REAL INSTANTONS, DIRAC OPERATORS
AND QUATERNIONIC CLASSIFYING SPACES

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Abstract. Let $M(k, SO(n))$ be the moduli space of based gauge equivalence classes of $SO(n)$ instantons on principal $SO(n)$ bundles over $S^4$ with first Pontryagin class $p_1 = 2k$. In this paper, we use a monad description (Y. Tian, The Atiyah-Jones conjecture for classical groups, preprint, S. K. Donaldson, Comm. Math. Phys. 93 (1984), 453–460) of these moduli spaces to show that in the limit over $n$, the moduli space is homotopy equivalent to the classifying space $BSp(k)$. Finally, we use Dirac operators coupled to such connections to exhibit a particular and quite natural homotopy equivalence.

1. Introduction

Let $M(k, SO(n))$ be the moduli space of based gauge equivalence classes of $SO(n)$ instantons on principal $SO(n)$ bundles over $S^4$ with first Pontryagin class $p_1 = 2k$. By adding a trivial connection on a trivial line bundle, there are natural maps $M(k, SO(n)) \hookrightarrow M(k, SO(n+1))$, and one can define the direct limit space $M(k, SO)$. In this paper we show that there is a homotopy equivalence $M(k, SO) \simeq BSp(k)$, where $Sp(k)$ denotes the symplectic group of norm preserving automorphisms of the quaternionic vector space $H^k$. We also show that this equivalence can be realized by a “Dirac-type” map, constructed by coupling a Dirac operator to an $SO(n)$ connection. More precisely, the coupling of a Dirac operator to a connection associates to each element of $M(k, SO(n))$ an operator acting on the space of sections of a certain vector bundle. Associated to each selfdual connection is the vector space of sections in the kernel of its associated operator. This procedure defines a complex vector bundle, which for $SO(n)$ connections has a symplectic structure, and this bundle is classified by a map which we shall refer to as the Dirac map, $\partial_{SO(n)} : M(k, SO(n)) \to BSp(k)$. The topological properties of these Dirac maps for $SU(n)$ connections were first studied by Atiyah and Jones [AJ], and more recently it was shown in [S] that the limit map $\partial_{SU} : M(k, SU) \to BU(k)$ realizes Kirwan’s [K] homology isomorphism $H_*(M(k, SU)) \cong H_*(BU(k))$, and is, therefore, a homotopy equivalence. It also makes sense to define such Dirac maps on the limit spaces $M(k, G)$, where $G$ is either $SO$ or $Sp$, and in [S] it was shown that the limit map $\partial_{Sp} : M(k, Sp) \to BO(k)$ is a homotopy equivalence. In this

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2193
paper we complete the picture for the classical groups by showing that the limit map \( \partial_{SO} : M(k, SO) \to BSp(k) \) is also a homotopy equivalence.

Our proof will be fairly direct. In Section 1 we review Tian’s [Ti] version of Donaldson’s [D] monad description of \( M(k, SO(n)) \). Tian exhibits this moduli space as the quotient of a set of triples of certain complex matrices by an action of \( Sp(k; C) \), the complex symplectic group. We shall show that this action is free, that there are natural maps \( M(k, SO(n)) \hookrightarrow M(k, SO(n+1)) \), and that in the limit over \( n \) the space of triples is contractible. Hence, \( M(k, SO) \) will be shown to be the quotient of a contractible space by a free \( Sp(k; C) \) action. In Section 2, we use a comparison between \( SO(n) \) and \( SU(n) \) connections to show that the Dirac map \( \partial_{SO(n)} : M(k, SO(n)) \to BSp(k) \) induces a surjection in integral homology through a range of dimension increasing with \( n \). Since \( H_\ast(M(k, SO); Z) \cong H_\ast(BSp(k); Z) \) by results of Section 1, the limit map \( \partial_{SO} \) must be a homology isomorphism and therefore a homotopy equivalence.

Notice that the \( Sp \) and \( SO \) duality in these moduli spaces is foreshadowed in Bott Periodicity. Since the entire space of based gauge equivalence classes of \( SO(n) \) connections is homotopy equivalent to \( \Omega^3 SO(n) \), the limit over \( n \) is homotopy equivalent to \( Z \times BSp \). Similarly, the space of \( Sp(n) \) connections is homotopy equivalent to \( \Omega^3 Sp(n) \) which, after passing to the limit, is homotopy equivalent to \( Z \times BO \). Alternatively, as we will see in Section 2, this duality comes from the fact that the bundle of real spinors over \( S^4 \) is naturally a symplectic vector bundle. Recently, in fact, Tian [Ti] has shown that by comparing the two possible limit processes which one can apply to these moduli spaces, viz., fixing \( k \) and taking the limit over \( n \) or fixing \( n \) and taking the limit over \( k \), one actually can prove Bott Periodicity. This consequence alone demonstrates the beauty and complexity of these moduli spaces.

