PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 7, July 1996

A DEGREE ESTIMATE FOR SUBDIVISION SURFACES OF HIGHER REGULARITY

ULRICH REIF

(Communicated by Peter Li)

Abstract. Subdivision algorithms can be used to construct smooth surfaces from control meshes of arbitrary topological structure. In contrast to tangent plane continuity, which is well understood, very little is known about the generation of subdivision surfaces of higher regularity. This work presents a degree estimate for piecewise polynomial subdivision surfaces saying that curvature continuity is possible only if the bi-degree d of the patches satisfies $d \geq 2k+2$, where k is the order of smoothness on the regular part of the surface. This result applies to any stationary or non-stationary scheme consisting of masks of arbitrary size provided that some generic symmetry and regularity assumptions are fulfilled.

1. Introduction

Subdivision algorithms for meshes of arbitrary topological type were introduced in [CC78] and [DS78] for quadrilateral meshes, and later on in [Loo87] and [DLG90] for triangular meshes. Permitting an efficient implementation and generating well-shaped surfaces these algorithms have found wide application in computer-aided geometric design and are considered to be a convenient tool to deal with the problem of constructing smooth surfaces of arbitrary topology or, equivalently, to fill n-sided holes in parametrically smooth spline surfaces. Recently, piecewise polynomial subdivision surfaces were used to solve the problem of interpolating irregularly structured surface data [DHK93] and to define a mulitresolution analysis for surfaces of general topological type [DLW93].

In contrast to the great success for practical purposes, the theoretical background of subdivision surfaces remained unclear for a long period of time and so far, only the question of convergence to tangent plane continuous limit surfaces has been answered for stationary schemes [Rei93], [Rei95]. A necessary condition for curvature continuity in terms of the leading eigenvalues of the subdivision matrix can already be found in the work of Doo and Sabin [DS78]. However, algorithms satisfying sufficient conditions have not yet been found despite a number of efforts in this direction. It is the purpose of this report to point out that the failure of all

Received by the editors December 6, 1994.
1991 *Mathematics Subject Classification.* Primary 65D17, 65D07, 68U07.
Key words and phrases. Subdivision, arbitrary topology, extraordinary vertex, curvature continuity, piecewise polynomial surface.
This work was supported by BMFT Projekt 03–HO7STU–2.

attempts is forced by inherent difficulties related to the use of spline surfaces based on low-degree polynomials.

There are at least two different ways to look at subdivision algorithms for meshes of arbitrary topology: Usually, subdivision is regarded as a process generating a sequence of finer and finer polyhedral meshes converging to some limit, ideally a smooth surface. Alternatively, and this is a more convenient approach for analytical purposes, the regular part of each mesh can be identified with its limit surface. In this way, an ascending sequence of smooth surfaces is produced, where each surface contains the preceding one as a proper subset since subdivision enlarges the regular regions and shrinks the irregular regions of a mesh. So, by considering only the new ring-shaped parts of the surfaces, subdivision is equivalent to adding smaller and smaller surface layers inside an n-sided hole of a given surface. A tangent plane continuous surface is obtained if the sequence of layers converges to a unique center point and if the normal vectors assigned to any sequence of points approaching this center are converging to a unique limit.

The classical Doo-Sabin algorithm shows that piecewise biquadratic spline surfaces, i.e. surfaces of minimal polynomial bi-degree, are suitable for modeling subdivision schemes which provide tangent plane continuity. Analogously, Catmull-Clark's algorithm, which is based on bicubic patches joining twice continuously differentiable, was expected to produce curvature continuous limit surfaces after some suitable modifications of the standard subdivision masks. However, all attempts – even choosing sequences of masks of arbitrary size – are doomed to failure. This is an immediate consequence of a more general result which, roughly speaking, reads as follows: Consider a subdivision scheme generating piecewise polynomial surfaces of bi-degree d and smoothness of order $k \geq 2$ on the regular part of a mesh. Provided that this scheme reflects the natural symmetry of an n-sided configuration and that it is non-degenerate in the sense that it does not enforce a flat spot at the center point, then global curvature continuity is possible only if $d \geq 2k + 2$.

