

## SCHOTTKY'S FORM AND THE HYPERELLIPTIC LOCUS

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ABSTRACT. We show that Schottky's modular form,  $J_g$ , has in every genus an irreducible divisor which contains the hyperelliptic locus. We also improve a corollary of Igusa concerning Siegel modular forms that must necessarily vanish on the hyperelliptic locus.

### §1. INTRODUCTION

In 1888 Schottky gave the famous modular cusp form  $J_4$  which vanishes on the Jacobian locus in  $\mathcal{H}_4$ , the Siegel upper half space of degree four. In 1981 Igusa represented  $J_4$  as a rational multiple of  $\vartheta_{D_8^+}^2 - \vartheta_{D_{16}^+}$  where  $\vartheta_{D_8^+}$  and  $\vartheta_{D_{16}^+}$  are the theta series associated to the lattices  $D_8^+$  and  $D_{16}^+$ , respectively. This representation could be "accidental" in that the dimension of cusp forms of weight 8 on  $\mathcal{H}_4$  is small enough to make the the proportionality of forms arising from different sources likely. On the other hand this representation could point to a deeper relationship between differences of theta series and geometrically interesting loci in  $\mathcal{H}_g$ .

This paper provides a piece of data which supports the hypothesis of a deeper relationship. We show that  $\vartheta_{D_8^+}^2 - \vartheta_{D_{16}^+}$  vanishes on the hyperelliptic locus in  $\mathcal{H}_g$  for every degree  $g$ . If we view  $\vartheta_{D_8^+}$  and  $\vartheta_{D_{16}^+}$  as *complete invariants* of their associated lattices, then we see that Jacobians of hyperelliptic curves of any genus cannot distinguish the  $D_8^+ \oplus D_8^+$  lattice from the  $D_{16}^+$  lattice. The proof we give is a simple modification of an argument due to Igusa in [3, page 845] that uses his homomorphism  $\rho_g : A(\Gamma_g) \rightarrow S(2, 2g + 2)$ . Theorem 3.8 shows that if  $f \in A(\Gamma_g)$  vanishes at the cusps of the hyperelliptic locus, then  $\rho_g(f)$  is divisible by the discriminant in  $S(2, 2g + 2)$ . Theorem 3.8 provides a brief proof of the more interesting Corollary 3.10 that the modular form  $\vartheta_{D_8^+}^2 - \vartheta_{D_{16}^+}$  always vanishes on the hyperelliptic locus. The author is presently investigating whether or not this form vanishes on the Jacobian locus for  $g \geq 5$ . I thank William Duke for the interesting talk at Columbia University on Siegel modular forms and codes which led me to this investigation. I also thank my colleague Armand Brumer for his explanations on these topics.

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§2. NOTATION

We first review the notation concerning modular forms and subvarieties of the moduli space of principally polarized abelian varieties. We let  $\mathcal{H}_g$  denote the Siegel upper half space of degree  $g \geq 1$ , and let  $\Gamma_g = \mathrm{Sp}_g(\mathbb{Z})$  denote the Siegel modular group which acts on  $\mathcal{H}_g$ . Let  $A_k(\Gamma_g)$  denote the Siegel modular forms of weight  $k$  for  $\Gamma_g$ , and let  $A(\Gamma_g) = \bigoplus_{k \geq 0} A_k(\Gamma_g)$  be the graded ring of Siegel modular forms. For  $g \geq 2$  a homomorphism of graded rings  $\Phi_g : A(\Gamma_g) \rightarrow A(\Gamma_{g-1})$  is defined for  $f \in A(\Gamma_g)$  and  $\Omega \in \mathcal{H}_{g-1}$  by

$$(2.1) \quad (\Phi_g f)(\Omega) = \lim_{\lambda \rightarrow +\infty} f \begin{pmatrix} \Omega & 0 \\ 0 & i\lambda \end{pmatrix}.$$