2. \( M(k, SO) \) and \( BSp(k) \)

The ADHM construction [ADHM] identifies the space of instantons with certain holomorphic bundles over complex projective space, and Donaldson [D] used a monad construction to characterize such bundles in terms of a quotient of a set of sequences of complex matrices by a natural group action. For \( SO(n) \) instantons, Tian [Ti] carried out this procedure explicitly.

Let \( \sigma \) denote the standard skew form on \( C^{2k} \),

\[
\sigma = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}
\]

where \( I_k \) is the \( k \times k \) identity matrix. The complexified symplectic group, \( Sp(k, C) \subset GL(2k, C) \), consists of those matrices \( g \) such that \( g^{-1} = -g^T \sigma \). The maximal compact subgroup of \( Sp(k, C) \) is the compact symplectic group \( Sp(k) \).

**Proposition 1** (Donaldson [D] and Tian [Ti]). Let \( A(k, SO(n)) \) be the space of triples of complex matrices \( (\gamma_1, \gamma_2, c) \), where \( \gamma_i \) is \( 2k \times 2k \) and \( c \) is \( n \times 2k \), satisfying:

a) \( \gamma_1^T = -\sigma \gamma_1 \sigma \),
b) \( \gamma_2^T = -\gamma_2 \),
c) \( 2(\gamma_1^T \gamma_2 + \gamma_2^T \gamma_1) + c^T c = 0 \),
d) \[
\begin{pmatrix}
\gamma_1 + xI_{2k} \\
\gamma_2 + y \sigma \\
c
\end{pmatrix}
\]

has rank \( 2k \) for all \( x, y \in C \).
Then there is a natural action of $Sp(k,\mathbb{C})$ on $A(k, SO(n))$ given by

$$g \cdot (\gamma_1, \gamma_2, c) = (g\gamma_1g^{-1}, (g^{-1})^T\gamma_2g^{-1}, cg^{-1}),$$

and $M(k, SO(n))$ is homeomorphic to the quotient $A(k, SO(n))/Sp(k, \mathbb{C})$.

It has been shown [S] that in the analogous description of $SU(n)$ and $Sp(n)$ instantons, this group action is free. This is, in some sense, already implicit in the monad construction. Not surprisingly, then, it is also true in the $SO(n)$ case. For the sake of completeness, however, we now give the proof.

**Lemma 2.** The natural action of $Sp(k, \mathbb{C})$ on $A(k, SO(n))$ is free.

**Proof.** Assume the converse, so we have

$$(g\gamma_1g^{-1}, (g^{-1})^T\gamma_2g^{-1}, cg^{-1}) = (\gamma_1, \gamma_2, c)$$

for a particular $g \neq I$ and triple $(\gamma_1, \gamma_2, c)$, and note that elements of $Sp(k, \mathbb{C})$ satisfy $g^{-1} = -sg^T\sigma$. Consider the subspace $im(g - I) = V \subset \mathbb{C}^{2k}$. By assumption it is non-empty. Thus, from the definition of the action we have

$$c(g - I) = 0,$$

$$\gamma_1(g - I) = (g - I)\gamma_1,$$

$$\sigma\gamma_2(g - I) = (g - I)\sigma\gamma_2.$$

This last fact is proved as follows:

$$\gamma_2 = (g^{-1})^T\gamma_2g^{-1}$$

$$\Rightarrow \gamma_2g = (g^{-1})^T\gamma_2$$

$$\Rightarrow \sigma\gamma_2g = -\sigma(g^{-1})^T\sigma\gamma_2$$

$$\Rightarrow \sigma\gamma_2g = g\sigma\gamma_2.$$

Equivalently $c$ annihilates $V$ and $\gamma_1$ and $\sigma\gamma_2$ preserve $V$. Using conditions a), b), and c), we see that on $V$

$$\gamma_1^T\gamma_2 + \gamma_2^T\gamma_1 = 0$$

$$\Rightarrow -\sigma\gamma_1\sigma\gamma_2 - \gamma_2\gamma_1 = 0$$

$$\Rightarrow \gamma_1\sigma\gamma_2 - \sigma\gamma_2\gamma_1 = 0.$$

Hence $\gamma_1$ and $\sigma\gamma_2$ have a common eigenvector in $V$.