2. Notions and basic concepts

Consider a subdivision algorithm near an extraordinary vertex of order $n \in \mathbb{N}\backslash\{1, 2, 4\}$. The process of identifying such an algorithm with joining a sequence of *surface layers* $\mathbf{x}_m, m \in \mathbb{N}$, smoothly was described in detail in [Rei93], [Rei95] and is only reviewed briefly here. A key observation for analyzing the *limit surface*

$$\mathbf{P} = \bigcup_{m=0}^{\infty} \mathbf{x}_m \tag{2.1}$$

is that all layers $\mathbf{x}_m$ have a common domain $\Omega := \omega \times \{1, \ldots, n\}$ consisting of n copies of the set $\omega := [0, 2]^2\backslash[1, 2]^2$; see Figure 1. So, the layers $\mathbf{x}_m$ consist of n L-shaped *patches*

$$\mathbf{x}_m = \bigcup_{i=1}^{n} \mathbf{x}_m^i \ , \tag{2.2}$$

where each patch

$$\mathbf{x}_m^i : \omega \ni (u, v) \longmapsto \mathbf{x}_m^i(u, v) \in \mathbb{R}^3 \tag{2.3}$$

is a surface segment over ω. Here and subsequently, the index i runs from 1 to n and has to be understood modulo n. Note that, in general, the patches $\mathbf{x}_m^i$ are

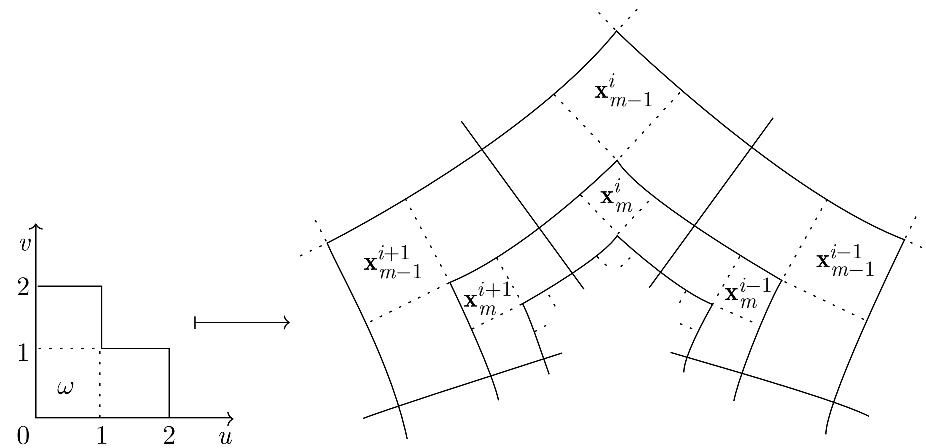

FIGURE 1. Structure of the surface layers

macro patches in the sense that they are formed by a certain number of elementary square patches in order to obtain the desired structure.

A case of particular interest is the use of B-spline knot insertion rules for subdividing the regular part of the mesh because then the limit surface admits a simple analytical description in terms of piecewise bivariate tensor product polynomials. So, we shall confine ourselves to this situation subsequently. Assume that the regular subdivision rules are derived from B-splines of degree d and smoothness of order k; then the patches $\mathbf{x}^i_m$ are C^k-functions and piecewise polynomial over some rectangular partition of ω not depending on m. Moreover, always two adjacent patches $\mathbf{x}^i_m$ and $\mathbf{x}^{i+1}_m$ have a common boundary curve, and along this curve the transversal derivatives up to order k are equal up to sign, that is

$$D^\nu_u \mathbf{x}^i_m(2,t) = (-1)^\nu \, D^\nu_v \mathbf{x}^{i+1}_m(t,2) \, , \quad \nu = 0,\ldots,k \, , \quad t \in [0,1] \, . \tag{2.4}$$