Elements in the kernel of  $\Phi_g$  are called *cusp forms*. We view  $\mathcal{A}_g = \mathcal{H}_g/\Gamma_g$  as the moduli space of principally polarized abelian varieties. The Torelli map sends a compact Riemann surface of genus  $g$  to its Jacobian’s class in  $\mathcal{A}_g$ . We let  $\mathcal{J}_g$  denote the closure in  $\mathcal{A}_g$  of the image of the Torelli map and refer to  $\mathcal{J}_g$  as the *Jacobian locus*. In the same way we let  $h_g$  denote the closure of the image of the restriction of the Torelli map to hyperelliptic Riemann surfaces, and call  $h_g$  the *hyperelliptic locus*. We say that a Siegel modular form  $f \in A(\Gamma_g)$  vanishes on  $h_g$  if for all  $\Omega \in \mathcal{H}_g$  such that  $[\Omega] \in h_g$  we have  $f(\Omega) = 0$ .

We now discuss lattices in  $\mathbb{R}^n$  and their associated theta series. A lattice  $\Lambda \subseteq \mathbb{R}^n$  is called *integral* if for any  $\ell_1, \ell_2 \in \Lambda$  the value of the inner product  $\langle \ell_1, \ell_2 \rangle$  is an integer. The following sequence of analytic functions  $\vartheta_\Lambda$  are invariant under isometries of the lattice  $\Lambda$ .

**2.2 Definition.** Let  $\Lambda$  be a lattice in  $\mathbb{R}^n$ . For each  $g \geq 1$  the *theta series* of  $\Lambda$ ,  $\vartheta_\Lambda : \mathcal{H}_g \rightarrow \mathbb{C}$ , is defined for  $\Omega \in \mathcal{H}_g$  by

$$\vartheta_\Lambda(\Omega) = \sum_{\ell_1, \dots, \ell_g \in \Lambda} \exp \left( i\pi \sum_{j,k=1}^g \Omega_{jk} \langle \ell_j, \ell_k \rangle \right).$$

An integral lattice  $\Lambda$  is called *even* if for all  $\ell \in \Lambda$  we have  $\langle \ell, \ell \rangle \in 2\mathbb{Z}$ . If  $\Lambda$  is an even self-dual lattice of dimension  $n$  we have  $\vartheta_\Lambda \in A_{n/2}(\Gamma_g)$  for each  $g$ . For such  $\Lambda$  we necessarily have that 8 divides  $n$ , and the examples relevant here are the lattices  $D_n^+$  for  $n \in 8\mathbb{Z}^+$  in the notation of Conway and Sloane [1, page 119]. For  $n = 8$  there is one isometry class of even self-dual lattices, given by  $D_8^+$ ; for  $n = 16$  there are two isometry classes, given by  $D_8^+ \oplus D_8^+$  and  $D_{16}^+$ . Theta series satisfy  $\Phi_g(\vartheta_\Lambda \text{ on } \mathcal{H}_g) = \vartheta_\Lambda \text{ on } \mathcal{H}_{g-1}$ .

**2.3 Definition.** For  $g \geq 1$ , define  $J_g \in A_8(\Gamma_g)$  by  $J_g = \vartheta_{(D_8^+ \oplus D_8^+)} - \vartheta_{D_{16}^+}$ .

From the work of Igusa in [4] we know that the vanishing of  $J_4$  defines  $\mathcal{J}_4$  in  $\mathcal{A}_4$ .

Finally, we shall use the standard notation  $\mathbb{C}[a_1, \dots, a_r]$ ,  $\mathbb{C}(a_1, \dots, a_r)$  for polynomial domains and their quotient fields in  $r$  variables. Let  $\mathbb{C}[a_1, \dots, a_r]^{\mathrm{sym}}$  and  $\mathbb{C}(a_1, \dots, a_r)^{\mathrm{sym}}$  denote, respectively, the polynomial domain and rational function field fixed under the action of the symmetric group  $S_r$  on  $\mathbb{C}(a_i)$  induced by permutations of  $a_1, \dots, a_r$ . For  $s \geq 0$ , we denote by  $\mathbb{C}[a_i]_s^{\mathrm{sym}}$  the polynomials in  $\mathbb{C}[a_i]^{\mathrm{sym}}$  of degree  $s$  in any one, hence in any, of the  $a_i$ . Elements of  $\mathbb{C}[a_i]_s^{\mathrm{sym}}$  are said to have *weight*  $s$ . We note that  $\mathbb{C}[a_i]^{\mathrm{sym}} = \bigoplus_{s \geq 0} \mathbb{C}[a_i]_s^{\mathrm{sym}}$  but that this is not the usual