Choose $v \in V$ satisfying $\gamma_1v = \lambda v$ and $\sigma\gamma_2v = \mu v$. Then

$$\begin{pmatrix}
\gamma_1 - \lambda I_{2k} \\
\gamma_2 + \mu\sigma \\
c
\end{pmatrix}v = 0,$$

contradicting condition d). Thus, the image of $g - I$ must be empty, so $g = I$ and the action is free.

We now construct the limit space $M(k, SO)$ and show that it is homotopy equivalent to $BSp(k)$. First notice that there is an $Sp(k, \mathbb{C})$ equivariant map from $A(k, SO(n)) \hookrightarrow A(k, SO(n + 1))$ which sends each $\gamma_i$ to itself and sends $c$ to the $(n + 1) \times k$ matrix made up of $c$ with an extra row of zeros on top. On the level of monads, this adds to the bundle over $CP^2$ the trivial holomorphic line bundle (see [Ti]). Thus this map induces the natural inclusion $M(k, SO(n)) \hookrightarrow M(k, SO(n+1))$ sending the connection $\omega$ to the connection $\omega \oplus d$, where $d$ is ordinary exterior differentiation. We now prove the main theorem of this section.
Theorem 3. \( A(k, SO) \) is a contractible space with a free \( Sp(k, \mathbb{C}) \) action. Thus, \( M(k, SO) \cong A(k, SO)/Sp(k, \mathbb{C}) \cong BS\mathfrak{p}(k) \).

Proof. To show that \( A(k, SO) \) is contractible it suffices to show that all of its homotopy groups are zero. To this end we show that for any \( k \) and \( n \) there is an \( r > n \) such that inclusion \( A(k, SO(n)) \hookrightarrow A(k, SO(r)) \) is homotopically trivial (cf. [S], sections 2 and 3).

For \( 0 \leq t \leq 1 \) define \( \tilde{I}_k(t) \) to be the \( 4k \times 2k \) matrix whose \( j^{th} \) column is the vector

\[
\begin{pmatrix}
0 \\
\vdots \\
t \\
it \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

where there are \( 2j - 2 \) zeroes before the \( t \). Note that \( (\tilde{I}_k(t))^T \cdot \tilde{I}_k(t) \) is the zero matrix. Now consider the homotopy \( H_t : A(k, SO(n)) \to A(k, SO(4k + n)) \) defined as follows:

\[
H_t(\gamma_1, \gamma_2, c) = ((1 - t)\gamma_1, (1 - t)\gamma_2, c_t)
\]

where

\[
c_t = \begin{pmatrix}
\tilde{I}_k(t) \\
(1 - t)c
\end{pmatrix}.
\]

It is easy to check that for any \( x \in A(k, SO(n)) \) we have \( H_t(x) \in A(k, SO(4k + n)) \) because \( c_t \) clearly has rank \( 2k \) and \( c_t^T \cdot c_t = c^T \cdot c(1 - t)^2 \). Finally, notice that \( H_0 \) is just the natural inclusion \( A(k, SO(n)) \hookrightarrow A(k, SO(4k + n)) \), and \( H_1 \) is a constant map. This finishes the proof of the theorem.

3. The Dirac map \( \partial_{SO(n)} : M(k, SO(n)) \to BSp(k) \)

In this section we review the construction of the Dirac map, and show that after passing to the limit over \( n \) it is a homotopy equivalence. To define this map it is instructive to first consider \( SU(n) \) instantons. Let \( \omega \) be a connection on the \( SU(n) \) vector bundle \( E_k \), where the second Chern class \( c_2(E_k) = k \), and let \( S \) denote the canonical bundle of complex spinors over \( S^4 \) with its canonical connection \( \nabla_s \). The tensor product bundle \( S \otimes E_k \) inherits a Clifford module structure from the one on \( S \), and we can view \( \nabla_s \otimes \omega \) as a connection on this bundle. This connection gives rise to a Dirac operator

\[
D_\omega : \Gamma(S \otimes E_k) \to \Gamma(S \otimes E_k),
\]

where \( \Gamma(S \otimes E_k) \) is the space of smooth sections of \( S \otimes E_k \). There is a splitting \( S \cong S^+ \oplus S^- \) and the Dirac operator interchanges the two summands. The operator