A subdivision algorithm based on such patches will be called *polynomial of type* (d,k). The space $\mathcal{L}^3$ of layers of the described type is linear, affine invariant, finite dimensional and independent of m. Thus, there exists a set of piecewise polynomial *basis functions* $\mathcal{B} := \{B_1,\ldots,B_J\}$, forming a partition of unity, with respect to which each layer $\mathbf{x}_m$ can be represented as

$$\mathbf{x}_m : \Omega \ni (u,v,i) \longmapsto \mathbf{x}^i_m(u,v) = \sum_{j=1}^{J} \mathbf{X}^{ij}_m \, B_j(u,v) \in \mathcal{L}^3 \, . \tag{2.5}$$

For later use we make $\mathcal{L}^r, r \in \{1,2,3\}$, a finite-dimensional Banach space by providing it with some norm $\|\cdot\|_{\mathcal{L}^r}$. The *control points*[1] $\mathbf{X}^{ij}_m \in \mathbb{R}^3$ are collected in vectors according to the structure of layers,

$$\mathbf{X}^i_m := [\mathbf{X}^{i1}_m,\ldots,\mathbf{X}^{iJ}_m], \tag{2.6}$$

$$\mathbf{X}_m := [\mathbf{X}^1_m,\ldots,\mathbf{X}^n_m] \, . \tag{2.7}$$

[1]This terminology follows the usage in B-spline theory, although the basis functions are not B-splines in the original sense but only segments of B-splines.

Now, a subdivision scheme is assumed to be a sequence of affine invariant maps $\{A_m\}_{m\in\mathbb{N}}$ acting on these vectors according to

$$\mathbf{X}_{m+1} := A_m\,\mathbf{X}_m\,. \tag{2.8}$$

Evidently, the maps A_m must be chosen such that consecutive layers are joined smoothly. But besides that any subdivision algorithm in the scope of interest should satisfy the following requirements:

i) *Convergence.* The sequence of layers $\mathbf{x}_m$ converges uniformly to a unique *center point* $\mathbf{p}\in\mathbb{R}^3$, that is

$$\lim_{m\to\infty}\|\mathbf{x}_m-\mathbf{p}\|_{\mathcal{L}^3}=0\,. \tag{2.9}$$

Note that $\mathbf{p}$ can be identified with an element of $\mathcal{L}^3$ via

$$\mathbf{p}(u,v,i) := \sum_{j=1}^{J}\mathbf{p}\,B_j(u,v)=\mathbf{p}\,. \tag{2.10}$$

ii) *Tangent Plane Continuity.* Consider any sequence of points $\mathbf{p}_m\in\mathbf{x}_m$ and the sequence $\mathbf{n}(\mathbf{p}_m)$ of normal vectors assigned to it. Then there exists a unique limit

$$\lim_{m\to\infty}\mathbf{n}(\mathbf{p}_m)=:\mathbf{n}(\mathbf{p})\,. \tag{2.11}$$

iii) *Regularity.* Tangent plane continuity implies the existence of a well-defined tangent plane at the center point $\mathbf{p}$. However, smoothness in the sense of differential geometry requires that the projection of the limit surface $\mathbf{P}$ to this plane is locally injective. That is, $\mathbf{P}$ can be parametrized regularly over the tangent plane, locally.

iv) *Symmetry.* The algorithm should not distinguish any side of the n-sided configuration, so $SA_m=A_mS$ with $S:[\mathbf{X}_m^1,\mathbf{X}_m^2,\dots,\mathbf{X}_m^n]\mapsto[\mathbf{X}_m^n,\mathbf{X}_m^1,\dots,\mathbf{X}_m^{n-1}]$ denoting the shift operator.

v) *Non-degeneracy.* Even in the regular case, there might always be some exceptional initial data producing cusps or flat spots. However, it is assumed that there exists at least one symmetric arrangement of initial data $\mathbf{X}_0$ corresponding to a limit surface $\mathbf{P}$ with a non-flat center point $\mathbf{p}$. $\mathbf{X}_0$ is called *symmetric* if it is invariant under a $2\pi/n$-rotation R_n^g about some straight line $g\subset\mathbb{R}^3$, i.e. $R_n^g\mathbf{X}_0=S\mathbf{X}_0$.