grading on  $\mathbb{C}[a.]^{\text{sym}}$  given by the homogeneous degree in the  $a_i$ ; rather it is the grading given by the homogeneous degree in the elementary symmetric functions of the  $a_i$ . If we let  $\Delta_r = \prod_{i < j} (a_i - a_j) \in \mathbb{C}[a_1, \dots, a_r]$  as usual, then the element  $\Delta_r^2 \in \mathbb{C}[a_1, \dots, a_r]_s^{\text{sym}}$  has weight  $s = 2(r - 1)$ .

§3.  $\mathbf{J}_g$  VANISHES ON  $\mathbf{h}_g$

In this section we prove that a hyperelliptic cusp form of weight less than  $8 + 4/g$  must vanish on the hyperelliptic locus,  $h_g$ . The main tools are Igusa's homomorphism  $\rho_g : A(\Gamma_g) \dashrightarrow S(2, 2g + 2)$  from a subring of Siegel modular forms to a graded ring of "binary invariants", and Tsuyumine's map  $T_g : S(2, 2g + 2) \dashrightarrow \mathbb{C}(a_1, \dots, a_{2g})$  that gives the  $\rho$ -induced action of  $\Phi_g$  on  $S(2, 2g + 2)$ .

**3.1 Definition.** For  $r \geq 1, s \geq 0$ , let  $S(2, r)_s$  be the set of  $f \in \mathbb{C}[a_1, \dots, a_r]$  such that both 1. and 2. hold.

1. For all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C})$  we have

$$f\left(\frac{aa_1 + b}{ca_1 + d}, \dots, \frac{aa_i + b}{ca_i + d}, \dots, \frac{aa_r + b}{ca_r + d}\right) = \left(\prod_{i=1}^r (ca_i + d)\right)^{-s} f(a_1, \dots, a_i, \dots, a_r).$$

2.  $f \in \mathbb{C}[a_1, \dots, a_r]_s^{\text{sym}}$ .

**3.2 Definition.** For  $r \geq 1$ , let  $S(2, r) = \bigoplus_{s \geq 0} S(2, r)_s$ .

**3.3 Remarks.** If we apply condition 1. of Definition 3.1 to  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$  we see that  $S(2, r) \subseteq \mathbb{C}[a_i - a_j \mid i, j \in \{1, \dots, r\}] \cap \mathbb{C}[a_1, \dots, a_r]_s^{\text{sym}}$ . Also notice that  $\Delta_r$  satisfies condition 1. with  $s = r - 1$ .

**3.4 Theorem** (Igusa, [3, page 844]). *Let  $g \geq 1$ . There exists a homomorphism of graded rings  $\rho_g : \text{Dom}(\rho_g) \subseteq A(\Gamma_g) \rightarrow S(2, 2g + 2)$  such that conditions 1.-3. hold.*

1.  $\text{Dom}(\rho_g)$  contains all modular forms of even weight.
2.  $\text{Ker}(\rho_g)$  is the ideal of  $\text{Dom}(\rho_g)$  vanishing on  $h_g$ .
3.  $\rho_g$  multiplies weights by  $\frac{1}{2}g$ .

**3.5 Definition** [7, page 762]. Let  $g, m \geq 1$ . Define  $T_g : \bigoplus_{m \geq 1} S(2, 2g + 2)_{gm} \rightarrow \mathbb{C}(a_1, \dots, a_{2g})$  by, for  $I \in S(2, 2g + 2)_{gm}$ ,

$$(T_g I)(a_1, \dots, a_{2g}) = \left(\prod_{i=1}^{2g} a_i\right)^{-m} I(a_1, \dots, a_{2g}, 0, 0).$$

