor's essential spectrum of* A.

In particular we have

Corollary 3. *For operators* A *and* B *on Banach space,* $(I - AB)$ *is Fredholm if and only if* $(I - BA)$ *is Fredholm and in this case*

$$\mathrm{ind}(I - AB) = \mathrm{ind}(I - BA), \qquad \dim\ker(I - AB) = \dim\ker(I - BA).$$

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