Definition 2.1. A subdivision algorithm satisfying i) – v) is called *proper.*

3. The degree estimate

For our purposes we need the following characterization of regular C^k-surfaces, which will be stated without proof.

Proposition 3.1. *Let* $\mathbf{Q}$ *be a regular* C^k*-surface,* $k\ge 1$*. Then, for any* $\mathbf{q}\in\mathbf{Q}$ *and any pair of orthogonal unit vectors* $\mathbf{t}_1,\mathbf{t}_2\in\mathbb{R}^3$ *spanning the tangent plane at* $\mathbf{q}$*, there exists a unique local representation of* $\mathbf{Q}$ *according to*

$$U\ni(u,v)\longmapsto\mathbf{q}+u\mathbf{t}_1+v\mathbf{t}_2+h(u,v)(\mathbf{t}_1\times\mathbf{t}_2)\in\mathbf{Q}\,, \tag{3.1}$$

where $U\subset\mathbb{R}^2$ *is some neighborhood of the origin and* $h(\underline{u})=\mathrm{o}(\|\underline{u}\|)$, $\underline{u}:=(u,v)^T$*, is a* C^k*-function. If* $\mathbf{Q}$ *is* curvature continuous, *i.e.* $k\ge 2$*, then* h *is of type*

$$h(\underline{u})=(1/2)\,\underline{u}^T H\underline{u}+\mathrm{o}(\|\underline{u}\|^2)\quad as\quad \|\underline{u}\|\to 0 \tag{3.2}$$

with the Hessian H being a symmetric (2×2)-matrix. The eigenvalues κ_1, κ_2 of H are the main curvatures *of* $\mathbf{Q}$ *at* $\mathbf{q}$, *and* $\mathbf{q}$ *is said to be* flat *if* $\kappa_1 = \kappa_2 = 0$.

Lemma 3.1. *Consider a proper subdivision algorithm generating a curvature continuous limit surface* $\mathbf{P}$ *from some symmetric initial value* $\mathbf{X}_0$, *and assume that the center point* $\mathbf{p}$ *is not flat. Further, let the coordinate system be chosen such that* $\mathbf{p}$ *is the origin and the xy-plane is the tangent plane at this point. Then* $\mathbf{P}$ *is invariant under* R_n^z, *the* $2\pi/n$*-rotation about the z-axis, and the main curvatures at* $\mathbf{p}$ *coincide.*

Proof. Affine invariance and symmetry of A_m imply that the composed maps

$$A^m := \prod_{\mu=0}^{m-1} A_\mu \ , \ m \in \mathbb{N} \ , \tag{3.3}$$

commute with both the rotation R_n^z and the shift S. Assuming that the initial data $\mathbf{X}_0$ is symmetric, i.e. $R_n^z\mathbf{X}_0 = S\mathbf{X}_0$, we obtain

$$\begin{aligned} R_n^z \mathbf{x}_m^i &= R_n^z \sum_{j=1}^J \mathbf{X}_m^{ij} B_j = \sum_{j=1}^J R_n^z (A^m \mathbf{X}_0)^{ij} B_j = \sum_{j=1}^J (R_n^z A^m \mathbf{X}_0)^{ij} B_j \\ &= \sum_{j=1}^J (A^m R_n^z \mathbf{X}_0)^{ij} B_j = \sum_{j=1}^J (A^m S\mathbf{X}_0)^{ij} B_j = \sum_{j=1}^J (SA^m \mathbf{X}_0)^{ij} B_j \\ &= \sum_{j=1}^J (S\mathbf{X}_m)^{ij} B_j = \sum_{j=1}^J \mathbf{X}_m^{(i+1)j} B_j = \mathbf{x}_m^{i+1} \ . \end{aligned} \tag{3.4}$$