In [7] the domain of  $T_g$  (designated by  $\Phi$  in [7]) is given as  $S(2, 2g + 2)$  and the range space of  $T_g$  is given as  $\mathbb{C}(a_1, \dots, a_{2g})$ , but the formula defining  $T_g$  gives  $T_g I$  in the algebraic extension  $\mathbb{C}(a_1, \dots, a_{2g})(\sqrt[2g]{a_1 \dots a_{2g}})$ . One easy way to mend this discrepancy is to define  $\text{Dom}(T_g) = \bigoplus_{m \geq 1} S(2, 2g + 2)_{gm}$  so that the range space of  $T_g$  is  $\mathbb{C}(a_1, \dots, a_{2g})$ . This is what has been done in Definition 3.5. The  $\text{Dom}(T_g)$  in Definition 3.5 includes the  $\rho_g$ -images of even weight modular forms; this inclusion is all we use here and all used in [7] to prove the following proposition.

**3.6 Proposition** (Tsuyumine [7, page 786]). *Let  $g \geq 1$ . There is a choice of  $\rho_g$  in Theorem 3.4 such that for even  $k$  and for all  $f \in A_k(\Gamma_g)$  we have  $(T_g \circ \rho_g)(f) = (\rho_{g-1} \circ \Phi_g)(f)$  in  $S(2, 2g)$ .*

**3.7 Definition.** Let  $g \geq 2$ , and let  $f \in A(\Gamma_g)$ . We say that  $f$  is a hyperelliptic cusp form when  $\Phi_g(f) \equiv 0$  on  $h_{g-1}$ .

**3.8 Theorem.** *Let  $k \in 2\mathbb{Z}^+$ . Let  $f \in A_k(\Gamma_g)$  be a hyperelliptic cusp form. Then we have  $\Delta_{2g+2}^2$  divides  $\rho_g(f)$  in  $S(2, 2g + 2)$ .*

*Proof.* Let  $f$  be a hyperelliptic cusp form so that we have  $\Phi_g(f) \equiv 0$  on  $h_{g-1}$  by Definition 3.7. From 2. of Theorem 3.4 we have  $\Phi_g(f) \in \text{Ker}(\rho_{g-1})$ . From Tsuyumine’s Proposition 3.6 we have  $T_g(\rho_g(f)) = \rho_{g-1}(\Phi_g(f)) = 0$  in  $\mathbb{C}(a_1, \dots, a_{2g})$ . From the definition of  $T_g$  we see that  $\rho_g(f)$  is in the ideal  $(a_{2g+1}, a_{2g+2})$  of the ring  $\mathbb{C}[a_1, \dots, a_{2g+2}]$ . Since  $\rho_g(f) \in S(2, 2g + 2) \subseteq \mathbb{C}[a_i - a_j \mid i, j \in \{1, \dots, 2g + 2\}]$ , we have  $(a_{2g+1} - a_{2g+2})$  divides  $\rho_g(f)$  in  $\mathbb{C}[a.]$ . Since  $\mathbb{C}[a.]$  is a unique factorization domain, the facts that  $\rho_g(f)$  is divisible by  $(a_{2g+1} - a_{2g+2})$  and that  $\rho_g(f) \in \mathbb{C}[a.]^{\text{sym}}$  imply that  $\Delta_{2g+2}$  divides  $\rho_g(f)$  in  $\mathbb{C}[a.]$ . Since  $\frac{\rho_g(f)}{\Delta_{2g+2}}$  is an alternating polynomial in the  $a.$ , we have that  $\Delta_{2g+2}$  divides  $\frac{\rho_g(f)}{\Delta_{2g+2}}$  in  $\mathbb{C}[a.]$ . Since  $\Delta_{2g+2}^2$  and  $\rho_g(f)$  are both in  $S(2, 2g + 2)$ , their quotient is as well, and we have that  $\Delta_{2g+2}^2$  divides  $\rho_g(f)$  in  $S(2, 2g + 2)$ .  $\square$

**3.9 Corollary.** *Let  $k \in 2\mathbb{Z}^+$ . Let  $f \in A_k(\Gamma_g)$  be a hyperelliptic cusp form. If  $k < 8 + \frac{4}{g}$ , then  $f$  vanishes identically on  $h_g$ .*