\[
D_\omega^+ : \Gamma(S^+ \otimes E_k) \to \Gamma(S^- \otimes E_k)
\]

is Fredholm, in an appropriate Sobolev completion, and of index \( k \) [AJ]. Furthermore, if \( \omega \) is selfdual, then \( \text{Coker}(D_\omega^+) = 0 \) [AHS]. Therefore, the space of sections in the kernel of \( D_\omega^+ \) gives a well-defined vector space associated to the connection.
\(\omega\). There is an equivariance of the kernel under gauge transformation in the sense that \(\sigma \in \text{Ker}(D^+_\omega)\) implies \(g\sigma \in \text{Ker}(D^+_\omega)\), for any \(g\) in the based gauge group of bundle automorphisms of \(E_k\). Passing to gauge equivalence classes gives a \(k\)-dimensional complex vector bundle over \(M(k, SU(n))\). This bundle is classified by a map, \(\partial_{SU(n)} : M(k, SU(n)) \rightarrow BU(k)\), which we shall refer to as the Dirac map.

A similar construction can be used to define the Dirac map for \(SO(n)\) connections. Given an \(SO(n)\) bundle \(E\) with \(p_1(E) = 2k\) and an \(SO(n)\) instanton \(\omega\) on \(E\), we can complexify the bundle and connection, denoted \(\omega_C\) and \(E_C\), and then use the unitary Dirac map to obtain

\[
M(k, SO(n)) \rightarrow M(2k, SU(n)) \stackrel{\partial_{SU(n)}}{\rightarrow} BU(2k).
\]

(Note that \(c_2(E_C) = 2k\).) However, because \(E_C\) has by definition an underlying real structure, given by some bundle involution \(J_E\), and the complex spinor bundle \(S\) has a quaternionic structure, given by some complex anti-linear bundle automorphism \(J_s\), where \(J_s \circ J_s = -1\), the tensor product bundle \(S \otimes E_C\) will also have a quaternionic structure. Moreover, the Dirac operator will respect this extra structure because the tensor product connection \(\nabla_s \otimes \omega_C\) will commute with \(J_s \otimes J_E\). Thus, the kernel bundle, defined by coupling a Dirac operator to a real \(SO(n)\) instanton, will be a \(k\)-dimensional quaternionic bundle over \(M(k, SO(n))\). In other words, the composition

\[
M(k, SO(n)) \rightarrow M(2k, SU(n)) \stackrel{\partial_{SU(n)}}{\rightarrow} BU(2k)
\]

factors through \(BSp(k)\). We denote this lifting by \(\partial_{SO(n)}\). In short, we have the homotopy commutative diagram

\[
\begin{array}{ccc}
M(k, SO(n)) & \stackrel{\partial_{SO(n)}}{\rightarrow} & BSp(k) \\
\downarrow & & \downarrow \\
M(2k, SU(n)) & \stackrel{\partial_{SU(n)}}{\rightarrow} & BU(2k)
\end{array}
\]

We now show that we can define the limit map \(\partial_{SO} : M(k, SO) \rightarrow BSp(k)\). From the matrix description of \(M(k, SO(n))\), we see that the natural inclusion \(M(k, SO(n)) \hookrightarrow M(k, SO(n+1))\), mapping \((\omega, E)\) to \((\omega \oplus d, E \oplus R)\), embeds \(M(k, SO(n))\) as a closed submanifold of \(M(k, SO(n+1))\). It follows that the direct limit \(M(k, SO)\) is homotopy equivalent to the homotopy direct limit \(M(k, SO)_b\). Thus, it suffices to define \(\partial_{SO}\) on \(M(k, SO)_b\). To this end, let \(A(k, SO(n))\) denote the space of instantons before passing to gauge equivalence classes, and let \(G_{k, SO(n)}\) denote the based gauge group of bundle automorphisms of the \(SO(n)\) bundle \(E\), where \(p_1(E) = 2k\). Let \(\eta(k, SO(n))\) denote the bundle classified by the map \(\partial_{SO(n)} : M(k, SO(n)) \rightarrow BSp(k)\). By definition,