Consequently, the limit surface $\mathbf{P}$ is symmetric, too. According to (3.1), $\mathbf{P}$ can be parametrized over the xy-plane near the origin by

$$z = (1/2)\, \underline{x}^T H \underline{x} + \mathrm{o}(\|\underline{x}\|^2) \quad \text{as} \quad \|\underline{x}\| \to 0 \tag{3.5}$$

with $\underline{x} := (x, y)^T$. By symmetry, an alternate representation of $\mathbf{P}$ is

$$z = (1/2)\, \underline{x}^T R_n^T H R_n \underline{x} + \mathrm{o}(\|\underline{x}\|^2) \quad \text{as} \quad \|\underline{x}\| \to 0 \tag{3.6}$$

with

$$R_n := \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix} \tag{3.7}$$

denoting the $2\pi/n$-rotation of the xy-plane about the origin. Comparison of (3.5) and (3.6) yields $HR_n = R_n H$ and

$$H = \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix} \tag{3.8}$$

by inspection. □

Since $\mathbf{p}$ was assumed not to be flat, the value of κ is non-zero and can be set to $\kappa = 2$ by a scaling of the z-axis. Thus, we obtain the representation

$$z_m = \underline{x}_m^T\, \underline{x}_m + \mathrm{o}(\|\underline{x}_m\|^2_{\mathcal{L}^2}) \quad \text{as} \quad \|\underline{x}_m\|_{\mathcal{L}^2} \to 0 \tag{3.9}$$

for $\mathbf{P}$ near the origin. This implies

$$z_m = \underline{x}_m^T\, \underline{x}_m + \mathrm{o}(\|\underline{x}_m\|^2_{\mathcal{L}^2}) \quad \text{as} \quad m \to \infty \tag{3.10}$$

since the layers $\mathbf{x}_m$ are converging to the origin. Introducing the rescaled quantities

$$\tilde{\underline{x}}_m := \underline{x}_m / \|\underline{x}_m\|_{\mathcal{L}^2} \tag{3.11}$$

$$\tilde{z}_m := z_m / \|\underline{x}_m\|^2_{\mathcal{L}^2} \tag{3.12}$$

we obtain

$$\tilde{z}_m = \tilde{\underline{x}}_m^T \, \tilde{\underline{x}}_m + \mathrm{o}(1) \quad \text{as} \quad m \to \infty \, . \tag{3.13}$$

$\tilde{\underline{x}}_m$ is a bounded sequence in $\mathcal{L}^2$ by construction, and consequently, by the latter equation, $\tilde{z}_m$ is bounded in $\mathcal{L}$, too. Hence, the theorem of Bolzano-Weierstraß guarantees the existence of a convergent subsequence $(\tilde{\underline{x}}_{\sigma(m)}, \tilde{z}_{\sigma(m)})$ and the limit

$$(\hat{\underline{x}}, \hat{z}) := \lim_{m\to\infty} (\tilde{\underline{x}}_{\sigma(m)}, \tilde{z}_{\sigma(m)}) \tag{3.14}$$

satisfies

$$\hat{z} = \hat{\underline{x}}^T \hat{\underline{x}} = \hat{x}^2 + \hat{y}^2 \, . \tag{3.15}$$

This formula is the key to our main result:

Theorem 3.1. *A proper subdivision algorithm of type (d, k) can generate curvature continuous surfaces only if $d \geq 2k + 2$.*

Proof. Denote the actual bi-degree of the piecewise polynomials $\hat{x}, \hat{y}, \hat{z}$ by $d_{\hat{x}}, d_{\hat{y}}, d_{\hat{z}}$, respectively. Then (3.15) implies

$$d \geq d_{\hat{z}} = 2 \max\{d_{\hat{x}}, d_{\hat{y}}\} =: 2d_* \, . \tag{3.16}$$