*Proof.* We have that  $\Delta_{2g+2}^2$  divides  $\rho_g(f)$  in  $S(2, 2g + 2)$  by Theorem 3.8. The weight of  $\Delta_{2g+2}^2$  is  $2(2g + 1)$  and the weight of  $\rho_g(f)$  is  $\frac{1}{2}gk$  when  $\rho_g(f)$  is nontrivial. However, we have  $\frac{1}{2}gk < 2(2g + 1)$  by hypothesis so that  $\rho_g(f)$  being divisible by  $\Delta_{2g+2}^2$  implies that  $\rho_g(f) = 0$ . Therefore we have  $f \equiv 0$  on  $h_g$  by Igusa’s Theorem 3.4.  $\square$

**3.10 Corollary.** *For all  $g \geq 1$ , the Siegel modular form  $J_g$  vanishes on the hyperelliptic locus  $h_g$ .*

*Proof.* For  $g$  such that  $1 \leq g \leq 4$  this is known from the work of Witt [8], Kneser and Igusa [4], [5]. Corollary 3.10 follows by induction on  $g$ . Suppose that we have  $J_g \equiv 0$  on  $h_g$ ; then we have  $\Phi_{g+1}(J_{g+1}) = J_g \equiv 0$  on  $h_g$  so that  $J_{g+1}$  is a hyperelliptic cusp form.  $J_{g+1}$  is of even weight 8 so that we may apply Corollary 3.9 to conclude that  $J_{g+1} \equiv 0$  on  $h_{g+1}$ .  $\square$

**3.11 Remark.** Corollary 3.10 may also be proven using Thomæ’s formula [3, pg. 838] and the theta identities in Lemma 1 of [5, pg. 354]. It then reduces to the following interesting polynomial identity which can be proven inductively by letting  $a_{2g+1} = a_{2g+2}$ . For  $g \geq 1$  we have the polynomial identity in  $\mathbb{Z}[a_1, \dots, a_{2g+2}]$ :

$$\begin{aligned} & \left( \sum_{\{T \cap T^c\}} \prod_{i < j; i, j \in T} (a_i - a_j)^2 \prod_{i < j; i, j \in T^c} (a_i - a_j)^2 \right)^2 \\ &= 2^g \sum_{\{T \cap T^c\}} \prod_{i < j; i, j \in T} (a_i - a_j)^4 \prod_{i < j; i, j \in T^c} (a_i - a_j)^4. \end{aligned}$$

The above sum is over the  $\frac{1}{2} \binom{2g+2}{g+1}$  partitions  $T \amalg T^c$  of  $\{1, 2, \dots, 2g+2\}$  for which both  $T$  and  $T^c$  have  $g+1$  elements. This formula was our original method of proof and was also noted by the referee.

Finally, we mention that  $J_g$  is irreducible in  $A(\Gamma_g)$ .

**3.12 Proposition.** *For all  $g \geq 4$ , the divisor of  $J_g$  in  $\mathcal{A}_g$  is irreducible.*

*Proof.* We will show by induction on  $g$  that the divisor of  $J_g$  is proper and irreducible in  $A(\Gamma_g)$  for  $g \geq 4$ . The case  $g = 4$  is due to Igusa [4]. We use a principle of Freitag which he deduces from his ‘‘Satz 2’’ in [2, page 256]. ‘‘For  $g \geq 3$  an  $f \in A(\Gamma_g)$  has an irreducible divisor,  $\text{div}(f)$ , if  $f$  may not be written as the product of modular forms of lower weight.’’ If we had  $J_g = \psi_1 \psi_2$  in  $A(\Gamma_g)$  where  $0 < \deg \psi_1, \deg \psi_2 < 8$ , then applying the map  $\Phi_g$  to  $J_g$  and using the induction hypothesis show that  $J_g$  is nontrivial on  $\mathcal{H}_g$  and that  $J_{g-1} = \Phi_g(\psi_1) \Phi_g(\psi_2)$  in  $A(\Gamma_{g-1})$  where  $0 < \deg \Phi(\psi_1), \deg \Phi(\psi_2) < 8$ . This is impossible because  $\text{div}(J_{g-1})$  is irreducible by the induction hypothesis. This shows that  $\text{div}(J_g)$  is both proper and irreducible in  $\mathcal{A}_g$ .  $\square$

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