\[
\eta(k, SO(n)) = \left\{[(\omega, \tau) : \tau \in \text{Ker}(D^+_\omega)] \right\} \subset A(k, SO(n)) \times_{G_{k, SO(n)}} \Gamma(S^+ \otimes E_C).
\]
Since the untwisted Dirac operator on $S^4$ has no kernel ($S^4$ has no harmonic spinors), the natural inclusion of bundles
\[ \eta(k, SO(n)) \hookrightarrow \eta(k, SO(n + 1)) \]
\[ M(k, SO(n)) \hookrightarrow M(k, SO(n + 1)) \]
defined by $(\omega, \tau) \mapsto (\omega \oplus d, \tau \oplus 0)$ is an isomorphism on fibers. Thus the pullback of $\eta(k, SO(n + 1))$ via the inclusion $M(k, SO(n)) \hookrightarrow M(k, SO(n + 1))$ is isomorphic to $\eta(k, SO(n))$. Hence, the diagram
\[ \begin{array}{c}
M(k, SO(n)) \hookrightarrow M(k, SO(n + 1)) \\
\partial_{SO(n)} \downarrow \quad \downarrow \partial_{SO(n+1)} \\
BSp(k) = BSp(k)
\end{array} \]
commutes up to homotopy. So there exists a map $\partial_{SO}: M(k, SO)_h \rightarrow BSp(k)$. Precomposing with the equivalence $M(k, SO) \simeq M(k, SO)_h$ gives a map
\[ \partial_{SO}: M(k, SO) \rightarrow BSp(k). \]
This map is not necessarily uniquely determined. Nevertheless, any two choices, when restricted to $M(k, SO(n))$, will classify the bundle $\eta(k, SO(n))$, and this is the only property of the limit map which we will use. In particular, any such choice will give a homotopy commutative diagram
\[ \begin{array}{c}
M(k, SO(n)) \rightarrow M(k, SO) \\
\partial_{SO(n)} \downarrow \quad \downarrow \partial_{SO} \\
BSp(k)
\end{array} \]
Now, since $H_*(M(k, SO)) \cong H_*(BSp(k))$ by Theorem 3, $\partial_{SO}$ will induce a homology isomorphism, and therefore be a homotopy equivalence, if and only if it induces a surjection in homology. By the homotopy commutativity of the previous diagram, it suffices to show that $\partial_{SO(n)}: M(k, SO(n)) \rightarrow BSp(k)$ induces a surjection in homology through a range increasing with $n$.

**Theorem 4.** The Dirac map $\partial_{SO(n)}$ induces a surjection in homology through dimension $2n - 4$. Thus, the limit map $\partial_{SO}: M(k, SO) \rightarrow BSp(k)$ is a homotopy equivalence.

We begin by proving the following lemma:

**Lemma 5.** There is a commutative diagram
\[ \begin{array}{ccc}
H_*(M(k, SU(n))) & \xrightarrow{(\partial_{SU(n)})_*} & H_*(BU(k)) \\
\downarrow & & \downarrow \\
H_*(M(k, SO(2n))) & \xrightarrow{(\partial_{SO(2n)})_*} & H_*(BSp(k))
\end{array} \]
where \( \text{Sp}(k) \subset U(2k) \) consists of all matrices of the form

\[
\begin{pmatrix}
A & B \\
-\overline{B} & \overline{A}
\end{pmatrix}
\]

for any \( A, B \in \text{End}(C^k) \), and the map \( BU(k) \to B\text{Sp}(k) \) is induced from the inclusion \( U(k) \hookrightarrow \text{Sp}(k) \) defined by

\[
A \mapsto \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}.
\]

Proof. First notice that the natural map of Lie algebras \( su(n) \hookrightarrow so(2n) \) induces a map \( M(k, SU(n)) \to M(k, SO(2n)) \). The self-duality condition is preserved because the Hodge star operator is complex linear. Locally, the connection matrix \( \gamma = \gamma_1 + i\gamma_2 \), where \( \gamma_j \) is a real matrix-valued one form, will map to the matrix

\[
\begin{pmatrix}
\gamma_1 & -\gamma_2 \\
\gamma_2 & \gamma_1
\end{pmatrix}.
\]

Also notice that, as mentioned previously, the complexification of a real connection on an \( SO(r) \) bundle induces a natural map \( M(k, SO(2n)) \to M(2k, SU(2n)) \). Locally, the composition of these two maps

\[
M(k, SU(n)) \to M(k, SO(2n)) \to M(2k, SU(2n))
\]

is given by

\[
\gamma = \gamma_1 + i\gamma_2 \mapsto \begin{pmatrix} \gamma_1 & -\gamma_2 \\ \gamma_2 & \gamma_1 \end{pmatrix} \mapsto \begin{pmatrix} \gamma_1 & -\gamma_2 \\ \gamma_2 & \gamma_1 \end{pmatrix},
\]