Assuming $d_* \leq k$ has the following consequences: Define the affine transformation τ in $\mathbb{R}^2$ by

$$\tau : (u, v) \longmapsto (4 - v, u), \tag{3.17}$$

$$\tau^{-1} : (u, v) \longmapsto (v, 4 - u) \, . \tag{3.18}$$

Then, any two adjacent patches $\hat{\underline{x}}^i$ and $\hat{\underline{x}}^{i+1}$ of $\hat{\underline{x}}$ can be combined to a single piecewise polynomial function $\hat{\underline{x}}^{i,i+1}$ over $\omega \cup \tau(\omega)$ by

$$\hat{\underline{x}}^{i,i+1}(u, v) := \begin{cases} \hat{\underline{x}}^i(u, v) & \text{if } (u, v) \in \omega, \\ \hat{\underline{x}}^{i+1}(\tau^{-1}(u, v)) & \text{if } (u, v) \in \tau(\omega) \end{cases} \tag{3.19}$$

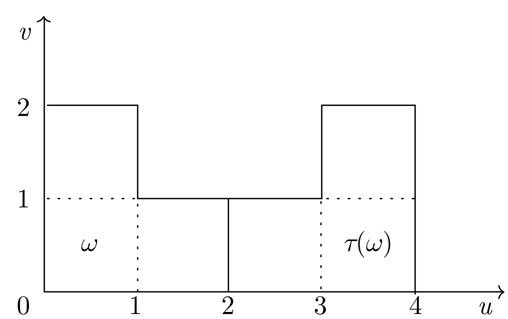

FIGURE 2. Domain of the combined function $\hat{\underline{x}}^{i,i+1}$

(see Figure 2). For the ν-fold partial derivative in u-direction we obtain using the chain rule

$$D_u^\nu \, \underline{\hat{x}}^{i,i+1}(u,v) := \begin{cases} D_u^\nu \, \underline{\hat{x}}^i(u,v) & \text{if } (u,v) \in \omega, \\ (-1)^\nu D_v^\nu \, \underline{\hat{x}}^{i+1}(\tau^{-1}(u,v)) & \text{if } (u,v) \in \tau(\omega) \, . \end{cases} \tag{3.20}$$

Evaluation at the common boundary $(2,t), t \in [0,1]$, yields

$$\lim_{\varepsilon \downarrow 0} D_u^\nu \, \underline{\hat{x}}^{i,i+1}(2-\varepsilon,t), = D_u^\nu \, \underline{\hat{x}}^i(2,t), \tag{3.21}$$

$$\lim_{\varepsilon \downarrow 0} D_u^\nu \, \underline{\hat{x}}^{i,i+1}(2+\varepsilon,t) = (-1)^\nu D_v^\nu \, \underline{\hat{x}}^{i+1}(t,2), \tag{3.22}$$

and by (2.4) both expressions coincide for $\nu = 0, \ldots, k$. So, on one hand, $\underline{\hat{x}}^{i,i+1}$ is a C^k-function and, on the other hand, it is piecewise polynomial of bi-degree $d_* \le k$. Consequently, $\underline{\hat{x}}^{i,i+1}$ as well as $\underline{\hat{x}}^i$ and $\underline{\hat{x}}^{i+1}$ are in fact not piecewise but *single* bivariate polynomials. Denote their extensions to $\mathbb{R}^2$ by $\underline{\hat{x}}_e^{i,i+1}$, $\underline{\hat{x}}_e^i$ and $\underline{\hat{x}}_e^{i+1}$, respectively; then both branches in (3.19) represent the same polynomial, i.e.

$$\underline{\hat{x}}_e^{i,i+1} = \underline{\hat{x}}_e^i = \underline{\hat{x}}_e^{i+1} \circ \tau^{-1} \, . \tag{3.23}$$

Moreover, symmetry implies $\underline{\hat{x}}_e^{i+1} = R_n \underline{\hat{x}}_e^i$ and thus

$$\underline{\hat{x}}_e^i = R_n \, \underline{\hat{x}}_e^i \circ \tau^{-1} \, . \tag{3.24}$$

Iterating this equation yields

$$\underline{\hat{x}}_e^i = R_n^4 \, \underline{\hat{x}}_e^i \circ \tau^{-4} = R_n^4 \, \underline{\hat{x}}_e^i \tag{3.25}$$

since τ^{-4} is the identity. R_n^4 has no real eigenvectors for $n \in \mathbb{N} \backslash \{1,2,4\}$ and so, we finally obtain $\underline{\hat{x}}^i = 0, i = 1, \ldots, n$. This is a contradiction since by construction $\|\underline{\hat{x}}\|_{\mathcal{L}^2} = 1$. Consequently, the assumption $d_* \le k$ was not correct and we obtain

$$d \ge 2d_* \ge 2(k+1) = 2k+2 \tag{3.26}$$

as stated. □

4. Conclusion

A necessary condition for constructing curvature continuous subdivision algorithms based on quadrilateral polynomial patches is given. It relates the polynomial degree d and the smoothness order k by the estimate $d \ge 2k+2$. Some simple consequences are:

- The simplest scheme providing curvature continuity is at least polynomial of type $(6,2)$ since, evidently, the minimal value for k is $k_{\min} = 2$.
- Being polynomial of type $(3,2)$, the algorithm of Catmull-Clark cannot be adjusted to converge to curvature continuous limit surfaces.

Naturally, the question arises whether the given estimate is sharp or not. The answer is affirmative and, moreover, constructive. This will be shown in a forthcoming publication by specifying a class of proper subdivision algorithms of type $(2k+2,k)$ generating C^k-limit surfaces.

When seeking non-polynomial methods, like interpolatory schemes, it should be taken into account that equation (3.15) is equally necessary since its deduction is independent of the particular type of basis functions. This implies that the space $\mathcal{L}^3$ of surface layers must be chosen such that it admits the representation of quadratic functions.

References

[CC78] E. Catmull and J. Clark, *Recursively generated B-spline surfaces on arbitrary topological meshes*, Computer Aided Design **10** (1978), 350 – 355.

[DHK93] T. DeRose, M. Halstead, and M. Kass, *Efficient, fair interpolation using Catmull-Clark surfaces*, Proceedings of Siggraph '93, Apple Computer, Inc., 1993, pp. 35 – 44.

[DLG90] N. Dyn, D. Levin, and J. A. Gregory, *A butterfly subdivision scheme for surface interpolation with tension control*, ACM Transactions on Graphics **9** (1990), 160 – 169.

[DLW93] T. DeRose, M. Lounsbery, and J. Warren, *Multiresolution analysis for surfaces of arbitrary topological type*, Tech. Report 93-10-05, University of Washington, 1993.

[DS78] D. Doo and M. A. Sabin, *Behaviour of recursive subdivision surfaces near extraordinary points*, Computer Aided Design **10** (1978), 356 – 360.

[Loo87] Ch. T. Loop, *Smooth subdivision for surfaces based on triangles*, Master's thesis, University of Utah, 1987.

[Rei93] U. Reif, *Neue Aspekte in der Theorie der Freiformflächen beliebiger Topologie*, Ph.D. thesis, Universität Stuttgart, 1993. MR **95f:**65042

[Rei95] U. Reif, *A unified approach to sudivision algorithms near extraordinary vertices*, Computer Aided Geometric Design **12** (1995), 153 – 174.

Mathematisches Institut A, Universität Stuttgart, Pfaffenwaldring 57, 70550 Stuttgart Germany

E-mail address: `reif@mathematik.uni-stuttgart.de`