where the last matrix is viewed as taking values in the Lie algebra \( su(2n) \). Since there is a \( g \in SU(2n) \) such that

\[
g^{-1} \begin{pmatrix} \gamma_1 & -\gamma_2 \\ \gamma_2 & \gamma_1 \end{pmatrix} g = \begin{pmatrix} \gamma_1 + i\gamma_2 & 0 \\ 0 & \gamma_1 - i\gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 0 & \overline{\gamma} \end{pmatrix},
\]

the connections represented by these matrix-valued one forms are gauge equivalent. Thus, the composition \( M(k, SU(n)) \to M(k, SO(2n)) \to M(2k, SU(2n)) \) sends the equivalence class of the selfdual connection \( \omega \) on the bundle \( E \) to the equivalence class of the selfdual connection \( \omega \oplus \overline{\omega} \) on the bundle \( E \oplus \overline{E} \).

Now consider the diagram

\[
\begin{array}{ccc}
M(k, SU(n)) & \xrightarrow{\partial_{SU(n)}} & BU(k) \\
\downarrow & & \downarrow \\
M(k, SO(2n)) & \xrightarrow{\partial_{SO(2n)}} & BSp(k) \\
\downarrow & & \downarrow \\
M(2k, SU(2n)) & \xrightarrow{\partial_{SU(2n)}} & BU(2k).
\end{array}
\]
By the definition of $\partial_{SO(2n)}$, the bottom square homotopy commutes. Since the map $BSp(k) \to BU(2k)$ induces an injection in homology, the top square will induce a commutative diagram in homology if the large outer “square”

\[
\begin{array}{ccc}
M(k, SU(n)) & \xrightarrow{\partial_{SU(n)}} & BU(k) \\
\downarrow & & \downarrow \\
M(2k, SU(2n)) & \xrightarrow{\partial_{SU(2n)}} & BU(2k)
\end{array}
\]

commutes in homology. Note that on the level of bundles the right vertical map sends a complex vector bundle $F$ to the complex bundle $F \oplus \bar{F}$. Let $\eta(r, SU(l))$ denote the Dirac bundle classified by the map $\partial_{SU(l)}: M(r, SU(l)) \to BU(r)$. The proof of the lemma will be complete if the composition

\[
\begin{array}{ccc}
M(k, SU(n)) & \xrightarrow{\partial_{SU(n)}} & BU(k) \\
\downarrow & & \downarrow \\
M(2k, SU(2n)) & \xrightarrow{\partial_{SU(2n)}} & BU(2k)
\end{array}
\]

classifies the bundle $\eta(k, SU(n)) \oplus \bar{\eta}(k, SU(n))$. There is a natural bundle map

\[
\eta(k, SU(n)) \oplus \bar{\eta}(k, SU(n)) \to \eta(2k, SU(2n))
\]

defined by

\[
[(\omega, \psi_1 \oplus \psi_2)] \to [(\omega \oplus \bar{\omega}, \psi_1 \oplus \bar{\psi}_2)]
\]

where $\bar{\psi}_2$ is the section $\psi_2$ viewed as a section of the conjugate bundle. Since $\psi$ is in the kernel of $D^+_n$ if and only if $\bar{\psi}$ is in the kernel of $D^+_n$, this bundle map is a surjection on fibers. Since the fibers have the same dimension, this map is an isomorphism. Thus $\eta(k, SU(n)) \oplus \bar{\eta}(k, SU(n))$ is isomorphic to the pullback of $\eta(2k, SU(2n))$, and the lemma is proved.

The proof of Theorem 4 is now easy. In [S], section 5, it was shown that the map

\[
(\partial_{SU(n)})_*: H_*(M(k, SU(n))) \to H_*(BU(k))
\]

is a surjection through dimension $2n - 4$. Furthermore, we know that the map $BU(k) \to BSp(k)$ induces a surjection in homology. Thus, by the commutativity of the diagram

\[
\begin{array}{ccc}
H_*(M(k, SU(n))) & \xrightarrow{(\partial_{SU(n)})_*} & H_*(BU(k)) \\
\downarrow & & \downarrow \\
H_*(M(k, SO(2n))) & \xrightarrow{(\partial_{SO(2n)})_*} & H_*(BSp(k))
\end{array}
\]

$(\partial_{SO(2n)})_*: H_*(M(k, SO(2n))) \to H_*(BSp(k))$ must also be a surjection through this range. In particular, then, the limit map $\partial_{SO} : M(k, SO) \to BSp(k)$ is a homotopy equivalence.
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